Non-E-overlapping and weakly shallow TRSs are confluent (Extended abstract)

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1 Introduction

Confluence of term rewriting systems (TRSs) is undecidable, even for flat TRSs [MOJ06] or length-two string rewrite systems [SW08]. Two decidable subclasses are known: right-linear and shallow TRSs by tree automata techniques [GT05] and terminating TRSs [KB70]. Most of sufficient conditions are for either terminating TRSs [KB70] (extended to TRSs with relative termination [HA11, KH12]) or left-linear non-overlapping TRSs (and their extensions) [Ros73, Hue80, Toy87, Oos95, Oku98, OO97]. For non-linear TRSs, a goal is RTA open problem **58** "*strongly non-overlapping and right-linear TRSs are confluent*". A best known result strengthens the *right-linear* assumption to *simple-right-linear* [TO95, OOT95], which means that each rewrite rule is right-linear and no left-non-linear variables appear in the right hand side. Other trials by depth-preserving conditions are found in [GOO98].

We have proposed a different methodology, called a *reduction graph* [SO10]. It has shown that "*weakly non-overlapping, shallow, and non-collapsing TRSs are confluent*". An original idea comes from observation that, when non-E-overlapping, peak-elimination uses only "*copies*" of reductions in an original rewrite sequences. Thus, if we focus on terms appearing in peak elimination, they are finitely many. We regard a rewrite relation over these terms as a directed graph, and we construct a confluent directed acyclic graph (DAG) in a bottom-up manner, in which the shallow assumption works. The keys are, a connected convergent DAG always has a unique normal form (if it is finite), and convergence is preserved if we add an arbitrary reduction starting from that normal form.

This paper briefly sketches that "*non-E-overlapping and weakly-shallow TRSs are confluent*" by extending *reduction graph* in our previous work [SO10] by introducing *constructor expansion.* A term is weakly shallow if each defined function symbol appears either at the root or in the ground subterms, and a TRS is weakly shallow if the both sides of rules are weakly shallow. The non-E-overlapping property is undecidable for weakly shallow TRSs [MOM12] and a decidable sufficient condition is the strongly non-overlapping condition. A Turing machine can be simulated by a weakly shallow TRS (p.27 in [Klo93]); thus the word problem is undecidable, in contrast to shallow TRSs [CHJ94].

Basic definitions and notations

We follow standard definitions and terminology of graphs and TRSs [BN98]. As notational convention, V for a finite set (often of terms), F is a finite set of function symbols, D and C are the sets of *defined* and *constructor symbols* in F, respectively. X is the set of variables. We use s, t, u, v, w for terms, x, y for variables, p, q for positions, σ, θ for substitutions, $\ell \to r$ for a rewrite rule, and R for a TRS.

An *abstract reduction system* (ARS) is a directed graph $G = \langle V, \to\rangle$ with $\to \subseteq V \times V$. For $V', V'' \subseteq V$, $\to|_{V' \times V''} = \to \cap (V' \times V'')$. We write V_G and $\to_G$ to emphasize G. An edge $v \to u$ is an *out-edge* of v and an *in-edge* of u. A node v is $\to$-normal if it has no out-edges. Let $G = \langle V, \to\rangle$ and $G' = \langle V', \to'\rangle$. The union $G \cup G'$ is $\langle V \cup V', \to \cup \to'\rangle$. We say G is *finite* if V is finite, G is *convergent* if G is confluent and terminating, G' *includes* G (denoted by $G' \supseteq G$) if $V' \supseteq V$ and $\to' \supseteq \to$, and G' *weakly subsumes* G (denoted by $G' \sqsupseteq G$) if $V' \supseteq V$ and $\leftrightarrow'^* \supseteq \to$.

