Semantics and Strong Sequentiality of Priority Term Rewriting Systems

Masahiko Sakai

Graduate School of Engineering, Nagoya University
Furo-cho, Chikusa, Nagoya 464-01, Japan
sakai@nuie.nagoya-u.ac.jp

Yoshihito Toyama

Japan Advanced Institute of Science and Technology, Hokuriku
Tatsunokuchi, Ishikawa 923-12, Japan
toyama@jaist.ac.jp

Abstract

This paper gives an operational semantics of priority term rewriting systems (PRSs) by using conditional systems, whose reduction relation is decidable and stable under substitution. We also define the class of strongly sequential PRSs and show that this class is decidable. Moreover, we show that the index rewriting of strongly sequential PRSs gives a normalizing strategy.

1 Introduction

Baeten, Bergstra, Klop and Weijland [1] proposed the formalism of priority term rewriting systems (PRSs) to capture the theoretical basis of order dependency in rules. A PRS differs from an ordinary TRS in that whenever a rule matches a given redex, it can only be chosen if none of the rules with higher priority can match the (internally rewritten) term, where a reduction step is internal if it proceeds entirely within the arguments of the leftmost symbol of the term. For example, the factorial function can be defined as follows:

$$\mathcal{R} = \left\{ \begin{array}{l} fac(0) \to 1 \\ \downarrow \\ fac(x) \to x * fac(x-1). \end{array} \right.$$

Here, the down arrow sign indicates that the top rule has higher priority than the bottom rule. If the argument of fac is 0 then the first rule is applicable.

However, if it is somehow known that the argument can never be reduced to 0, then the priority of the rules makes it possible to choose the second rule without actually reducing the argument. Hence, the second rule can be applied to $fac(1+1)$ but not to $fac(1-1)$.

This situation can be naively expressed as follows by using conditions like conditional term rewriting system (CTRS):

$$\mathcal{R}_{naive} = \begin{cases} fac(0) \to 1 \\ fac(x) \to x * fac(x-1) \ \Leftarrow \ \neg(x \xrightarrow{*} 0). \end{cases}$$

Unfortunately, since the negation of conditional part sometimes makes paradoxical situation, the semantics is not well defined. For example,

$$\mathcal{R}' = \begin{cases} 0 \to f(0) \\ f(0) \to 0 \\ f(1) \to 0 \\ f(x) \to 1 \end{cases}$$

$$\mathcal{R}'_{naive} = \begin{cases} 0 \to f(0) \\ f(0) \to 0 \\ f(1) \to 0 \\ f(x) \to 1 \ \Leftarrow \ \neg(x \xrightarrow{*} 1). \end{cases}$$

Consider $f(0)$. If $\neg(0 \xrightarrow{*}_{\mathcal{R}'} 1)$, then we have $f(0) \to_{\mathcal{R}'} 1$. Thus $0 \to_{\mathcal{R}'} f(0) \to_{\mathcal{R}'} 1$: contradiction. Assume $0 \xrightarrow{*}_{\mathcal{R}'} 1$. Then, we cannot apply the last rule to $f(0)$, $f(f(0))$, $f(f(f(0)))$, ... Hence, it is clear that $\neg(0 \xrightarrow{*}_{\mathcal{R}'} 1)$: contradiction.

On the other hand, most functional languages admit priorities in rules and evaluation methods such as the functional strategy [12,17], an implementation method for lazy evaluation [4,7], possess the ability to handle priorities implicitly. However, it is hard to use PRSs as a theoretical computation model for analyzing the functional strategy. The main reason is that the PRSs in [1,11,18] have limitations, for example "boundedness", to ensure well-defined operational semantics.

This paper develops a new semantics for PRSs. This new semantics is natural as a model of actual computation systems such as the functional strategy [12,17]. The key idea is translating PRS $\mathcal{R}$ to CTRS $\mathcal{R}'$ by using a decidable approximation such that if the condition of a rule in $\mathcal{R}'$ holds then

the corresponding condition of the rule in $\mathcal{R}_{naive}$ also holds. This technique allows us to discuss behaviors of PRSs both theoretically and in practice. By using our framework of PRSs, we discuss a normalizing strategy of PRSs and provide a class of PRSs with a decidable normalizing strategy.

2 Term Rewriting Systems

We mainly follow the notation of [8] and assume readers are familiar with abstract reduction systems and CTRSs [6,8,3].

A *reduction system* is a structure $A = \langle D, \to \rangle$ for some set D and binary relation $\to$ on D called a reduction relation. A *reduction* is a finite sequence $x_0 \to x_1 \to \cdots \to x_n$ $(n \geq 0)$ or an infinite sequence $x_0 \to x_1 \to \cdots$ of reduction steps. The identity of elements $x, y \in D$ is denoted by $x \equiv y$. $\overset{+}{\to}$ is the transitive closure of $\to$, $\overset{*}{\to}$ is the reflexive and transitive closure of $\to$, and $=$ is the equivalence relation generated by $\to$. $\overset{k}{\to}$ denotes a reduction of k steps. If there is no element y such that $x \to y$, then we say x is a *normal form (with respect to $\to$)*; let $NF_{\to}$ be the set of normal forms. If y is a normal form such that $x \overset{*}{\to} y$ then we say y is a normal form of x (with respect to $\to$). If there is a normal form of x, we say x has a normal form.

We say a reduction system $A = \langle D, \to \rangle$ is *terminating* if and only if there are no infinite reductions. A is *confluent* if and only if $x \overset{*}{\to} y \ \wedge \ x \overset{*}{\to} z$ imply $\exists w \in D, y \overset{*}{\to} w \ \wedge \ z \overset{*}{\to} w$ for all $x, y, z \in D$.

Let $A = \langle D, \to \rangle$ and $\to_s$ be a sub-relation of $\to$. If $NF_{\to_s} = NF_{\to}$, we say that $\to_s$ is a *(one step) reduction strategy for* $\to$. A reduction strategy $\to_s$ is *normalizing* if and only if there exists no infinite sequence $x \equiv x_0 \to_s x_1 \to_s \cdots$ for all x having a normal form with respect to $\to$.

Let F and V be a set of function symbols with fixed arities and a set of variables, respectively. $T_{F \cup V}$ (or simply T) denotes the set of terms constructed from symbols in $F \cup V$. $|t|$ denotes the size of a term t. Letting $\Box$ be an extra constant, a term $C \in T_{F \cup V \cup \{\Box\}}$ is called a *context* denoted by $C[\ , \ldots, \ ]$. For a $C[\ , \ldots, \ ]$ which contains n $\Box$s and for $t_1, \ldots, t_n \in T_{F \cup V}$, $C[t_1, \ldots, t_n]$ denotes the obtained term by replacing $\Box$s with $t_1, \ldots, t_n$ from left to right. A context that possesses exactly one $\Box$ is denoted by $C[\ ]$.

Let σ be a substitution. We denote the application of σ to term t by $t\sigma$. $t \trianglelefteq s$ indicates that t is a subterm of s. If a term t has some symbol e, we write $e \in t$. The list $t_1, \ldots, t_n$ of terms for some $n \geq 0$ is often abbreviated by $\vec{t}$. $\vec{x}$ denotes the list $x_1, \ldots, x_n$ of variables.

A *conditional term rewriting system* (CTRS) R is a finite set of *conditional rewrite rules*, each of which is $l \to r \Leftarrow P(\vec{x})$, where l and r are terms and $P(\vec{x})$ is a condition. A conditional rewrite rule must satisfy three requirements: (a) l is not a variable, (b) variables in r must appear in l, (c) variables in $P(\vec{x})$ must appear in l. A term $l\sigma$ is called a *redex* if $P(x_1\sigma, \ldots, x_n\sigma)$ holds.

We say that the term t *reduces* to s by rule $l \to r \Leftarrow P(\vec{x})$ if and only if there exist a substitution σ and context $C[\,]$ such that $t \equiv C[l\sigma]$, $s \equiv C[r\sigma]$, and $l\sigma$ is a redex. We write $t \to_R s$ when t reduces to s. Sometimes we write $t \xrightarrow{\Delta}_R s$ to describe redex occurrence $\Delta \equiv l\sigma$ explicitly.

A CTRS is called *left-linear*, if every variable of l occurs only once for every rule $l \to r \Leftarrow P(\vec{x})$. We say that rules $l \to r \Leftarrow P(\vec{x})$ and $l' \to r' \Leftarrow P'(\vec{x'})$ are *overlapping*, if there exist a term $s \trianglelefteq l'$ $(s \notin V)$ and substitutions σ and σ' such that $l\sigma \equiv s\sigma'$ (except the trivial case, i.e., the rules are the same and $l \equiv s$). There is *root overlap* if $s \equiv l'$. A CTRS has overlap if it has overlapping rules. A CTRS has root overlap if it has only root overlapping rules.

A *rewrite rule* is a rule $l \to r \Leftarrow P(\vec{x})$ whose condition $P(\vec{x})$ always holds (i.e. $P(\vec{x}) \equiv true$), and we abbreviate rewrite rules as $l \to r$. A *term rewriting system* (TRS) is a CTRS that has only rewrite rules.

3 Priority Term Rewriting Systems and their Operational Semantics

The semantics of PRSs was first studied by Baeten et.al. [1]. Since the semantics is not always well defined, they gave a fix-point semantics and introduced the notion of "boundedness" as a sufficient condition for executable class of PRS. In order to clarify the class of PRSs having well-defined semantics, van de Pol [18] gave an operational semantics by translating PRS to transition system specification (TSS) and relate well-definedness of PRS and completeness of TSS. On the other hand, as "boundedness" is too strong restriction, Mohan [11] introduced the notion of I-rewrite, which is a decidable definition of reduction. However, I-rewrite is restricted into innermost reduction strategy. Thus, a term may not be reduced to its normal form by I-rewrite, even if it has a normal form by outer reduction. Mohan [11] also introduced P-rewriting. However, it is not practical because it can rewrite the only redex that is reducible by the rule having the highest priority in all redexes of the object term.

In this section, we give an operational semantics of PRSs that is decidable

and is independent from the notion of strategy.

Definition 1 (PRS) *A priority term rewriting system (PRS) $\mathcal{R}$ is a pair $\langle R_0, \sqsupset \rangle$ of an underlying TRS R_0 and an acyclic relation $\sqsupset$ on R_0.*

Example 2 *The "parallel or" is described in PRS as follows:*

$$POR = \left\{ \begin{array}{l} or(tt, x) \to tt \\ or(x, tt) \to tt \\ or(x, y) \to ff. \end{array} \right.$$

Before we define a semantics of PRS, we need the notion of an approximation of PRS by ignoring priorities and right-hand sides.

Definition 3 (Ω-terms [5,8,9]) *Letting Ω be an extra constant, we represent prefixes of terms by Ω-terms in $T_{F \cup V \cup \{\Omega\}}$ (also denoted by T_Ω simply).*

(a) t_Ω denotes the Ω-term obtained from a term t by replacing each variable with Ω.
(b) The prefix ordering $\succeq$ on T_Ω is defined as follows:

$$\begin{array}{ll} t \succeq \Omega & \text{for all } t \in T_\Omega, \\ t \succeq t & \text{for each variable or constant } t, \\ f(t_1, \ldots, t_n) \succeq f(s_1, \ldots, s_n) & \text{if } t_i \succeq s_i \text{ for } i = 1, \ldots, n. \end{array}$$

(c) t and s are compatible, written by $t \uparrow s$, if $u \succeq t$ and $u \succeq s$ for some u; otherwise they are incompatible, denoted by $t \# s$.
(d) $s \sqcup t$ denotes a minimal Ω-term u such that $u \succeq s$ and $u \succeq t$ if it exists.
(e) We say an Ω-term t and a set S of Ω-terms are incompatible, denoted by $t \# S$, if t and s are incompatible for any $s \in S$.

Proposition 4

(a) If $s \preceq t$ and $t \uparrow w$ then $s \uparrow w$ and $s \sqcup w \preceq t \sqcup w$.
(b) If $s \preceq t$ and $s \# u$ then $t \# u$.

PROOF. Trivial. □

Definition 5 (ω-reduction) *Let R_0 be a TRS. Let s and t be Ω-terms. Then, $s \to_\omega t$ if and only if there exist a context $C[\ ]$, an Ω-term $u \not\equiv \Omega$ and a rule $l \to r \in R_0$ such that $s \equiv C[u]$, $t \equiv C[\Omega]$ and $u \uparrow l_\Omega$. The reduction relation $\to_\omega$ on T_Ω is called ω-reduction.*

For example, POR in example 2 has $f(or(a, tt)) \rightarrow_\omega f(\Omega)$.

Lemma 6 ([9]) *ω-reduction is confluent and terminating.*

Definition 7

(a) The direct approximant $\omega(t)$ of an Ω-term t is the normal form of t with respect to ω-reduction.
(b) $\overline{\omega}(f(t_1, \ldots, t_n))$ denotes $f(\omega(t_1), \ldots, \omega(t_n))$.