We use $\mathrm{sub}(t)$ for the set of *direct subterms* of a term t defined as $\mathrm{sub}(t) = \emptyset$ if t is a variable and $\mathrm{sub}(t) = \{t_1, \ldots, t_n\}$ if $t = f(t_1, \ldots, t_n)$. $s \xrightarrow[R]{p} t$ is a *top reduction* if $p = \varepsilon$. Otherwise, it is a *non-top* *reduction*, written as $s \overset{\varepsilon<}{\underset{R}{\to}} t$. We use $T|_f$ to denote the subset of $T \subseteq \mathrm{T}(F, X)$ and $f \in F$ that consists of the terms in T with the root symbol f. For $F' \subseteq F$, we use $T|_{F'}$ to denote $\cup_{f \in F'} T|_f$.

A *weakly shallow term* is a term in which defined function symbols appear only either at the root or in the ground subterms (i.e., $p \neq \varepsilon$ and $\mathrm{root}(s|_p) \in D$ imply that $s|_p$ is ground). A rewrite rule $\ell \to r$ is *weakly shallow* if ℓ and r are weakly shallow. A TRS is *weakly shallow* if each rewrite rule is weakly shallow. We assume that a TRS has finitely many rewrite rules.

Let $\ell_1 \to r_1, \ell_2 \to r_2 \in R$. If there exist substitutions θ_1, θ_2 for $p \in \mathrm{Pos}_X(\ell_1)$ such that $\ell_1|_p\theta_1 = \ell_2\theta_2$ (resp. $\ell_1|_p\theta_1 \overset{\varepsilon<}{\underset{R}{\leftrightarrow}}{}^* \ell_2\theta_2$), $(r_1\theta_1, (\ell_1\theta_1)[r_2\theta_2]_p)$ is a *critical pair* (resp. *E-critical pair*) except that $p = \varepsilon$ and the two rules are identical (up to renaming variables). A TRS R is *overlapping* (resp. *E-overlapping*, strongly overlapping) if there exists a critical pair (resp. E-critical pair, critical pair of linearizatoin of R). Note that when a TRS is left-linear, they are equivalent.

2 Extensions of convergent abstract reduction systems

Definition 2.1. For ARSs $G_1 = \langle V_1, \to_1\rangle$ and $G_2 = \langle V_2, \to_2\rangle$, we say that $G_1 \cup G_2$ is the *hierarchical combination of G_2 with G_1*, denoted by $G_1 \gtrdot G_2$, if $\to_1 \subseteq (V_1 \setminus V_2) \times V_1$.

Lemma 2.2. *Let $G_1 \gtrdot G_2$ be a convergent hierarchical combination of ARSs. If a convergent ARS G_3 weakly subsumes G_2 and $G_1 \gtrdot G_3$ is a hierarchical combination, then $G_1 \gtrdot G_3$ is convergent.*

Definition 2.3. Let $G = \langle V, \to\rangle$ be a convergent ARS and $v \neq v'$. Let G' be obtained by:

$$\begin{cases} \langle V \cup \{v'\}, \to \cup \{(v, v')\}\rangle & \text{if } v \in V \text{ is } \to\text{-normal, and } v' \notin V \\ \langle V, \to \cup \{(v, v')\}\rangle & \text{if } v \in V \text{ is } \to\text{-normal, } v' \in V \text{ and } v' \not\leftrightarrow^* v \\ \langle V, \to \setminus \{(v', v'') \mid v' \to v''\} \cup \{(v, v')\}\rangle & \text{if } v \in V \text{ is } \to\text{-normal, } v' \in V\text{, and } v' \leftrightarrow^* v \\ \langle V \cup \{v, v'\}, \to \cup \{(v, v')\}\rangle & \text{if } v \notin V \\ \text{Undefined} & \text{otherwise} \end{cases}$$

We denote G' by $\langle V, \to\rangle \multimap (v \to v')$ if G' is defined (i.e., the first four cases). We denote $G \multimap (v_0 \to v_1) \multimap (v_1 \to v_2) \multimap \cdots \multimap (v_{n-1} \to v_n)$ as $G \multimap (v_0 \to v_1 \to \cdots \to v_n)$.