The following properties are used in the sequel. Their simple proofs have been omitted.

Lemma 8

(a) If $t \succeq s$ then $\omega(t) \succeq \omega(s)$ [5].
(b) If $C[\Omega], t \in NF_{\rightarrow_\omega}$ then $C[t] \in NF_{\rightarrow_\omega}$.

Definition 9 *Let $\mathcal{R} = \langle R_0, \sqsupset \rangle$. The set $Hlhs(e)$ of left-hand-side skeletons of rules each of which has higher priority than a rule e is defined by:*

$$Hlhs(e) = \{l_\Omega \mid l \rightarrow r \in R_0,\ l \rightarrow r \sqsupset e\}$$

Now we can define an operational semantics of PRSs.

Definition 10 (a semantics of PRS) *Let $\mathcal{R} = \langle R_0, \sqsupset \rangle$. Let s and t be terms. Then, $s \rightarrow_{\mathcal{R}} t$ if and only if there exist a context $C[\ ]$, a rule $l \rightarrow r \in R_0$, and a substitution σ such that $s \equiv C[l\sigma]$, $t \equiv C[r\sigma]$ and $\overline{\omega}((l\sigma)_\Omega) \# Hlhs(l \rightarrow r)$.*

From the above definition, we may treat PRS $\mathcal{R} = \langle R_0, \sqsupset \rangle$ as CTRS formulation

$$\mathcal{R} = \{l \rightarrow r \ \Leftarrow\ \overline{\omega}(l_\Omega) \# Hlhs(l \rightarrow r) \ \mid\ l \rightarrow r \in R_0\}$$

that is much more convenient to illustrate conditions for rewriting relation explicitly. From here on, we sometimes use this conventional notation.

Example 11 *For POR defined in example 2,*

$$POR = \begin{cases} or(tt, y) \to tt \;\Leftarrow\; \overline{\omega}(or(tt, y_\Omega))\#\{\} \\ or(x, tt) \to tt \;\Leftarrow\; \overline{\omega}(or(x_\Omega, tt))\#\{or(tt, \Omega)\} \\ or(x, y) \to ff \;\Leftarrow\; \overline{\omega}(or(x_\Omega, y_\Omega))\#\{or(tt, \Omega), or(\Omega, tt)\} \end{cases}$$

$$= \begin{cases} or(tt, y) \to tt \\ or(x, tt) \to tt \;\Leftarrow\; \omega(x_\Omega)\#tt \\ or(x, y) \to ff \;\Leftarrow\; \omega(x_\Omega)\#tt \wedge \omega(y_\Omega)\#tt \end{cases}$$

Note that x_Ω and y_Ω in the above rules do not denote Ω. Instead, x_Ω and y_Ω should take place after they are substituted, i.e. $(x\sigma)_\Omega$ and $(y\sigma)_\Omega$, because of definition 10.

- *$or(ff, tt) \to_{POR} tt$ by using the second rule. But, the last rule cannot be applied to $or(ff, tt)$.*
- *$or(x, tt)$ is a normal form.*
- *$or(ff, ff) \to_{POR} ff$ by using the last rule. On the other hand, $or(ff, y)$ is a normal form.*
- *$or(or(ff, ff), tt) \to_{POR} or(ff, tt) \to_{POR} tt$ by using the last rule and the second rule. But $or(or(ff, ff), tt)$ cannot directly be reduced to tt.*

Remark 12 *(i) $\mathcal{R}$ does not have limitations like "boundedness" in [1] or "ground" in [11] for $\mathcal{R}$. Thus, $\mathcal{R}$ can give well-defined semantics to all finite PRSs.*

(ii) $\to_\omega$ was originally introduced by [5] as an easy-to-handle reduction of arbitrary reduction $\to_!$ ($\supseteq \to_{\mathcal{R}}$). Thus, our semantics of PRS is essentially equal to the naive semantics that uses $\to_!$ instead of $\to_{\mathcal{R}}$ in conditional parts. However, the use of $\to_!$ is not essential for our framework. Indeed, our semantics might be improved by replacing $\to_!$ with appropriate approximation $\to_{\mathcal{R}'}$ such that $\to_! \supseteq \to_{\mathcal{R}'} \supseteq \to_{\mathcal{R}}$.

Proposition 13 *Let $\mathcal{R} = \langle R_0, \sqsupset \rangle$ be a PRS. Then, $\to_{\mathcal{R}}$ is stable under substitution, i.e., $s \to_{\mathcal{R}} t$ implies $s\sigma \to_{\mathcal{R}} t\sigma$ for all substitution σ.*

PROOF. Let $s \equiv C[l\theta] \to_{\mathcal{R}} C[r\theta] \equiv t$ for some context $C[\ ]$, substitution θ and $l \to r \in R_0$ such that $\overline{\omega}((l\theta)_\Omega)\#Hlhs(l \to r)$. Since $l_\Omega \preceq (l\theta)_\Omega \preceq (l\theta\sigma)_\Omega$, we can show $\overline{\omega}((l\theta)_\Omega) \preceq \overline{\omega}((l\theta\sigma)_\Omega)$ by lemma 8 (a) and the definition of $\overline{\omega}$. It follows $\overline{\omega}((l\theta\sigma)_\Omega)\#Hlhs(l \to r)$, which concludes $(C\sigma)[l\theta\sigma] \to_{\mathcal{R}} (C\sigma)[r\theta\sigma]$. $\square$

Example 14 *Consider the following PRS:*

$$\mathcal{R} = \left\{ \begin{array}{l} f(a) \to b \\ f(x) \to c \\ d \to d \end{array} \right.$$

The conditional TRS giving semantics of above $\mathcal{R}$ *is*

$$\mathcal{R} = \left\{ \begin{array}{l} f(a) \to b \ \Leftarrow \ \overline{\omega}(f(a))\#\{\} \\ f(x) \to c \ \Leftarrow \ \overline{\omega}(f(x_\Omega))\#\{f(a)\} \\ d \to d \ \Leftarrow \ \overline{\omega}(d)\#\{\} \end{array} \right.$$

$$= \left\{ \begin{array}{l} f(a) \to b \\ f(x) \to c \ \Leftarrow \ \omega(x_\Omega)\#a \\ d \to d \end{array} \right.$$

We have $f(g(y)) \to_{\mathcal{R}} c$. *Thus,* $f(g(d)) \to_{\mathcal{R}} c$ *by proposition 13.*

The reduction relation $\to_B$ given by [1] is not stable under substitution. I-rewrite proposed by [11] is essentially defined on ground terms. That is, rules can be applied only to ground terms. For example, consider the PRS in example 14. It is natural that $f(g(y))$ can be reduced to c, since g of $g(y)$ is not changed even if any term is substituted to y. But, $f(g(y))$ cannot be reduced to c by I-rewriting.

For further comparison with other works, see appendix A.

We give formal definition of reduction relations of PRSs proposed in other papers, and compare them.

We write $s \to^{>\varepsilon} t$ if $s \xrightarrow{\Delta} t$ and $s \not\equiv \Delta$.

Definition 15 *Let* $\mathcal{R} = \langle R_0, \sqsupset \rangle$ *be a PRS. Let* Q *be a predicate. Then, we denote* $s \to_Q t$ *if there exist a context* $C[\,]$, *a rule* $l \to r \in R_0$ *and a substitution* σ *such that* $s \equiv C[l\sigma]$, $t \equiv C[r\sigma]$ *and* Q.

From predicates shown in table 1, we get reduction relations; Baeten's reduction $\to_B$ in [1], Mohan's I-reduction $\to_I$ in [11], and the reduction $\to_{our}$ proposed in this paper.

Table 1
Predicates defining reduction relations of PRSs

$\to_Q$	definition of Q
$\to_B$	$\neg(\exists l' \to r' \in R_0, \exists \sigma'[l' \to r' \sqsupset l \to r \ \wedge \ l\sigma \stackrel{*}{\to}_B^{>\varepsilon} l'\sigma'])$
$\to_{our}$	$\neg(\exists l' \to r' \in R_0[l' \to r' \sqsupset l \to r \ \wedge \ \overline{\omega}((l\sigma)_\Omega) \uparrow l'_\Omega])$
$\to_I$	$l\sigma \in NF_{\to_I^{>\varepsilon}}$, $\neg(\exists l' \to r' \in R_0, \exists \sigma'[l' \to r' \sqsupset l \to r \ \wedge \ l\sigma \equiv l'\sigma'])$ and $l\sigma$ is ground

We need to prepare lemmas before proceeding.

Lemma 16 *Let $\mathcal{R} = \langle R_0, \sqsupset \rangle$ be a PRS. Let $s \stackrel{*}{\to}_{R_0} t$ and $s \succeq s'$ for terms s, t and an Ω-term s'. Then, there exists an Ω-term t' such that $s' \stackrel{*}{\to}_\omega t'$ and $t \succeq t'$.*

PROOF. By induction on the number of transitions in $s \stackrel{*}{\to}_{R_0} t$, we prove the lemma. In case of $s \equiv t$, the lemma trivially holds. Next, consider the case $s \to_{R_0} u \stackrel{*}{\to}_{R_0} t$. Let $s \equiv C[l\sigma] \to_{R_0} C[r\sigma] \equiv u$ for some $l \to r \in R_0$, context $C[\ ]$ and substitution σ. Then, it follows from $s \succeq s'$ that $s' \equiv C'[v]$ for some $C'[\ ] \preceq C[\ ]$ and $v \preceq l\sigma$. Since $v \uparrow l$, we have $C'[v] \to_\omega C'[\Omega] \preceq u$. Thus, we have $s' \equiv C'[v] \to_\omega C'[\Omega] \stackrel{*}{\to}_\omega t'$ for some $t' \preceq t$ by induction hypothesis.

Lemma 17 *Let $\mathcal{R} = \langle R_0, \sqsupset \rangle$ be a PRS. Then, $NF_{\to_{our}} = NF_{\to_\omega}$ on Ω-free ground terms.*

PROOF. Since $NF_{\to_{our}} \supseteq NF_{\to_\omega}$ is trivial, we prove by induction on $|s|$ that $s \in NF_{\to_{our}}$ implies $s \in NF_{\to_\omega}$ for Ω-free ground term s.

Let $s \equiv C[t] \to_\omega C[\Omega]$ for some t and context $C[\ ]$. Let $l \to r \in R_0$ be a maximal rule with respect to $\sqsupset$ such that $l_\Omega \preceq t$. If $t \notin NF_{\to_{our}^{>\varepsilon}}$, then t is reducible with respect to $\to_{our}$. Consider the case $t \in NF_{\to_{our}^{>\varepsilon}}$. We have $t \in NF_{\to_\omega^{>\varepsilon}}$ by induction hypothesis. It follows from ground-ness of t that $\overline{\omega}(t_\Omega) \equiv \overline{\omega}(t) \equiv t$. Assuming $t \uparrow l'_\Omega$ for some $l' \to r' \sqsupset l \to r$, we have $t \succeq l'_\Omega$, which is a contradiction to the maximality of $l \to r$. Hence, $s \equiv C[t]$ is reducible with respect to $\to_{our}$. $\square$

Theorem 18 *Let $\mathcal{R} = \langle R_0, \sqsupset \rangle$ be a PRS.*

(a) $NF_{\to_B} = NF_{\to_I}$ on ground terms.
(b) $NF_{\to_I} = NF_{\to_{our}}$ on ground terms for left-linear PRS $\mathcal{R}$.
(c) $\to_B \supseteq \to_I$. Moreover, there is a PRS $\mathcal{R}$ such that $\to_B \supset \to_I$.

(d) $\to_{our} \supseteq \to_I$ *on ground terms for left-linear PRS* $\mathcal{R}$*. Moreover, there is a left-linear PRS* $\mathcal{R}$ *such that* $\to_{our} \supset \to_I$ *on ground terms.*

(e) $\to_B \supseteq \to_{our}$*. Moreover, there is a PRS* $\mathcal{R}$ *such that* $\to_B \supset \to_{our}$*.*

PROOF.

(a) Let s be a ground term. We prove $s \in NF_{\to_B} \iff s \in NF_{\to_I}$ by induction on $|s|$.

Let $s \equiv C[l\sigma] \to_I C[r\sigma]$ for some $l \to r \in R_0$, context $C[\ ]$ and substitution σ. Then, $l\sigma \in NF^{>\varepsilon}_{\to_I}$ and $\neg(\exists l' \to r' \in R_0, \exists \sigma'[l' \to r' \sqsupset l \to r \ \wedge\ l\sigma \equiv l'\sigma'])$. Since $l\sigma \in NF^{>\varepsilon}_{\to_B}$ by induction hypothesis, we have $C[l\sigma] \to_B C[r\sigma]$.

Let $s \equiv C[l\sigma] \to_B C[r\sigma]$ for some $l \to r \in R_0$, context $C[\ ]$ and substitution σ. Then, $\neg(\exists l' \to r' \in R_0, \exists \sigma'[l' \to r' \sqsupset l \to r \ \wedge\ l\sigma \overset{*}{\to}^{>\varepsilon}_B l'\sigma'])$. Hence, $\neg(\exists l' \to r' \in R_0, \exists \sigma'[l' \to r' \sqsupset l \to r \ \wedge\ l\sigma \equiv l'\sigma'])$. In case of $l\sigma \notin NF^{>\varepsilon}_{\to_I}$, $s \equiv C[l\sigma]$ is reducible with respect to $\to_I$. In case of $l\sigma \in NF^{>\varepsilon}_{\to_I}$, we have $s \equiv C[l\sigma] \to_I C[r\sigma]$.

(b) Let s be a ground term. We prove $s \in NF_{\to_{our}} \iff s \in NF_{\to_I}$ by induction on $|s|$.

Let $s \equiv C[l\sigma] \to_I C[r\sigma]$ for some $l \to r \in R_0$, context $C[\ ]$ and substitution σ. Then, $l\sigma \in NF^{>\varepsilon}_{\to_I}$ and $\neg(\exists l' \to r' \in R_0, \exists \sigma'[l' \to r' \sqsupset l \to r \ \wedge\ l\sigma \equiv l'\sigma'])$. Since $l\sigma \in NF^{>\varepsilon}_{\to_{our}}$ by induction hypothesis, we have $l\sigma \in NF^{>\varepsilon}_{\to^{\omega}}$ by lemma 17. Hence, $\overline{\omega}((l\sigma)_\Omega) \equiv \overline{\omega}(l\sigma) \equiv l\sigma$. Assuming $l\sigma \uparrow l'_\Omega$ for some $l' \to r' \sqsupset l \to r$, there exists σ' such that $l\sigma \equiv l'\sigma'$ from left-linearity: contradiction. Thus, $C[l\sigma] \to_{our} C[r\sigma]$.

Let $s \equiv C[l\sigma] \to_{our} C[r\sigma]$ for some $l \to r \in R_0$, context $C[\ ]$ and substitution σ. Then, $\neg(\exists l' \to r' \in R_0[l' \to r' \sqsupset l \to r \ \wedge\ \overline{\omega}((l\sigma)_\Omega) \uparrow l'_\Omega])$. In case of $l\sigma \notin NF^{>\varepsilon}_{\to_I}$, $s \equiv C[l\sigma]$ is reducible with respect to $\to_I$. Consider the case $l\sigma \in NF^{>\varepsilon}_{\to_I}$ and assume $l\sigma \equiv l'\sigma'$ for some $l' \to r' \sqsupset l \to r$. Since $l\sigma \in NF^{>\varepsilon}_{\to^{\omega}}$ by induction hypothesis and lemma 17, it follows from ground-ness of s that $\overline{\omega}((l\sigma)_\Omega) \equiv \overline{\omega}(l\sigma) \equiv l\sigma \uparrow l'_\Omega$, which is a contradiction. Hence, $s \equiv C[l\sigma] \to_I C[r\sigma]$.

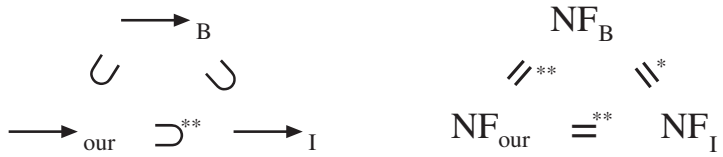
(c) Let $C[l\sigma] \to_I C[r\sigma]$ for some $l \to r \in R_0$, context $C[\ ]$ and substitution σ. Then, $l\sigma \in NF^{>\varepsilon}_{\to_I}$, $\neg(\exists l' \to r' \in R_0, \exists \sigma'[l' \to r' \sqsupset l \to r \ \wedge\ l\sigma \equiv l'\sigma'])$ and $l\sigma$ is ground. Since $l\sigma \in NF^{>\varepsilon}_{\to_B}$ by (a), we have $C[l\sigma] \to_B C[r\sigma]$.

To prove that $\neg(\to_B \subseteq \to_I)$, consider the following PRS:

$$\mathcal{R} = \begin{cases} a \to b \\ f(x) \to c \end{cases}$$

Then, we have $f(a) \to_B c$ but $\neg(f(a) \to_I c)$.

(d) We have $\to_{our} \supseteq \to_I$ from the proof of (b). To prove that $\neg(\to_{our} \subseteq \to_I)$, consider the PRS in (c). We have $f(a) \to_{our} c$ but $\neg(f(a) \to_I c)$.



$*$: relation on ground terms,
$**$: relation on ground terms for a left-linear system

Fig. 1. Comparison with other works

(e) Let $C[l\sigma] \to_{our} C[r\sigma]$ for some $l \to r \in R_0$, context $C[\,]$ and substitution σ. To prove that $C[l\sigma] \to_B C[r\sigma]$, it is enough to show that $l\sigma \overset{*}{\to}{}^{>\varepsilon}_B l'\sigma'$ implies $\overline{\omega}((l\sigma)_\Omega) \uparrow l'_\Omega$.

Let $l\sigma \overset{*}{\to}{}^{>\varepsilon}_B l'\sigma'$. Then, we have $l\sigma \overset{*}{\to}{}^{>\varepsilon}_{R_0} l'\sigma'$. From $l\sigma \succeq (l\sigma)_\Omega$ and lemma 16, there exists u such that $(l\sigma)_\Omega \overset{*}{\to}{}^{>\varepsilon}_\omega u$ and $l'\sigma' \succeq u$. Since $u \overset{*}{\to}{}^{>\varepsilon}_\omega \overline{\omega}((l\sigma)_\Omega)$, we have $\overline{\omega}((l\sigma)_\Omega) \uparrow l'_\Omega$.

To prove that $\neg(\to_B \subseteq \to_{our})$, consider the PRS in example 14. We have $f(d) \to_B c$ but $\neg(f(d) \to_{our} c)$. $\square$

Note that left-linearity in (b) and (d) is essential, because $f(a,b) \to_I a$ but $f(a,b) \in NF_{\to_{our}}$ considering the following PRS:

$$\mathcal{R} = \left\{ \begin{array}{l} f(x,x) \to a \\ f(x,y) \to b \end{array} \right.$$

These results are summarized in figure 1.

We next introduce the concept for orthogonality for PRSs in similar way to TRSs.

Definition 19 (orthogonality)

(a) We say that $\mathcal{R} = \langle R_0, \sqsupset \rangle$ has complete priority, if $e \sqsupset e'$ or $e' \sqsupset e$ for all root overlapping rules e and e' in R_0.
(b) $\mathcal{R} = \langle R_0, \sqsupset \rangle$ is orthogonal, if it has complete priority, R_0 is left-linear and R_0 possibly has root overlap.

An orthogonal PRS $\mathcal{R} = \langle R_0, \sqsupset \rangle$ may have root overlapping rules in R_0, but two different rules cannot apply to the same redex because of the priority between the rules. From this 'non-overlapping property' and the stability under substitution indicated in the above proposition, PRS $\mathcal{R}$ works like orthogonal (i.e., left-linear and non-overlapping) TRSs. Thus, the confluent property of orthogonal PRSs can be easily proven in similar way as for orthogonal TRSs [14,6] as follows.

Lemma 20 *Let $\mathcal{R} = \langle R_0, \sqsupset \rangle$ be orthogonal. Let $t \xrightarrow{\Delta}_{\mathcal{R}} s$ and $t \xrightarrow{\Delta}_{\mathcal{R}} s'$. Then $s \equiv s'$.*

PROOF. If we assume $s \not\equiv s'$, we have different rules $e\colon l \to r$ and $e'\colon l' \to r'$ in R_0 such that $\overline{\omega}(\Delta_\Omega)\#Hlhs(e)$, $\overline{\omega}(\Delta_\Omega)\#Hlhs(e')$ and $\Delta \equiv l\sigma \equiv l'\sigma'$.

From orthogonality, we can assume $e \sqsupset e'$, i.e., $l_\Omega \in Hlhs(e')$ without loss of generality. It is clear $\Delta \equiv l\sigma \succeq l_\Omega$. Since $\Delta \succeq \Delta_\Omega \succeq \overline{\omega}(\Delta_\Omega)$, we have $\Delta \uparrow \overline{\omega}(\Delta_\Omega)$. It follows $\overline{\omega}(\Delta_\Omega) \uparrow l_\Omega$ contradicting to $\overline{\omega}(\Delta_\Omega)\#l_\Omega$. □

The parallel reduction $t \twoheadrightarrow s$ is defined with $t \equiv C[\Delta_1, \ldots, \Delta_n] \xrightarrow{\Delta_1} \cdots \xrightarrow{\Delta_n} s \equiv C[s_1, \ldots, s_n]$ $(n \geq 0)$.

Theorem 21 *Orthogonal PRSs are confluent.*

PROOF. If $\twoheadrightarrow_{\mathcal{R}}$ is confluent, we can get the theorem, since $\xrightarrow{*}_{\mathcal{R}}$ and $\overset{*}{\twoheadrightarrow}_{\mathcal{R}}$ are the same relation.

Let $t \overset{\Delta_1,\ldots,\Delta_n}{\twoheadrightarrow}_{\mathcal{R}} s$ and $t \overset{\Delta'_1,\ldots,\Delta'_m}{\twoheadrightarrow} s'$. From lemma 20, the cases we have to consider are that Δ_i and Δ'_j are disjoint or nested. This is done as the similar way as the TRS case, since $\rightarrow_{\mathcal{R}}$ is stable under substitution. □

4 Strong Sequentiality of PRSs

The fundamental concept of strong sequentiality for orthogonal TRSs was introduced by Huet and Lévy [5]. In this section, we first explain the basic notions related to strong sequentiality, according to [5], for orthogonal PRSs. We next describe an useful decision procedure that determines the index of a given Ω-term with respect to an approximation $R_?$ of a PRS $\mathcal{R}$.

From here on, we assume PRSs $\mathcal{R}$ are orthogonal.

Definition 22 (arbitrary-reduction) *Let $\mathcal{R}$ be a PRS $\langle R_0, \sqsupset \rangle$. Let s and t be Ω-terms. Then, $s \rightarrow_? t$ if and only if there exist a context $C[\,]$, a Ω-term u, a rule $l \to r \in R_0$, and a substitution σ such that $s \equiv C[l\sigma]$, $t \equiv C[u]$ and $\overline{\omega}((l\sigma)_\Omega)\#Hlhs(l \to r)$.*

Definition 23 (strongly sequential PRS)

(a) Displayed occurrence Ω in term $C[\Omega]$ is an index (denoted by $C[\Omega_I]$), if the following condition holds for all t and context $C'[\,]$ such that $C'[\,] \succeq C[\,]$:

If $C'[t]$ has an Ω-free normal form with respect to ?-reduction then $t \not\equiv \Omega$.

$C[\Omega_{NI}]$ denotes that the displayed occurrence Ω in $C[\Omega]$ is not an index.

(b) *An orthogonal PRS $\mathcal{R}$ is strongly sequential if every Ω-term t with Ω that is in normal form with respect to $\rightarrow_{\mathcal{R}}$ has an index.*

For analyzing the decidability of strong sequentiality of orthogonal TRSs, Klop and Middeldorp [9] proposed an useful procedure that determines whether a given Ω occurrence in a term t is an index, by using Ω-reduction and the test symbol $\bullet$. We now develop a similar procedure that determines index with respect to $R_?$. The most difficult problem in constructing Ω-reduction for PRSs is how to handle the test symbol $\bullet$. Although $\bullet$ is introduced for preventing reductions, $\bullet$ has an opposite effect in conditions of rules. Hence, wrong application of rewriting rule occurs if we handle $\bullet$ as a mere constant symbol as in [9]. For example, consider POR in example 2. Then we have $or(\Omega_I, tt)$ by definition 23. However, $or(\bullet, tt)$ is reduced to Ω by Ω-reduction with constant treatment of $\bullet$. To overcome this difficulty, we exchange $\bullet$ and Ω in a term t when we compute normal forms with respect to ω-reduction for deciding applicability of rules.