Proposition 2.4. *Let $G = \langle V, \to\rangle$ be a convergent ARS. Let $v_0, v_1, \ldots, v_n$ satisfy $v_i \neq v_j$ (for $i \neq j$), and the following conditions:*

i) if $v_0 \in V$, then v_0 is $\to$-normal and $v_i \in V$ implies $v_i \leftrightarrow^* v_0$ for each $i(< n)$,

ii) if $v_0 \notin V$, then $v_1, \cdots, v_{n-1} \notin V$.

Then, $G' = G \multimap (v_0 \to v_1 \to \cdots \to v_n)$ is convergent, and satisfies $G' \sqsupseteq G$.

3 Reduction graphs

Definition 3.1 ([SO10])**.** A finite ARS $G = \langle V, \to\rangle$ is an *R-reduction graph* if $V \subseteq \mathrm{T}(F, X)$ and $\to \subseteq \underset{R}{\to}$.

For an R-reduction graph $G = \langle V, \to\rangle$, *top-edges*, *inner-edges*, and *strict inner-edges* are given as $\overset{\varepsilon}{\to} = \to \cap \overset{\varepsilon}{\underset{R}{\to}}$, $\overset{\varepsilon<}{\to} = \to \cap \overset{\varepsilon<}{\underset{R}{\to}}$, and $\overset{\neq\varepsilon}{\to} = \to \setminus \overset{\varepsilon}{\underset{R}{\to}}$, respectively. We use G^{ϵ}, $G^{\varepsilon<}$, and $G^{\neq\varepsilon}$ to denote $\langle V, \overset{\varepsilon}{\to}\rangle$ $\langle V, \overset{\varepsilon<}{\to}\rangle$, and $\langle V, \overset{\neq\varepsilon}{\to}\rangle$, respectively. Remark that an edge $(s, t) \in \to$ may be both $\overset{\varepsilon}{\to}$ and $\overset{\varepsilon<}{\to}$, e.g., $(f(a, a), f(b, a))$ for $R = \{a \to b, f(x, x) \to f(b, a)\}$. For an R-reduction graph $G = \langle V, \to\rangle$ and $F' \subseteq F$, we represent $G|_{F'} = \langle V, \to|_{F'}\rangle$ where $\to|_{F'} = \to|_{V|_{F'} \times V}$.

Definition 3.2. Let $G = \langle V, \to \rangle$ be an R-reduction graph. The *direct-subterm reduction-graph* $\mathrm{sub}(G)$ of G is $\langle \mathrm{sub}(V), \mathrm{sub}(\to) \rangle$ where $\langle \mathrm{sub}(V), \mathrm{sub}(\to) \rangle = \langle \bigcup_{t \in V} \mathrm{sub}(t), \{(s_i, t_i) \mid f(s_1, \ldots, s_n) \overset{\varepsilon\leq}{\to} f(t_1, \ldots, t_n),\ s_i \neq t_i\} \rangle$. An R-reduction graph $G = \langle V, \to \rangle$ is *subterm-closed* if $sub(V) \subseteq V$ and $\mathrm{sub}(\overset{\neq\varepsilon}{\to}) \subseteq \leftrightarrow^*$.

Lemma 3.3. *Let $G = \langle V, \to \rangle$ be a subterm-closed R-reduction graph. Assume that $p \in \mathrm{Pos}(s)$ for a term s and $s[t]_p \leftrightarrow^* s[t']_p$, in which any reductions do not occur above p. Then $t \leftrightarrow^* t'$.*

Definition 3.4. Let $G = \langle V, \to \rangle$ be an R-reduction graph and F' $(\subseteq F)$. The *F'-monotonic extension* is

$$M_{F'}(G) = \langle V_1, \to_1 \rangle \quad \text{for} \quad \begin{cases} V_1 &= \{f(s_1, \ldots, s_n) \mid f \in F',\ s_1, \ldots, s_n \in V\}, \\ \to_1 &= \{(f(\cdots s \cdots), f(\cdots t \cdots)) \in V_1 \times V_1 \mid s \to t\}. \end{cases}$$

When G is subterm-closed, an *C-expansion* $\overline{M_C}(G)$ is the hierarchical combination $G|_D \gtrdot M_C(G)$ $(= G|_D \cup M_C(G))$. The k-times application of $\overline{M_C}$ to G is denoted by $\overline{M_C}^k(G)$.