Definition 24 (Ω-reduction) *Let $\mathcal{R}$ be a PRS.*

(a) *Let $\bullet$ be a new constant. $\overline{t}$ denotes the Ω-term obtained from an Ω-term t by exchanging $\bullet$ and Ω.*

(b) *Let s and t be Ω-terms. Then, $s \rightarrow_\Omega t$ if and only if there exist a context $C[\ ]$, an Ω-term $u \not\equiv \Omega$ and a rule $l \rightarrow r$ in $\mathcal{R}$ such that $s \equiv C[u]$, $t \equiv C[\Omega]$, $u \uparrow l_\Omega$ and $\overline{\omega}((\overline{u \sqcup l_\Omega})_\Omega) \# Hlhs(l \rightarrow r)$. The reduction relation $\rightarrow_\Omega$ on T_Ω is called Ω-reduction.*

Example 25 *For POR defined in example 2, $\rightarrow_\Omega$ can be defined by the fol-*

lowing CTRS POR_Ω.

$$POR_\Omega = \left\{ \begin{array}{lcl} or(x,y) \to \Omega & \Leftarrow & or(x,y) \uparrow or(tt,\Omega) \\ & & \wedge\ \overline{\omega}((\overline{or(x,y) \sqcup or(tt,\Omega)})_\Omega)\#\{\} \\ or(x,y) \to \Omega & \Leftarrow & or(x,y) \uparrow or(\Omega,tt) \\ & & \wedge\ \overline{\omega}((\overline{or(x,y) \sqcup or(\Omega,tt)})_\Omega)\#\{or(tt,\Omega)\} \\ or(x,y) \to \Omega & \Leftarrow & or(x,y) \uparrow or(\Omega,\Omega) \\ & & \wedge\ \overline{\omega}((\overline{or(x,y) \sqcup or(\Omega,\Omega)})_\Omega) \\ & & \quad \#\{or(tt,\Omega), or(\Omega,tt)\} \end{array} \right.$$

$$= \left\{ \begin{array}{lcl} or(x,y) \to \Omega & \Leftarrow & x \uparrow tt \\ or(x,y) \to \Omega & \Leftarrow & y \uparrow tt \ \wedge\ \omega((\overline{x})_\Omega)\#tt \\ or(x,y) \to \Omega & \Leftarrow & \omega((\overline{x})_\Omega)\#tt \ \wedge\ \omega((\overline{y})_\Omega)\#tt \end{array} \right.$$

Note that every operation in conditional parts should be applied after substitutions to variables x and y.

- $or(ff,\Omega) \to_\Omega \Omega$ *by the second or third rule.*
- $or(\bullet,\Omega)$ *is a normal form with respect to* $\to_\Omega$.
- $or(ff,\bullet)$ *is a normal form with respect to* $\to_\Omega$.
- $or(\Omega,\bullet) \to_\Omega \Omega$ *by the first rule.*
- $or(ff,ff) \to_\Omega \Omega$ *by the third rule.*

Proposition 26 *Let s, t and u be Ω-terms.*

(a) If s is $\bullet$-free and $s \preceq t$, then $s \preceq \overline{t}$.
(b) Let P be a set of $\bullet$-free Ω-terms. If $s \preceq t$ and $\overline{\omega}((\overline{t})_\Omega)\#P$, then $\overline{\omega}((\overline{s})_\Omega)\#P$.

PROOF.

(a) We can assume $s \equiv C[\Omega]$ and $t \equiv C[u]$ for $\bullet$-and-Ω-free context $C[\ ]$ and Ω-term u without loss of generality. Then we clearly have $s \equiv C[\Omega] \preceq C[\overline{u}] \equiv \overline{t}$.

(b) It is enough to show that $s \preceq t$ and $\omega((\overline{t})_\Omega)\#u \Rightarrow \omega((\overline{s})_\Omega)\#u$ for $\bullet$-free Ω-term u. Hence, we prove that $\omega((\overline{s})_\Omega) \uparrow u \ \wedge\ (s \preceq t) \Rightarrow \omega((\overline{t})_\Omega) \uparrow u$ by induction on $|s|$. If $|s| = 1$ the claim is shown easily, because u is $\bullet$-free. Next, let $s \equiv f(s_1,\ldots,s_n)$. We can assume $t \equiv f(t_1,\ldots,t_n)$ from $s \preceq t$.

We have two sub-cases.

- Consider the case that $\omega((\overline{s})_\Omega) \equiv f(\omega((\overline{s_1})_\Omega), \ldots, \omega((\overline{s_n})_\Omega))$. If the top symbol of u is not f, the claim clearly holds. Thus, we assume $u \equiv f(u_1, \ldots, u_n)$. Since $\omega((\overline{t_i})_\Omega) \uparrow u_i$ by induction hypothesis, we get $\omega((\overline{t})_\Omega) \uparrow u$.
- If $\omega((\overline{s})_\Omega) \equiv \Omega$, we have $(\overline{s})_\Omega \xrightarrow{*}_\omega f(\omega((\overline{s_1})_\Omega), \ldots, \omega((\overline{s_n})_\Omega)) \rightarrow_\omega \Omega$. Hence, we have $f(\omega((\overline{s_1})_\Omega), \ldots, \omega((\overline{s_n})_\Omega)) \uparrow l$ for some rule $l \rightarrow r$ in $\mathcal{R}$. Letting $l \equiv f(l_1, \ldots, l_n)$, we have $\omega((\overline{s_i})_\Omega) \uparrow (l_i)_\Omega$. Then we have $\omega((\overline{t_i})_\Omega) \uparrow (l_i)_\Omega$ by induction hypothesis by taking $(l_i)_\Omega$ as u. Since $f(\omega((\overline{t_1})_\Omega), \ldots, \omega((\overline{t_n})_\Omega)) \uparrow l_\Omega$, we have $(\overline{t})_\Omega \xrightarrow{*}_\omega f(\omega((\overline{t_1})_\Omega), \ldots, \omega((\overline{t_n})_\Omega)) \rightarrow_\omega \Omega$. Therefore, we get $\omega((\overline{t})_\Omega) \equiv \Omega$. $\square$

Lemma 27 *If $t \xrightarrow{*}_\Omega s$ then $t \succeq s$.*

PROOF. Trivial from definition of $\rightarrow_\Omega$. $\square$

Lemma 28 *Let s and t be Ω-terms.*

(a) $s \preceq t$ and $t \rightarrow_\Omega \Omega$ imply $s \equiv \Omega$ or $s \rightarrow_\Omega \Omega$.
(b) $s \preceq t$ and $t \xrightarrow{*}_\Omega u$ imply $s \xrightarrow{*}_\Omega v$ for some $v \preceq u$.

PROOF.

(a) In case of $s \equiv \Omega$, the lemma holds trivially. Let $s \not\equiv \Omega$. It follows from $t \rightarrow_\Omega \Omega$ that $t \uparrow l_\Omega$ and $\overline{\omega}((\overline{t \sqcup l_\Omega})_\Omega) \# Hlhs(l \rightarrow r)$ for some rule $l \rightarrow r$ in $\mathcal{R}$. Since $s \preceq t$, we have $s \uparrow l_\Omega$. On the other hand, we have $s \sqcup l_\Omega \preceq t \sqcup l_\Omega$ by proposition 4 (a). It follows from $\overline{\omega}((\overline{t \sqcup l_\Omega})_\Omega) \# Hlhs(l \rightarrow r)$ and proposition 26 (b) that $\overline{\omega}((\overline{s \sqcup l_\Omega})_\Omega) \# Hlhs(l \rightarrow r)$. Therefore, we have $s \rightarrow_\Omega \Omega$, which concludes the proof.

(b) Prove by induction on k in $t \xrightarrow{k}_\Omega u$. In case of $k = 0$, trivially the lemma holds by taking s as v. We next consider $t \xrightarrow{*}_\Omega u' \rightarrow_\Omega u$. Then there exist $C[\,]$ and u'' such that $u' \equiv C[u'']$, $u \equiv C[\Omega]$ and $u'' \rightarrow_\Omega \Omega$. By induction hypothesis, we have $s \xrightarrow{*}_\Omega v'$ for some $v' \preceq u'$. If $v' \preceq C[\Omega] \equiv u$, the lemma holds by taking v' as v. Otherwise, since $v' \preceq u' \equiv C[u'']$, we have $v' \equiv C'[v'']$ for some $C'[\,]$ and v'' such that $C'[\,] \preceq C[\,]$ and $v'' \preceq u''$. It follows from (a) that $v'' \rightarrow_\Omega \Omega$. By taking $C'[\Omega]$ as v, we have $v \equiv C'[\Omega] \preceq C[\Omega] \equiv u$ and $s \xrightarrow{*}_\Omega v' \equiv C'[v''] \rightarrow_\Omega C'[\Omega] \equiv v$. $\square$

We can show the following lemma similarly to the proof of lemma 6.

Lemma 29 *Ω-reduction is confluent and terminating.*

Definition 30 *The approximant $\Omega(t)$ of an Ω-term t is the normal form of t with respect to Ω-reduction.*

Lemma 31 *Let s and t be Ω-terms. Then $s \preceq t$ implies $\Omega(s) \preceq \Omega(t)$.*

PROOF. Consider the sequence $t \xrightarrow{*}_\Omega \Omega(t)$. It follows by lemma 28 (b) that $\exists v \preceq \Omega(t)$, $s \xrightarrow{*}_\Omega v \xrightarrow{*}_\Omega \Omega(s)$. We can conclude that $\Omega(s) \preceq v \preceq \Omega(t)$ by lemma 27. $\Box$

Lemma 32 *Let q be a $\bullet$-free Ω-term. If $p \xrightarrow{*}_\Omega q$, then $\overline{p'} \xrightarrow{*}_? \overline{q}$ for some Ω-free term $p' \succeq p$.*

PROOF. Prove by induction on k in $p \xrightarrow{k}_\Omega q$. In case of $k = 0$, the lemma holds by taking $p' \equiv p$. Next, we consider $p \to_\Omega q_1 \xrightarrow{*}_\Omega q$. By induction hypothesis, there exists an Ω-free term $q_2 \succeq q_1$ such that $\overline{q_2} \xrightarrow{*}_? \overline{q}$. Since $p \equiv C[u] \to_\Omega C[\Omega] \equiv q_1$ for some $C[\,]$, f and u, there exist a context $C'[\,] \succeq C[\,]$ and an Ω-free term q_3 such that $q_2 \equiv C'[q_3]$.

On the other hand, since $u \to_\Omega \Omega$, we have $u \uparrow l_\Omega$ and $\overline{\omega}(\overline{(u \sqcup l_\Omega)_\Omega})\#Hlhs(l \to r)$ for some rule $l \to r$ in $\mathcal{R}$. It follows from $u \sqcup l_\Omega \succeq l_\Omega$ and proposition 26 (a) that $\overline{u \sqcup l_\Omega} \succeq l_\Omega$. Hence, we have $\overline{u \sqcup l_\Omega} \to_? \overline{q_3}$ by applying a rule $l \to r$ in $\mathcal{R}$, because $\overline{\omega}(\overline{(u \sqcup l_\Omega)_\Omega})\#Hlhs(l \to r)$. Therefore, we can take $p' \equiv C'[u \sqcup l_\Omega]$, which leads $\overline{p'} \equiv \overline{C'}\left[\overline{u \sqcup l_\Omega}\right] \to_? \overline{C'[q_3]} \equiv \overline{q_2} \xrightarrow{*}_? \overline{q}$. $\Box$

Lemma 33 *Let $\mathcal{R}$ be a PRS.*

(a) If Δ is a redex of $\mathcal{R}$, then $\overline{\Delta} \to_\Omega \Omega$.
(b) If Δ is $\bullet$-free redex of $\mathcal{R}$, then $\Delta \to_\Omega \Omega$.
(c) $p \to_? q$ implies $\exists q' \preceq \overline{q}$, $\overline{p} \to_\Omega q'$.
(d) $p \xrightarrow{*}_? q$ implies $\exists q' \preceq \overline{q}$, $\overline{p} \xrightarrow{*}_\Omega q'$.

PROOF.

(a) We have $\Delta \succeq l_\Omega$ and $\overline{\omega}(\Delta_\Omega)\#Hlhs(l \to r)$ for some rule $l \to r$ in $\mathcal{R}$. Since $\Delta \succeq l_\Omega$ and l_Ω is $\bullet$-free, we have $\overline{\Delta} \succeq l_\Omega$ by proposition 26 (a). Hence, $\overline{\Delta} \equiv \overline{\Delta} \sqcup l_\Omega$. Since $\overline{\omega}(\Delta_\Omega)\#Hlhs(l \to r)$, we have $\overline{\omega}(\Delta_\Omega) \equiv \overline{\omega}(\overline{(\overline{\Delta})_\Omega}) \equiv \overline{\omega}(\overline{(\overline{\Delta} \sqcup l_\Omega)_\Omega})\#Hlhs(l \to r)$. Therefore, $\overline{\Delta} \to_\Omega \Omega$ by a rule $l \to r$.

(b) As Δ is $\bullet$-free, we have $\Delta \preceq \overline{\Delta}$ by proposition 26 (a). We have $\overline{\Delta} \to_\Omega \Omega$ from (a). Thus, the lemma follows from lemma 28 (a).

(c) We can assume $p \equiv C[\Delta] \to_? C[q_1] \equiv q$ for some $C[\,]$ and q_1. From (a), we have $\overline{p} \equiv \overline{C}\left[\overline{\Delta}\right] \to_\Omega \overline{C}[\Omega] \preceq \overline{C}[\overline{q_1}] \equiv \overline{q}$ that concludes the lemma by taking $q' \equiv \overline{C}[\Omega]$.

(d) We prove by induction on k in $p \xrightarrow{k}_? q$. In case of $k = 0$, the lemma holds by taking $q' \equiv \overline{p}$. Next, we consider $p \to_? q_1 \xrightarrow{*}_? q$. We have $\exists q_2 \preceq \overline{q}$, $\overline{q_1} \xrightarrow{*}_\Omega q_2$. We also have $\exists q_1', q_1' \preceq \overline{q_1}, \overline{p} \to_\Omega q_1'$ by (c). Hence, there exists q' such that $q' \preceq q_2$, $q_1' \xrightarrow{*}_\Omega q'$ by lemma 28 (b). Therefore, we can get $q' \preceq q_2 \preceq \overline{q}$. □

Lemma 34 *Let $C[\ ]$ be $\bullet$-free context. Then, the followings are equivalent.*

(a) $C[\Omega_I]$.
(b) $\Omega(C[\bullet]) \not\equiv \Omega(C[\Omega])$.
(c) $\bullet \in \Omega(C[\bullet])$.

PROOF.

(a)⇒(b) Assuming $\Omega(C[\bullet]) \equiv \Omega(C[\Omega])$, we have $C[\bullet] \xrightarrow{*}_\Omega \Omega(C[\Omega])$. Letting $t \equiv \Omega(C[\Omega])$, t is $\bullet$-free Ω-term in normal form with respect to Ω-reduction. By lemma 32, there is an Ω-free context $C'[\ ]$ such that $C'[\ ] \succeq C[\ ]$ and $\overline{C'[\bullet]} \equiv \overline{C'}[\Omega] \xrightarrow{*}_? \overline{t}$. If $\overline{t}$ is a normal form with respect to ?-reduction, we can conclude that $C[\Omega_I]$ does not hold, since $\overline{t}$ is Ω-free and $\overline{C'}[\ ] \succeq C[\ ]$ by proposition 26 (a). Next we must show $\overline{t}$ is a normal form with respect to ?-reduction. If we assume $\overline{t} \to_? u$ for some u, there exists $u' \preceq \overline{u}$ such that $\overline{\overline{t}} \equiv t \to_\Omega u'$ by lemma 33 (c), which contradicts the fact that t is the normal form with respect to Ω-reduction.

(b)⇒(c) If we assume $\bullet \notin \Omega(C[\bullet])$, we have $\Omega(C[\bullet]) \preceq C[\Omega]$. Since $\Omega(\Omega(t)) \equiv \Omega(t)$ for all t, we have $\Omega(C[\bullet]) \preceq \Omega(C[\Omega])$ from lemma 31. On the other hand, it follows from $C[\Omega] \preceq C[\bullet]$ and lemma 31 that $\Omega(C[\Omega]) \preceq \Omega(C[\bullet])$. Therefore, we have $\Omega(C[\bullet]) \equiv \Omega(C[\Omega])$.

(c)⇒(a) If we assume that $C[\Omega_I]$ does not hold, we have $C'[\Omega] \xrightarrow{*}_? n$ for some $C'[\ ] \succeq C[\ ]$ and Ω-free term n in normal form with respect to ?-reduction. Then we have $\overline{C'[\Omega]} \xrightarrow{*}_\Omega n'$ for some $n' \preceq \overline{n}$ by Lemma 33 (d). Moreover, $\overline{C'}[\ ] \succeq C[\ ]$ by proposition 26 (a). Hence, it follows by lemma 28 (b) that $C[\bullet] \xrightarrow{*}_\Omega n'' \xrightarrow{*}_\Omega \Omega(n'')$ for some $n'' \preceq n'$. Since $\Omega(n'')$ is $\bullet$-free from the fact that $\overline{n}$ is $\bullet$-free and $\overline{n} \succeq n' \succeq n'' \succeq \Omega(n'')$, we get that $\Omega(C[\bullet])$ is also $\bullet$-free, contradicting to $\bullet \in \Omega(C[\bullet])$. □

Example 35 *Consider POR defined in example 2.*

- $or(\Omega_I, \Omega_{NI})$ *because* $or(\bullet, \Omega)$ *is a normal form with respect to* Ω*-reduction and* $or(\Omega, \bullet) \xrightarrow{*}_\Omega \Omega$.
- $or(ff, \Omega_I)$ *because* $or(ff, \bullet)$ *is a normal form with respect to* Ω*-reduction.*

From lemma 34, we can write $C[\Delta_I]$ for some redex Δ instead of $C[\Omega_I]$. $t \xrightarrow{\Delta}_\mathcal{R} s$ is an index reduction if Δ is an index of t. We write $t \to_I s$ if there exists an index reduction $t \xrightarrow{\Delta}_\mathcal{R} s$ for some index Δ.

5 The normalizability of index reduction

We will now show the normalizing property of index reduction for orthogonal PRSs by using the balanced weak Church-Rosser property proposed in [16]. We first explain the basic notions and properties related to the balanced weak Church-Rosser property according to [16].

$\rightarrow$ is *balanced weakly Church-Rosser* (BWCR), if for any term t, s and s', $t \rightarrow s \ \wedge \ t \rightarrow s'$ implies $\exists w, \exists k \geq 0, s \stackrel{k}{\rightarrow} w \ \wedge \ s' \stackrel{k}{\rightarrow} w$.

We write $t \longleftarrow\!\!\!\!\rightarrow s$ if there exists a connection $t \stackrel{m_1}{\longrightarrow} \cdot \stackrel{n_1}{\longleftarrow} \cdot \stackrel{m_2}{\longrightarrow} \cdot \stackrel{n_2}{\longleftarrow} \cdots \stackrel{m_p}{\longrightarrow} \cdot \stackrel{n_p}{\longleftarrow} s$ such that $\Sigma m_i > \Sigma n_i$. We also write $s \longleftarrow\!\!\!\!\rightarrow t$ instead of $t \longleftarrow\!\!\!\!\rightarrow s$

Lemma 36 ([16]) *Let $\rightarrow_s$ be a reduction strategy for $\rightarrow$ such that:*

(a) $\rightarrow_s$ is balanced weakly Church-Rosser,
(b) If $t \rightarrow s$ then $t =_s s$ or $t \longleftarrow\!\!\!\!\rightarrow_s \cdot \leftrightarrow \cdot \longleftarrow\!\!\!\!\rightarrow_s s$.

Then $\rightarrow_s$ is a normalizing strategy.

Now let's show that $\rightarrow_I$ is a normalizing strategy for orthogonal PRSs.

Lemma 37 *Let $C[\Delta_I, \Delta']$. Then $C[\Delta_I, t]$ for all t.*

PROOF. $\bullet \in \Omega(C[\bullet, \Delta']) \equiv \Omega(C[\bullet, \Omega]) \preceq \Omega(C[\bullet, t])$ by lemmas 33 (b) and 31. $\Box$

Lemma 38 *Let $\mathcal{R}$ be orthogonal. Let $\Delta \equiv C[\Delta']$ and $\Delta \not\equiv \Delta'$. Then, $C[t]$ is a redex for all t.*

PROOF. Since Δ is a redex, we have $l_\Omega \preceq \Delta$ and $\overline{\omega}(\Delta_\Omega) \# Hlhs(l \rightarrow r)$ for some rule $l \rightarrow r$ in $\mathcal{R}$. Since $\omega(\Delta') \equiv \Omega$ and Δ and Δ' are not overlapping, we can show $\overline{\omega}(C[\Delta']_\Omega) \preceq \overline{\omega}(C[t]_\Omega)$ and $l_\Omega \preceq C[t]$. Hence $C[t]$ is a redex. $\Box$

Lemma 39 *Let $\mathcal{R}$ be orthogonal. Let $\Delta' \lhd \Delta \trianglelefteq t$. Then, Δ' is not an index of t.*

PROOF. Trivial from definition of index and lemma 38. $\Box$

Lemma 40 *Let $\mathcal{R}$ be orthogonal and strongly sequential. If $t \rightarrow\!\!\!\!\!\!+ s$ then $t =_I s$ or $t \longleftarrow\!\!\!\!\rightarrow_I \cdot \rightarrow\!\!\!\!\!\!+ \cdot \longleftarrow\!\!\!\!\rightarrow_I s$.*

PROOF. Let $t \overset{\Delta_1 \cdots \Delta_n}{\twoheadrightarrow} s$ $(n \geq 0)$. Prove by induction on n. If $n = 0$, then we have $t =_I s$ trivially. Next we consider $n > 0$:

- In case that one of Δ_i, say Δ_1 is an index, we have $t \xrightarrow{\Delta_1}_I t' \overset{\Delta_2 \cdots \Delta_n}{\twoheadrightarrow} s$. By induction hypothesis, we can show the lemma.
- Consider the case that none of Δ_i is an index. From strong sequentiality, there exists an index Δ in t. From orthogonality, Δ is not overlapping to any Δ_i. If Δ is disjoint from all Δ_i, we obtain the lemma by lemma 37. In case that Δ_i appears in Δ for some i, we can assume $\Delta \equiv C[\Delta_1, \ldots, \Delta_m] \overset{\Delta_1, \ldots, \Delta_m}{\twoheadrightarrow} C[t_1, \ldots, t_m]$ for some $m \leq n$ without loss of generality. From lemma 38, $\Delta' \equiv C[t_1, \ldots, t_m]$ is a redex. We can show $t \xrightarrow{\Delta}_I \cdot \twoheadrightarrow \cdot \xleftarrow{\Delta'}_I s$ by using lemma 37. $\square$

Lemma 41 *Let $\mathcal{R}$ be an orthogonal PRS. Let $t \xrightarrow{\Delta}_I s$, $t \xrightarrow{\Delta'}_I s'$ and $s \not\equiv s'$. Then, we have $s \rightarrow_I u$ and $s' \rightarrow_I u$ for some term u.*

PROOF. From orthogonality, $\Delta \not\equiv \Delta'$. Hence, the redexes Δ and Δ' are disjoint from lemma 39. Thus, the lemma holds trivially by lemma 37. $\square$

Theorem 42 *Let $\mathcal{R}$ be orthogonal and strongly sequential. Then index reduction $\rightarrow_I$ is a normalizing strategy.*

PROOF. Let $t \twoheadrightarrow' s$ be $t \equiv C[\Delta_1, \ldots, \Delta_n] \xrightarrow{\Delta_1} \cdots \xrightarrow{\Delta_n} s \equiv C[s_1, \ldots, s_n]$ $(n > 0)$. Since $\rightarrow_I$ is a reduction strategy of $\rightarrow$, it is also a reduction strategy of $\twoheadrightarrow'$. Taking $\rightarrow_I$ and $\twoheadrightarrow'$ as $\rightarrow_s$ and $\rightarrow$ respectively, the premises of lemma 36 are satisfied by lemmas 41 and 40. Hence, we have $\rightarrow_I$ is a normalizing strategy for $\twoheadrightarrow'$. Since $\overset{*}{\rightarrow} = \overset{*}{\twoheadrightarrow}'$, the theorem follows. $\square$

By using the same proof technique in [16], theorem 42 can be extended to hyper index reduction defined as $\overset{*}{\rightarrow}_{NI} \rightarrow_I$, where $\rightarrow_{NI} = \rightarrow - \rightarrow_I$.

Corollary 43 *Let $\mathcal{R}$ be orthogonal and strongly sequential. Then hyper index reduction $\overset{*}{\rightarrow}_{NI} \rightarrow_I$ is a normalizing strategy.*

6 Decidability

In this section, we show strong sequentiality of orthogonal PRS is decidable according to Comon's method [2]. This method depends on two results relating to weak second-order monadic logic (WSkS) [13,15]. One is that WSkS is

decidable. The other is that a set of finite trees is definable in WSkS if and only if it is recognized by a finite tree automaton. By using these results, Comon showed decidability of several kind of sequentialities.

Proposition 44 ([2]) *If a predicate P is definable in a tree automaton, the sequentiality of P is decidable.*

According to proposition 44, if we have a finite tree automaton that accepts terms having Ω-free normal form with respect to ?-reduction, we can prove the decidability of sequentiality of PRS.

Definition 45 *A finite tree automaton is $A = \langle \Sigma, Q, Q_f, \mathcal{T} \rangle$, where Σ is a finite set of symbols with arity, Q is a finite set of states, Q_f is a subset of Q called final states, and $\mathcal{T}$ is a set of transition rules.*

A transition rule has one of the following forms:

- $f(q_1, \ldots, q_n) \to q$ *with* $f \in \Sigma$, $q_1, \ldots, q_n, q \in Q$,
- $q \to q'$ *with* $q, q' \in Q$.