Example 3.5. As a running example, we use a TRS $R_2 = \{f(x, g(x)) \to g^3(x),\ c \to g(c)\ \}$ with $C = \{g\}$ and $D = \{c, f\}$. Consider a subterm-closed R_2-reduction graph $G = \langle \{c, g(c), g^2(c)\}, \{(c, g(c))\} \rangle$. For easy description, we also denote as $G = \{c \to g(c),\ \ g^2(c)\}$. Then, $M_C(G) = \{g(c) \to g^2(c),\ \ g^3(c)\}$, $\overline{M_C}(G) = \{c \to g(c) \to g^2(c),\ \ g^3(c)\}$, $\overline{M_C}^3(G) = \{c \to g(c) \to g^2(c) \to g^3(c) \to g^4(c),\ \ g^5(c)\}$.

Lemma 3.6. *For a subterm-closed R-reduction graph G and $m > k \geq 0$, (1) $G \sqsubseteq \overline{M_C}^k(G)$, (2) $\overline{M_C}^k(G)$ is subterm-closed, (3) $\overline{M_C}^k(G)$ is convergent, if G is convergent, and (4) $\overline{M_C}^k(G) \sqsubseteq \overline{M_C}^m(G)$.*

4 Constructor expansion

In Section 4 and 5, given an R-reduction graph G_0, we show how to inductively construct a convergent and subterm-closed R-reduction graph G_4 with $G_0 \sqsubseteq G_4$. Note that Section 5 assumes that a TRS R is non-E-overlapping and weakly shallow. Throughout these sections, we fix the notations.

- Given an R-reduction graph $G_0 = \langle V_0, \to_0 \rangle$ as an input.
- $G = \langle V, \to \rangle$ is used to denote a convergent and subterm-closed R-reduction graph that weakly subsumes $\mathrm{sub}(G_0)$ (by induction hypothesis).
- $G_1 = \langle V_1, \to_1 \rangle$ denotes a convergent R-reduction graph with $M_F(G) \sqsubseteq G_1$ (by Lemma 4.1).
- $G_{2_i} = \langle V_{2_i}, \to_{2_i} \rangle$ denotes $M_F(\overline{M_C}^i(G))$ for $i \geq 0$.
- T denotes a subgraph of $(G_0^\epsilon \cup G^\epsilon) \setminus (G_0^{\epsilon<} \cup G^{\epsilon<})$ such that T modulo $\leftrightarrow_1^*$ is acyclic and preserves connectivity of $(G_0^\epsilon \cup G^\epsilon) \setminus (G_0^{\epsilon<} \cup G^{\epsilon<})$ modulo $\leftrightarrow_1^*$.
- We repeatedly expand G_1 (by Lemma 5.1 and 5.2) by adding edges of T from nodes with out-edges only to sink order, and construct a convergent and subterm-closed G_4 with $G_0 \sqsubseteq G_4$.

If $G_i = \langle V_i, \to_i \rangle$ is convergent, we refer the normal form (in G_i) of a term $u (\in V_i)$ by $u{\downarrow_i}$.