Reduction relation $\to_A$ is naturally defined on $T_{\Sigma \cup Q} \times T_{\Sigma \cup Q}$ by regarding Q as a set of constants. A tree automaton accepts $t \in T_\Sigma$ if and only if there is a reduction $t \stackrel{+}{\to}_A q \in Q_f$.

First we construct a tree automaton A^ω that simulates ω-reduction with respect to PRS $\mathcal{R} = \langle R_0, \sqsupset \rangle$ on $T_{F \cup V}$. We can assume that every variable contained in object terms is only x, since rewrite rules are linear. Thus, in the following we fix $\Sigma = F \cup \{x, \Omega\}$.

Since ω-reduction requires the functions for matching and for compatibility check, we prepare the following finite sets MR_0 and MR_n respectively as states of tree automaton.

$$\begin{aligned} MR_0 &= \{t \mid t \lhd l_\Omega, l \to r \in R_0\} \\ MR_{n+1} &= MR_n \cup \{t \mid t \preceq (s \sqcup s'),\ s, s' \in MR_n \ s.t.\ s \uparrow s'\}, \end{aligned}$$

Let $MR_\Omega = \bigcup_{i \geq 0} MR_i$. Since the height of every term in MR_Ω is restricted to the maximal height of l_Ωs, it is clear that MR_Ω is finite. Let $\bot$ be a fresh constant. $\bot$ represents a term which no subterm of the left-hand sides of rewriting rules matches. Since we have to treat Ω-term as input of tree automaton, we need to distinguish Ω and $\bot$. MR is the set of terms each of which is obtained from $t \in MR_\Omega$ by replacing arbitrary number of Ω occurrences in t with $\bot$. $MR_\bot$ is the set of Ω-free terms in MR. $t \preceq_\bot s$ displays that s can be obtained from t by replacing occurrences of $\bot$ in t with (possibly different) terms.

Example 46 *For POR defined in example 2,*

$$\begin{aligned}\Sigma &= \{or, tt, ff, x, \Omega\},\\ MR &= \{\Omega, \bot, tt\},\\ MR_\bot &= \{\bot, tt\}.\end{aligned}$$

Lemma 47

(a) Let s be maximal in MR with respect to $\preceq_\bot$ such that $s \preceq_\bot t$. Then s is unique.
(b) Let s be maximal in $MR_\bot$ with respect to $\preceq_\bot$ such that $s \preceq_\bot t$. Then s is unique.

PROOF. Let s and r be maximal in MR ($MR_\bot$) with respect to $\preceq_\bot$ such that $s \preceq_\bot t$ and $r \preceq_\bot t$. By replacing all $\bot$ occurring in s and r with Ω we get s' and r' respectively. Since $s' \uparrow r'$, $p' = s' \sqcup r' \in MR$. Thus, by replacing appropriate (all) Ω occurrences in p' with $\bot$, we can obtain $p \in MR$ ($MR_\bot$) such that $s \preceq_\bot p \preceq_\bot t$ and $r \preceq_\bot p \preceq_\bot t$. From the maximality of s and r, it follows that $s \equiv p \equiv r$. $\Box$

Definition 48 *Tree automaton A^ω is $\langle \Sigma, Q^\omega, Q^\omega, \mathcal{T}^\omega \rangle$, where $Q^\omega = \{\langle t \rangle \mid t \in MR\}$, and $\mathcal{T}^\omega$ consists of the following transition rules:*

(a) $\Omega \to \langle \Omega \rangle$.
(b) $x \to \langle \Omega \rangle$.
(c) $f(\langle t_1 \rangle, \ldots, \langle t_n \rangle) \to \langle \Omega \rangle$, *if* $f(t_1, \ldots, t_n) \uparrow l_\Omega$ *for some* $l \to r$ *in* $\mathcal{R}$.
(d) $f(\langle t_1 \rangle, \ldots, \langle t_n \rangle) \to \langle t \rangle$, *if* $f(t_1, \ldots, t_n) \# l_\Omega$ *for all* $l \to r$ *in* $\mathcal{R}$, *and* t *is maximal in* MR *with respect to* $\preceq_\bot$ *such that* $t \preceq_\bot f(t_1, \ldots, t_n)$.

Note that A^ω is deterministic by lemma 47 (a) and completely defined.

Example 49 *For POR defined in example 2,*

$$\begin{aligned}Q^\omega &= \{\langle \Omega \rangle, \langle \bot \rangle, \langle tt \rangle\}\\ \mathcal{T}^\omega &= \{\, \Omega \to \langle \Omega \rangle, x \to \langle \Omega \rangle, tt \to \langle tt \rangle, ff \to \langle \bot \rangle\}\\ &\quad \cup \{or(p_1, p_2) \to \langle \Omega \rangle \mid p_1, p_2 \in Q^\omega\}\end{aligned}$$

Then, we have $or(tt, ff) \xrightarrow{+}_{A^\omega} or(\langle tt \rangle, \langle \bot \rangle) \to_{A^\omega} \langle \Omega \rangle$.

To show properties on A^ω, we need several definitions and lemmas.

Proposition 50 *Let u be $\bot$-free. Then,*

(a) $u \preceq s \preceq_\bot t$ *implies* $u \preceq t$.
(b) $u \uparrow s \preceq_\bot t$ *implies* $u \uparrow t$.

PROOF.

(a) In case of $u \equiv \Omega$, it clearly holds. Let $u \equiv f(u_1, \ldots, u_n)$. Since $u \preceq s \preceq_\bot t$, we can write $s \equiv f(s_1, \ldots, s_n)$ and $t \equiv f(t_1, \ldots, t_n)$, where $u_i \preceq s_i \preceq_\bot t_i$ for all i. Since $u_i \preceq t_i$ from induction hypothesis, we have $u \preceq t$.
(b) In case of $u \equiv \Omega$, it clearly holds. Consider the case $u \equiv f(u_1, \ldots, u_n)$. Then, we have two sub-cases. If $s \equiv \Omega$ then $t \equiv \Omega$ concluding the lemma. If $s \equiv f(s_1, \ldots, s_n)$ then $t \equiv f(t_1, \ldots, t_n)$ and $u_i \uparrow s_i \preceq_\bot t_i$ for all i. Since $u_i \uparrow t_i$ from induction hypothesis, we have $u \uparrow t$. $\Box$

Lemma 51 *Let $u \in MR_\Omega$ and $u \preceq t$. Then, $\exists s \in MR_\bot[u \preceq s \preceq_\bot t]$.*

PROOF. Let s be the term obtained from u by replacing every Ω with $\bot$. From definition of MR, we have $s \in MR_\bot$. We can also show that $u \preceq s \preceq_\bot t$. $\Box$

Corollary 52 *Let s be maximal in $MR_\bot$ with respect to $\preceq_\bot$ such that $s \preceq_\bot t$. Then $\forall u \in MR_\Omega[u \preceq s \iff u \preceq t]$.*

PROOF. As $\Rightarrow$ is trivial by proposition 50 (a), we show $\Leftarrow$. There exists $s' \in MR_\bot$ such that $u \preceq s' \preceq_\bot t$ by lemma 51. Since s is maximal in $MR_\bot$ such that $s \preceq_\bot t$, we have $s' \preceq_\bot s$ by lemma 47 (b). Hence $u \preceq s$ by proposition 50 (a). $\Box$

Lemma 53 *Let $u \in MR_\Omega$ and $u \uparrow t$. Then, $\exists s \in MR[u \uparrow s \preceq_\bot t]$.*

PROOF. Let $s \diamond t$ denote a term defined by: i) $s \diamond \Omega \equiv \Omega$, ii) $s \diamond \bot \equiv \bot$, iii) $\Omega \diamond t \equiv \bot$ for $t \not\equiv \Omega$, iv) $f(s_1, \ldots, s_n) \diamond f(t_1, \ldots, t_n) \equiv f(s_1 \diamond t_1, \ldots, s_n \diamond t_n)$. Consider $u \diamond t$. Since $u \diamond t$ is a term obtained from u by replacing appropriate subterm occurrences with Ω or $\bot$, it follows from $u \in MR$ that $u \diamond t$ is in MR.

Prove by induction on $|u|$ that $u \uparrow (u \diamond t)$ and $(u \diamond t) \preceq_\bot t$. In case of $u \equiv \Omega$, we have $u \uparrow (u \diamond t)$ trivially. Moreover, if $t \equiv \Omega$, $(u \diamond t) \equiv \Omega \equiv t$. Otherwise, we have $(u \diamond t) \equiv \bot \preceq_\bot t$.

Consider the case of $u \equiv f(u_1, \ldots, u_n)$. If $t \equiv \Omega$, it holds clearly because $(u \diamond t) \equiv \Omega$. Next, let $t \equiv f(t_1, \ldots, t_n)$. Since $u_i \uparrow (u_i \diamond t_i)$ and $(u_i \diamond t_i) \preceq_\perp t_i$ for all i from induction hypothesis, the claim follows. $\Box$

Corollary 54 *Let s be maximal in MR with respect to $\preceq_\perp$ such that $s \preceq_\perp t$. Then $\forall u \in MR_\Omega[u \uparrow s \iff u \uparrow t]$.*

PROOF. We can prove it similarly to corollary 52 by using proposition 50 (b) and lemma 53. $\Box$

Lemma 55 *Let s be maximal in MR with respect to $\preceq_\perp$ such that $s \preceq_\perp f(p_1, \cdots, p_n)$, and let every p_i be maximal in MR with respect to $\preceq_\perp$ such that $p_i \preceq_\perp t_i$. Then s is maximal in MR with respect to $\preceq_\perp$ such that $s \preceq_\perp f(t_1, \cdots, t_n)$.*

PROOF. Suppose that there exists some $r \in MR$ such that $s \prec_\perp r \preceq_\perp f(t_1, \cdots, t_n)$. Let $r \equiv f(r_1, \cdots, r_n)$. Then $r_i \preceq_\perp p_i \preceq_\perp t_i$ for all i because of $r_i \in MR$ and the maximality of p_i. Thus, $r \preceq_\perp f(p_1, \cdots, p_n)$ holds; contradiction to the maximality of s. $\Box$

Lemma 56 *Let $t \in T_\Sigma$ and $t \xrightarrow{+}_{A^\omega} \langle s \rangle$. Then, s is maximal in MR with respect to $\preceq_\perp$ such that $s \preceq_\perp \omega(t_\Omega)$ and $\forall u \in MR_\Omega[u \uparrow s \iff u \uparrow \omega(t_\Omega)]$.*

PROOF. By induction on $|t|$, we prove the lemma. According to corollary 54, we have only to show the claim that s is maximal in MR with respect to $\preceq_\perp$ such that $s \preceq_\perp \omega(t_\Omega)$.

Consider the case $|t| = 1$. If t is Ω or x then $\omega(t_\Omega) \equiv s \equiv \Omega$ by the transition rules (a) or (b). If $t \equiv c \not\equiv \Omega$ is a constant not in normal form, then $\omega(t_\Omega) \equiv s \equiv \Omega$ by (c). If $t \equiv c \not\equiv \Omega$ is a constant in normal form, then $s \equiv \perp$ if $c \notin MR$ or $s \equiv c$ otherwise, and $\omega(t_\Omega) \equiv c$ by (d). Thus, in all the cases the claim follows.

Next, we consider the case $|t| > 1$. Let $t \equiv f(t_1, \ldots, t_n) \xrightarrow{+}_{A^\omega} f(\langle p_1 \rangle, \ldots, \langle p_n \rangle) \rightarrow_{A^\omega} \langle s \rangle$.

- If $f(p_1, \cdots, p_n) \uparrow l_\Omega$ for some $l \rightarrow r$ in $\mathcal{R}$, then $s \equiv \Omega$. Letting $l_\Omega \equiv f(l_1, \ldots, l_n)$, we have $\omega((t_i)_\Omega) \uparrow l_i$ for all i by induction hypothesis. It follows from $f(\omega((t_1)_\Omega), \ldots, \omega((t_n))_\Omega) \uparrow l_\Omega$ that $\omega(t_\Omega) \equiv \Omega$. Thus, the claim holds.
- Otherwise, s is maximal in MR with respect to $\preceq_\perp$ such that $s \preceq_\perp f(p_1, \ldots, p_n)$. Suppose $f(\omega((t_1)_\Omega), \ldots, \omega((t_n)_\Omega)) \uparrow l_\Omega \equiv f(l_1, \ldots, l_n)$ for some $l \rightarrow r$ in $\mathcal{R}$. Then, it follows from induction hypothesis that $p_i \uparrow l_i$ for all i: contradiction to $f(p_1, \ldots, p_n) \# l_\Omega$. Thus, we have $\omega(f(t_1, \ldots, t_n)_\Omega) \equiv$

$f(\omega((t_1)_\Omega), \ldots, \omega((t_n)_\Omega))$. From induction hypothesis, every p_i is maximal in MR such that $p_i \preceq_\perp \omega((t_i)_\Omega)$. Thus, by lemma 55 the claim follows. $\Box$

We next define a tree automaton A^{NF} that accepts Ω-free normal forms with respect to ?-reduction. Since each rule in $R_?$ has a condition described by ω-reduction, the states in A^{NF} must keep extra information associated to ω-reduction. Hence, we design each state as a triple $\langle s, r, q\rangle$ associated to a term t, where s and r keep the structures of u and $\omega(u_\Omega)$, respectively. q shows whether t is an Ω-free normal form or not.

Definition 57 *Let A^ω be $\langle \Sigma, Q^\omega, Q^\omega, \mathcal{T}^\omega\rangle$. Then, tree automaton A^{NF} is defined by $\langle \Sigma, Q^{NF}, Q_f^{NF}, \mathcal{T}^{NF}\rangle$, where $Q^{NF} = \{\langle s, r, q\rangle \mid s \in MR_\perp, r \in MR, q \in \{true, false\}\}$, $Q_f^{NF} = \{\langle s, r, true\rangle \in Q^{NF}\}$, and $\mathcal{T}^{NF}$ consists of the following transition rules;*

(a) $\Omega \to \langle \perp, \Omega, false\rangle$.
(b) $x \to \langle \perp, \Omega, true\rangle$.
(c) $f(\langle s_1, r_1, q_1\rangle, \ldots, \langle s_n, r_n, q_n\rangle) \to \langle s, r, q\rangle$, *if all of the following conditions hold:*
  i. *s is maximal in $MR_\perp$ with respect to $\preceq_\perp$ such that $s \preceq_\perp f(s_1, \ldots, s_n)$,*
  ii. *$f(\langle r_1\rangle, \ldots, \langle r_n\rangle) \to \langle r\rangle \in \mathcal{T}^\omega$;*
  iii. *$q \equiv true$, if $q_i \equiv true$ for all i and $\neg(l_\Omega \preceq f(s_1, \ldots, s_n) \wedge f(r_1, \ldots, r_n)\#Hlhs(l \to w)$ for all rule $l \to w$ in $\mathcal{R}$. Otherwise, $q \equiv false$.*

In above definition iii., $q \equiv true$ means that the corresponding term t is an Ω-free normal form. Thus, q should be set to $true$ if all subterms of t are in Ω-free normal form and t is not a redex of $\mathcal{R}$.

Note that A^{NF} is deterministic by lemma 47 (b) and completely defined.

Example 58 *For POR defined in example 2,*

$$
\begin{aligned}
Q^{NF} &= \{\langle s, r, true\rangle, \langle s, r, false\rangle \mid s \in \{\perp, tt\}, r \in \{\Omega, \perp, tt\}\} \\
Q_f^{NF} &= \{\langle s, r, true\rangle \mid s \in \{\perp, tt\}, r \in \{\Omega, \perp, tt\}\} \\
\mathcal{T}^{NF} &= \{\Omega \to \langle \perp, \Omega, false\rangle, x \to \langle \perp, \Omega, true\rangle, \\
&\qquad tt \to \langle tt, tt, true\rangle, ff \to \langle \perp, \perp, true\rangle\} \\
&\quad \cup \mathcal{T}'
\end{aligned}
$$

where, $\mathcal{T}'$ is a set of rules for or tabulated as follows (unreachable states and

meaningless rules are omitted).

2nd arg. / *1st arg.*	$\langle \bot, \Omega, false\rangle$	$\langle \bot, \Omega, true\rangle$	$\langle tt, tt, true\rangle$	$\langle \bot, \bot, true\rangle$
$\langle \bot, \Omega, false\rangle$	$\langle \bot, \Omega, false\rangle$	$\langle \bot, \Omega, false\rangle$	$\langle \bot, \Omega, false\rangle$	$\langle \bot, \Omega, false\rangle$
$\langle \bot, \Omega, true\rangle$	$\langle \bot, \Omega, false\rangle$	$\langle \bot, \Omega, true\rangle$	$\langle \bot, \Omega, true\rangle$	$\langle \bot, \Omega, true\rangle$
$\langle tt, tt, true\rangle$	$\langle \bot, \Omega, false\rangle$	$\langle \bot, \Omega, false\rangle$	$\langle \bot, \Omega, false\rangle$	$\langle \bot, \Omega, false\rangle$
$\langle \bot, \bot, true\rangle$	$\langle \bot, \Omega, false\rangle$	$\langle \bot, \Omega, true\rangle$	$\langle \bot, \Omega, false\rangle$	$\langle \bot, \Omega, false\rangle$

Then, we have $or(x, tt) \stackrel{+}{\rightarrow}_{A^{NF}} or(\langle \bot, \Omega, true\rangle, \langle tt, tt, true\rangle) \rightarrow_{A^{NF}} \langle \bot, \Omega, true\rangle \in Q_f^{NF}$*, which shows that* $or(x, tt)$ *is an* Ω*-free normal form with respect to* $?$*-reduction.*

Lemma 59 *Let* $t \in T_\Sigma$ *and* $t \stackrel{+}{\rightarrow}_{A^{NF}} \langle s, r, q\rangle$*. Then,*

(i) s *is maximal in* $MR_\bot$ *with respect to* $\preceq_\bot$ *such that* $s \preceq_\bot t$*,*
(ii) $t \stackrel{+}{\rightarrow}_{A^\omega} \langle r\rangle$*,*
(iii) t *is an* Ω*-free normal form if and only if* $q \equiv true$*.*

PROOF. From the construction of A^{NF}, (ii) is trivial. Prove by induction on $|t|$.

Consider the case $|t| = 1$. If t is Ω then $t \rightarrow_{A^{NF}} \langle \bot, \Omega, false\rangle$ by (a). If t is x then $t \rightarrow_{A^{NF}} \langle \bot, \Omega, true\rangle$ by (b). Let $t \equiv c \not\equiv \Omega$. If c is a normal form then $t \rightarrow_{A^{NF}} \langle s, r, true\rangle$; otherwise, $t \rightarrow_{A^{NF}} \langle s, r, false\rangle$ by (c), which satisfies the lemma.

Consider the case $|t| > 1$. Let $t \equiv f(t_1, \ldots, t_n) \stackrel{+}{\rightarrow}_{A^{NF}} f(\langle s_1, r_1, q_1\rangle, \ldots, \langle s_n, r_n, q_n\rangle) \rightarrow_{A^{NF}} \langle s, r, q\rangle$, where s, r and q satisfy the conditions in rule (c). Then, (i) follows from induction hypothesis and lemma 55. Next, consider (iii).

Case 1. t_i is in Ω-free normal form for all i. From induction hypothesis, we have $q_i \equiv true$ for all i.

- If t is not in Ω-free normal form, there exists some rule $l \rightarrow w$ in $\mathcal{R}$ such that $l_\Omega \preceq t$ and $\overline{\omega}(t_\Omega) \# Hlhs(l \rightarrow w)$. Let $l \equiv f(l_1, \ldots, l_n)$. Since $t_i \stackrel{+}{\rightarrow}_{A^{NF}} \langle s_i, r_i, q_i\rangle$ for all i, s_i is maximal in $MR_\bot$ with respect to $\preceq_\bot$ such that $s_i \preceq_\bot t_i$ by induction hypothesis. It follows from $(l_i)_\Omega \preceq t_i$ and corollary 52 that $(l_i)_\Omega \preceq s_i$. Thus, we have $l_\Omega \preceq f(s_1, \ldots, s_n)$. Moreover, it follows from lemma 56 that $f(r_1, \ldots, r_n) \# Hlhs(l \rightarrow w)$. Hence, we have $q \equiv false$ from the definition of (c).

- If t is in Ω-free normal form. We can show that $\neg(l_\Omega \preceq f(s_1, \ldots, s_n) \wedge f(r_1, \ldots, r_n)\#Hlhs(l \to w))$ for all rule $l \to w$ in $\mathcal{R}$. Hence, $q \equiv true$.

Case 2. t_i is not in Ω-free normal form for some i. From induction hypothesis, we have $q_i \equiv false$. Hence, $q \equiv false$ from the definition of (c). $\Box$

Corollary 60 *Let* $f(t_1, \cdots, t_n) \in T_\Sigma$ *and* $f(t_1, \cdots, t_n) \xrightarrow{+}_{A^{NF}} f(\langle s_1, r_1, q_1\rangle, \cdots, \langle s_n, r_n, q_n\rangle)$. *Then the following holds for every rule* $l \to w$.

(a) $l_\Omega \preceq f(t_1, \cdots, t_n)$ *if and only if* $l_\Omega \preceq f(s_1, \cdots, s_n)$.
(b) $\overline{\omega}(f(t_1, \cdots, t_n)_\Omega)\#Hlhs(l \to w)$ *if and only if* $f(r_1, \cdots, r_n)\#Hlhs(l \to w)$.

PROOF.

(a) From $t_i \xrightarrow{+}_{A^{NF}} \langle s_i, r_i, q_i\rangle$ and lemma 59 (i), s_i is maximal in $MR_\perp$ such that $s_i \preceq_\perp t_i$. Hence, we have $(l_i)_\Omega \preceq s_i \iff (l_i)_\Omega \preceq t_i$ by corollary 52. Thus, we have $l_\Omega \preceq f(s_1, \ldots, s_n) \iff l_\Omega \preceq f(t_1, \ldots, t_n)$.
(b) From $t_i \xrightarrow{+}_{A^{NF}} \langle s_i, r_i, q_i\rangle$ and lemma 59 (ii), we have $t_i \xrightarrow{+}_{A^\omega} \langle r_i\rangle$. From lemma 56, we have $p_i \uparrow r_i \iff p_i \uparrow \omega((t_i)_\Omega)$ for any $p_i \in MR_\Omega$. Thus, $f(p_1, \ldots, p_n) \uparrow f(r_1, \ldots, r_n) \iff f(p_1, \ldots, p_n) \uparrow \overline{\omega}(f(t_1, \ldots, t_n)_\Omega)$. Hence, $f(r_1, \ldots, r_n)\#Hlhs(l \to w) \iff \overline{\omega}(f(t_1, \ldots, t_n)_\Omega)\#Hlhs(l \to w)$. $\Box$

Now, we construct a tree automaton $A^?$ that recognizes terms having Ω-free normal form with respect to ?-reduction. We start to show some preliminary properties of $R_?$.

Lemma 61 *Let* $t \equiv C[\Delta] \xrightarrow{\Delta}_? C[t'] \xrightarrow{\Delta'}_? s$ *and* $\Delta' \trianglelefteq t'$. *Then* $t \xrightarrow{\Delta}_? s$.

PROOF. Trivial from the definition of $\to_?$. $\Box$

We define a tree automaton $A^?$ that recognizes terms having Ω-free normal form with respect to ?-reduction as follows.

Definition 62 *Let* A^{NF} *be* $\langle \Sigma, Q^{NF}, Q_f^{NF}, \mathcal{T}^{NF}\rangle$. *Then, tree automaton* $A^?$ *is defined by* $\langle \Sigma, Q^?, Q_f^?, \mathcal{T}^?\rangle$, *where* $Q^? = Q^{NF}$, $Q_f^? = Q_f^{NF}$. $\mathcal{T}^?$ *is a set both of transition rules in* $\mathcal{T}^{NF}$ *and the following rules:*

(d) $f(\langle s_1, r_1, q_1\rangle, \ldots, \langle s_n, r_n, q_n\rangle) \to \langle s, r, q\rangle$, *if all of the following conditions hold:*
  i. $l_\Omega \preceq f(s_1, \ldots, s_n)$ *and* $f(r_1, \ldots, r_n)\#Hlhs(l \to w)$ *for some rule* $l \to w$ *in* $\mathcal{R}$.

ii. $u \xrightarrow{+}_{A^{NF}} \langle s, r, q\rangle$ *for some term* u *in* T_Σ.