Lemma 4.1. *For a convergent and subterm-closed R-reduction graph G, there exist k (≥ 0) and an R-reduction graph G_1 satisfying the following conditions: i) G_1 is convergent, and consists of non-top edges, ii) $G_1 \sqsubseteq G_{2_k}$, iii) $u \leftrightarrow_{2_i}^* v$ implies $u \leftrightarrow_1^* v$ for each $u, v \in V_1$ and i (≥ 0), and iv) $M_F(G) \sqsubseteq G_1$.*

Example 4.2. Consider R_2 in Example 3.5. Let $G_0 = \{f(g(c), c) \leftarrow f(c, c) \to f(c, g(c)) \overset{\varepsilon}{\to} g^3(c)\}$. The subterm graph $\mathrm{sub}(G_0)$ is equal to G in Example 3.5, and is convergent and subterm-closed. Then, Lemma 4.1 starts from $M_F(G)$, which is displayed by the solid arrows in Figure 1. An example of G_1 is constructed by augmenting the dashed edges with $k = 1$.

$$
\begin{array}{ccccccc}
c & & f(c,c) & \rightarrow & f(g(c),c) & & f(g^2(c),c) \\
\dot{\downarrow} & & \downarrow & & \downarrow & & \downarrow \\
g(c) & & f(c,g(c)) & \rightarrow & f(g(c),g(c)) & \dashrightarrow & f(g^2(c),g(c)) \\
\downarrow & & & & & & \vdots\downarrow \\
g^2(c) & & f(c,g^2(c)) & \rightarrow & f(g(c),g^2(c)) & \dashrightarrow & f(g^2(c),g^2(c)) \\
\vdots\downarrow & & & & & & \| \\
g^3(c) & \dot{\rightarrow} & g^4(c) & \dot{\rightarrow} & g^5(c) & \dot{\leftarrow} & f(g^2(c),g^2(c))
\end{array}
$$

Figure 1: G_4 (dot arrows) and G_1 (dashed arrows) starting from G_0 (solid arrows) in Example 3.5

5 Merging top edges to the direct-subterm graph

Let $G_1 = \langle V_1, \rightarrow_1 \rangle$ and $T_1 = \langle V_{T_1}, \rightarrow_{T_1} \rangle$ be R-reduction graphs with $V_{T_1} \subseteq V_1$. The *component graph*, denoted by T_1/G_1, of T_1 with G_1, is the graph $\langle V, \rightarrow \rangle$ having connected components of G_1 as nodes and $\rightarrow_{T_1/G_1}$ as edges such that $V = \{[v]_{\leftrightarrow^*_1} \mid v \in V_1\}$ and $\rightarrow_{T_1/G_1} = \{([u]_{\leftrightarrow^*_1}, [v]_{\leftrightarrow^*_1}) \mid (u,v) \in \rightarrow_{T_1}\}$. If clear from the context, we simply denote $[v]$ instead of $[v]_{\leftrightarrow^*_1}$.

Lemma 5.1. *Let G be a convergent subterm-closed R-reduction graph, $G_1 = \langle V_1, \rightarrow_1 \rangle$, and k as in Lemma 4.1. Let $\rightarrow_S, \rightarrow_T \subseteq V_1 \times V_1$ such that $\rightarrow_S = \xrightarrow{\epsilon}_S$, $\rightarrow_T = \xrightarrow{\epsilon}_T$, $\xrightarrow{\varepsilon}_G \subseteq (\leftrightarrow_S \cup \leftrightarrow_T \cup \leftrightarrow_1)^\varepsilon$, and*

v) The component graph $(S \cup T)/G_1$ is acyclic, where out-edges are at most one for each node. Moreover, if $[u]_{\leftrightarrow^*_1}$ has an in-edge in T/G_1 then it has no edges in S/G_1.

vi) u is $\rightarrow_1$-normal for each $(u,v) \in S$.

If $T \neq \emptyset$, there is a tuple (S', T', G_1', k') such that $|T| > |T'|$ and the conditions i) to vi), (1) $\leftrightarrow^*_1 \subseteq \leftrightarrow^*_{1'}$ and (2) $(\leftrightarrow_T \cup \leftrightarrow_S)^* \subseteq (\leftrightarrow_{T'} \cup \leftrightarrow_{S'} \cup \leftrightarrow_{1'})^*$ hold. We denote it by $(S, T, G_1, k) \vdash (S', T', G_1', k')$.