Remark 63 *(i) The above condition i. means that some rule in* $\mathcal{R}$ *is applicable.*

(ii) The above condition ii. is decidable for a given $\langle s, r, q\rangle$ *by the decidability of emptiness problem for tree automata as the set of* u *is accepted by a finite tree automata.*

(iii) $A^?$ *is obviously non-deterministic because of the condition ii. in (d), and non-deterministic selection between (c) and (d).*

Example 64 *The set of reachable states of* A^{NF} *in example 58 is*

$$Q' = \{\langle \bot, \Omega, false\rangle, \langle \bot, \Omega, true\rangle, \langle tt, tt, true\rangle, \langle \bot, \bot, true\rangle\}.$$

Then, transition rules $\mathcal{T}^?$ *for POR defined in example 2 consists of* $\mathcal{T}^{NF}$ *and the following rules (unreachable states and meaningless rules are omitted):*

$$\begin{aligned} &\{or(\langle tt, tt, true\rangle, p) \to p' \mid p, p' \in Q'\} \\ \cup\ &\{or(\langle \bot, \bot, true\rangle, \langle tt, tt, true\rangle) \to p \mid p \in Q'\} \\ \cup\ &\{or(\langle \bot, \bot, true\rangle, \langle \bot, \bot, true\rangle) \to p \mid p \in Q'\} \end{aligned}$$

For example, $or(or(tt, x), \Omega)$ *has an* Ω*-free normal form with respect to ?-reduction, because it has a reduction*

$$\begin{aligned} or(or(tt, x), \Omega) &\xrightarrow{+}_{A^?} or(or(\langle tt, tt, true\rangle, \langle \bot, \Omega, true\rangle), \langle \bot, \Omega, false\rangle) \\ &\to_{A^?} or(\langle tt, tt, true\rangle, \langle \bot, \Omega, false\rangle) \\ &\to_{A^?} \langle \bot, \bot, true\rangle \in Q^?_f. \end{aligned}$$

Lemma 65 *Let* $t \in T_\Sigma$ *and* $t \xrightarrow{+}_{A^?} \langle s, r, true\rangle$. *Then there exists some* Ω*-free normal form* t' *such that* $t \xrightarrow{*}_? t'$.

PROOF. By induction on the number k of transitions by rule (d) occurring in $t \xrightarrow{+}_{A^?} \langle s, r, true\rangle$, we prove the claim.

In case of $k = 0$, since no transitions by rule (d) occur in $t \xrightarrow{+}_{A^?} \langle s, r, true\rangle$, we have $t \xrightarrow{+}_{A^{NF}} \langle s, r, true\rangle$. Thus, t is Ω-free normal form by lemma 59. Take $t' \equiv t$.

In case of $k > 0$, we can write $t \equiv C[f(t_1, \cdots, t_n)] \xrightarrow{+}_{A^?} C[f(\langle s_1, r_1, q_1\rangle, \ldots, \langle s_n, r_n, q_n\rangle)] \rightarrow_{A^?} C[\langle s', r', q'\rangle] \xrightarrow{*}_{A^?} \langle s, r, true\rangle$ without loss of generality, where the first transition by rule (d) is $C[f(\langle s_1, r_1, q_1\rangle, \ldots, \langle s_n, r_n, q_n\rangle)] \rightarrow_{A^?} C[\langle s', r', q'\rangle]$. Then, we have $l_\Omega \preceq f(s_1, \cdots, s_n)$ and $\overline{\omega}(f(r_1, \cdots, r_n)) \# Hlhs(l \rightarrow w)$ for some $l \rightarrow w$ in $\mathcal{R}$, and $u \xrightarrow{+}_{A^{NF}} \langle s', r', q'\rangle$ for some u in T_Σ. As $\rightarrow_{A^{NF}} \subseteq \rightarrow_{A^?}$, $u \xrightarrow{+}_{A^?} \langle s', r', q'\rangle$. Hence, the number of transitions by rule (d) in $C[u] \xrightarrow{+}_{A^?} C[\langle s', r', q'\rangle] \xrightarrow{*}_{A^?} \langle s, r, true\rangle$ is less than that in $t \xrightarrow{+}_{A^?} \langle s, r, true\rangle$. From induction hypothesis, it follows that $C[u] \xrightarrow{*}_? t'$ for some Ω-free normal form t'. Since $f(t_1, \cdots, t_n) \xrightarrow{+}_{A^{NF}} f(\langle s_1, r_1, q_1\rangle, \ldots, \langle s_n, r_n, q_n\rangle)$, $l_\Omega \preceq f(s_1, \cdots, s_n)$, and $\overline{\omega}(f(r_1, \cdots, r_n)) \# Hlhs(l \rightarrow w)$, we have $l_\Omega \preceq f(t_1, \cdots, t_n)$ and $\overline{\omega}(f(t_1, \cdots, t_n)_\Omega) \# Hlhs(l \rightarrow w)$ by corollary 60. Thus, $C[f(t_1, \cdots, t_n)] \rightarrow_? C[u]$. Therefore, $t \equiv C[f(t_1, \cdots, t_n)] \rightarrow_? C[u] \xrightarrow{*}_? t'$. $\square$

Lemma 66 *Let $t, t' \in T_\Sigma$ and $t \xrightarrow{*}_? t'$. Then there exist some s, r, q such that $t \xrightarrow{+}_{A^?} \langle s, r, q\rangle$ and $t' \xrightarrow{+}_{A^{NF}} \langle s, r, q\rangle$.*

PROOF. By lemma 61, we can suppose that $t \xrightarrow{*}_? t'$ is a reduction from the inside to the outside possibly with some jumps. By induction on $|t|$ we prove the claim.

Consider the case $|t| = 1$. If $t \equiv t'$, then the claim holds as $t \equiv t' \xrightarrow{+}_{A^{NF}} \langle s, r, q\rangle$. Let $t \equiv c \not\equiv t'$. Then there exists some rule $c \rightarrow w$ in $\mathcal{R}$ such that $c \# Hlhs(c \rightarrow w)$. Hence, by applying transition rule (d) at c we have $c \rightarrow_{A^?} \langle s, r, q\rangle$ where $t' \xrightarrow{+}_{A^{NF}} \langle s, r, q\rangle$.

Consider the case $|t| > 1$.

Case 1. Let $t \equiv f(t_1, \cdots, t_n) \xrightarrow{*}_? t' \equiv f(t'_1, \cdots, t'_n)$ where $t_i \xrightarrow{*}_? t'_i$ for all i. Then, from induction hypothesis we have $t_i \xrightarrow{+}_{A^?} \langle s_i, r_i, q_i\rangle$ and $t'_i \xrightarrow{+}_{A^{NF}} \langle s_i, r_i, q_i\rangle$ for some s_i, r_i, q_i for all i. Let $f(\langle s_1, r_1, q_1\rangle, \ldots, \langle s_n, r_n, q_n\rangle) \rightarrow_{A^{NF}} \langle s, r, q\rangle$. Then, we have $t \equiv f(t_1, \cdots, t_n) \xrightarrow{+}_{A^?} f(\langle s_1, r_1, q_1\rangle, \ldots, \langle s_n, r_n, q_n\rangle) \rightarrow_{A^{NF}} \langle s, r, q\rangle$ and $t' \equiv f(t'_1, \cdots, t'_n) \xrightarrow{+}_{A^{NF}} \langle s, r, q\rangle$.

Case 2. Let $t \equiv f(t_1, \cdots, t_n) \xrightarrow{*}_? f(u_1, \cdots, u_n) \xrightarrow{\epsilon}_? t'$ (ϵ denotes the root position) where $t_i \xrightarrow{*}_? u_i$ for all i. Then, from induction hypothesis we have $t_i \xrightarrow{+}_{A^?} \langle s_i, r_i, q_i\rangle$ and $u_i \xrightarrow{+}_{A^{NF}} \langle s_i, r_i, q_i\rangle$ for some s_i, r_i, q_i for all i. Since $f(u_1, \cdots, u_n) \xrightarrow{\epsilon}_? t'$, there exists some rule $l \rightarrow w$ such that $l_\Omega \preceq f(u_1, \cdots, u_n)$ and $\overline{\omega}(f(u_1, \cdots, u_n)_\Omega) \# Hlhs(l \rightarrow w)$. Then by corollary 60 we have $l_\Omega \preceq f(s_1, \cdots, s_n)$ and $f(r_1, \cdots, r_n) \# Hlhs(l \rightarrow w)$. Hence, by applying transition rule (d) it holds that $f(t_1, \cdots, t_n) \xrightarrow{+}_{A^?} f(\langle s_1, r_1, q_1\rangle, \ldots, \langle s_n, r_n, q_n\rangle) \rightarrow_{A^?} \langle s, r, q\rangle$ where $t' \xrightarrow{+}_{A^{NF}} \langle s, r, q\rangle$. $\square$

Corollary 67 *Let $t, t' \in T_\Sigma$ and $t \xrightarrow{*}_? t'$. If t' is in Ω-free normal form, then there exist some s, r such that $t \xrightarrow{+}_{A^?} \langle s, r, true \rangle$.*

PROOF. Straightforward from lemma 59 and 66. □

Theorem 68 *The strong sequentiality of PRS is decidable.*

PROOF. It follows from lemma 65, corollary 67, and proposition 44. □

7 Conclusion

We have proposed an operational semantics of PRSs by using conditional systems, whose reduction relation is decidable and stable under substitution. By using this framework of PRSs, we have discussed a normalizing strategy of PRSs and have provided the class of strongly sequential PRSs with a decidable normalizing strategy. The complexity for deciding strong sequentiality is still left open. The presented normalizing strategy may be extended to more general PRSs having overlapping rules without priority between them. Because, our proof technique based on balanced weakly Church-Rosser property can be applied to overlapping situation [16].

For lazy evaluation in functional programming, the root-normalization property [10] is also important, because it is useful to calculate an infinite normal form. Root-normalization for strongly sequential PRSs can be shown by using the notion of descendants of redex position. We believe that further research in this direction will exploit the potential of PRSs. It is also an interesting open problem to find a subclass of strongly sequential PRSs for which an efficient normalizing strategy exists.

Acknowledgement

This work is partially supported by Grants from Ministry of Education, Science and Culture of Japan, #07680350, #07680347 and #09245212. We would like to thank all the referees for their valuable comments and suggestions on an earlier draft of this paper, which have resulted in significantly improving the contents of this article.

References

[1] J. C. M. Baeten, J. A. Bergstra, J. W. Klop, and W. P. Weijland, Term rewriting systems with rule priorities, *Theoretical Computer Science*, **67** (1989) 283–301.

[2] H. Comon, Sequentiality, second order monadic logic and tree automata, In *Proc. of the 10th annual IEEE Symposium on Logic in Computer Science* (1995) 508–517.

[3] N. Dershowitz and J.-P. Jouannaud, Rewrite Systems, In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, vol.B, (North-Holland, 1990) 243–320.

[4] P. Henderson, *Functional Programming*, (Prentice Hall International, 1980).

[5] G. Huet and J.-J. Lévy, Computations in Orthogonal Rewriting Systems, I and II, in *Computational Logic, Essays in Honor of Alan Robinson*, (The MIT Press, 1991) 396–443.

[6] G. Huet, Confluent Reductions: Abstract Properties and Applications to Term Rewriting Systems, *Journal of Association for Computing Machinery*, Vol.27, No.4 (1980) 797–821.

[7] S. L. P. Jones, *The Implementation of Functional Programming Languages*, (Prentice Hall International, 1987).

[8] J. W. Klop, Term rewriting systems, In S. Abramsky, D. Gabbay, and T. Maibaum, editors, *Handbook of Logic in Computer Science*, vol. I, (Oxford University Press, 1992).

[9] J. W. Klop and A. Middeldorp, Sequentiality in orthogonal term rewritng systems, *J. Symbolic Computation*, **12** (1991) 161–195.

[10] A. Middeldorp, Call by Need Computations to Root-Stable Form, *Proc. of the 24th Annual ACM Symposium on Principles of Programming Languages* (1997) 94–105.

[11] C. K. Mohan, Priority rewriting: Semantics, confluence, and conditionals, In *Proc. of the 3rd International Conference, RTA-89, LNCS*, vol.355, (Springer-Verlag, 1989) 278–291.

[12] R. Plasmeijer and M. van Eekelen, *Functional Programming and Parallel Graph Rewriting*, (Addison-Wesley, 1993).

[13] M. Rabin, Decidable theories, In J. Barwise, editor, *Handbook of Mathematical Logic* (North-Holland, 1977) 595–629.

[14] B. K. Rosen, Tree-manipulating systems and Church-Rosser theorems, *JACM*, Vol.20, (1973) 160–187.

[15] W. Thomas, Automata on infinite objects, In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science* (Elsevier, 1990) 134–191.

[16] Y. Toyama, Strong sequentiality of left-linear overlapping term rewriting systems, In *Proc. of the 7th annual IEEE Symposium on Login in Computer Science* (1992) 274–284.

[17] Y. Toyama, S. Smetsers, M. van Eekelen, and R. Plasmeijer, The Functional Strategy and Transitive Term Rewriting Systems, In Ronan Sleep, Rinus Plasmeijer, and Marko van Eelkelen, editors, *Term Graph Rewriting: Theory and Practice* (John Wiley & Sons Ltd, 1993) 61–75.

[18] J. van de Pol, Operational semantics of term rewriting with priorities, Technical Report 162, Dep. of Philosophy, Utrecht University, 1996.