A convergent reduction graph $G_4 = \langle V_4, \rightarrow_4 \rangle$ with $G_0 \sqsubseteq G_4$ is obtained from $S = \phi$, T (after $\vdash$ in Lemma 5.1 is preprocessed), and G_1 by repeated applications of $\vdash_l$, $\vdash_r$, and $\vdash_e$ below. For $(\ell\sigma, r\sigma) \in T$, there are $h \geq k$ and a substitution θ with $(\ell\sigma)\downarrow_1 = u_0(\xrightarrow[R]{\varepsilon<} \cap \leftrightarrow^*_{2_h})u_1(\xrightarrow[R]{\varepsilon<} \cap \leftrightarrow^*_{2_h}) \cdots (\xrightarrow[R]{\varepsilon<} \cap \leftrightarrow^*_{2_h})u_n = \ell\theta$.

$$
\textit{Let} \quad \begin{cases}
(S, T, G_1, k) \vdash_l (S, T, G_{1^l}, h) & \text{by } G_{1^l} = G_1 \multimap (u_0 \rightarrow \cdots \rightarrow u_n). \\
(S, T, G_{1^l}, h) \vdash_r (S, T, G_{1'}, k') & \text{for } w \in V_1 \text{ such that } w \text{ is } \rightarrow_{1^l}\text{-normal, and } w \leftrightarrow^*_{2_{k'}} r\theta. \\
(S, T, G_{1'}, k') \vdash_e (S', T', G_{1'}, k') & \text{for } S' = S \cup \{(\ell\theta, r\theta)\} \text{ and } T' = T \setminus \{(\ell\sigma, r\sigma)\}.
\end{cases}
$$

Lemma 5.2. *Let $G_0 = \langle V_0, \rightarrow_0 \rangle$ be an R-reduction graph. Then, there exists a convergent and subterm-closed R-reduction graph G_4 with $G_0 \sqsubseteq G_4$.*

Example 5.3. Let us consider to apply Lemma 5.2 on G_0 in Example 4.2. First, we take a convergent subterm-closed R_2-reduction graph that weakly subsumes $\mathrm{sub}(G_0)$. This graph is essentially the same as G in Example 3.5, containing some garbage. For simplicity, we use G in Example 3.5. As in Example 4.2, we obtain G_1 and $k = 1$. Let $T = (G_0^\varepsilon \cup G^\varepsilon) \setminus (G_0^{\varepsilon<} \cup G^{\varepsilon<})$, where G_0^ε and G^ε have the only edges $f(c, g(c)) \rightarrow g^3(c)$ and $c \rightarrow g(c)$, respectively.

The conversion $\vdash$ is applied twice, corresponding to two edges in T. The edge $c \to g(c)$ in T is simply moved to S. For the edge $f(c, g(c)) \to g^3(c)$ in T, $\vdash_l$ adds $f(g^2(c), g^2(c)) \to f(g^2(c), g^3(c))$ to G_1. $\vdash_r$ adds $g^3(c) \to g^4(c) \to g^5(c)$ to G_1 and increases k to 3. $\vdash_e$ adds $f(g^2(c), g^3(c)) \to g^5(c)$ to S. They are denoted by dotted arrows. Since $M_C(\overline{M_C}^3(G))$ is $\{g(c) \to g^3(c) \to \cdots \to g^4(c) \to g^5(c),\ \ g^6(c)\}$, $G_4 = (S \cup G_1|_D) \gg M_C(\overline{M_C}^2(G))$ is as in Figure 1, in which some garbage nodes are not presented.

Main Theorem *Non-E-overlapping and weakly-shallow TRSs are confluent.*

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