PAPER

Conditional Linearization of Non-Duplicating Term Rewriting Systems*

Yoshihito TOYAMA† *and* **Michio OYAMAGUCHI**††, *Members*

SUMMARY We propose a new conditional linearization based on *left-right separated* conditional term rewriting systems, in which the left-hand side and the right-hand side of a rewrite rule have separate variables. By developing a concept of weight decreasing joinability we first present a sufficient condition for the Church-Rosser property of left-right separated conditional term rewriting systems. Applying this result to conditional linearization, we next show sufficient conditions for the unique normal form property and the Church-Rosser property of non-duplicating (unconditional) term rewriting systems even if they are non-left-linear or overlapping.

key words: term rewriting system, linearization, Church-Rosser, unique normal form, non-duplicating

1. Introduction

The original idea of the conditional linearization of non-left-linear term rewriting systems was introduced by Klop and De Vrijer [5], De Vrijer [4] for giving a simpler proof of Chew's theorem [3], [7], [10]. They developed an interesting method for proving the unique normal form property for some non-Church-Rosser, non-left-linear term rewriting system R. The method is based on the fact that the unique normal form property of the original non-left-linear term rewriting system R follows the Church-Rosser property of an associated left-linear conditional term rewriting system R^L which is obtained from R by *linearizing* a non-left-linear rule, for example $Dxx \to x$, into a left-linear conditional rule $Dxy \to x \Leftarrow x = y$. Bergstra and Klop [2] proved that non-overlapping left-linear semi-equational conditional term rewriting systems are Church-Rosser. Hence, combining these two results, Klop and De Vrijer [5], De Vrijer [4] showed that the term rewriting system R has the unique normal form property if R^L is non-overlapping. However, as their conditional linearization technique is based on the Church-Rosser property for the traditional conditional term rewriting system R^L [2], its application is restricted in non-overlapping R^L (though this limitation may be slightly relaxed with R^L containing only trivial critical pairs).

Manuscript received 2000
Manuscript revised 2000

†The author is with Research Institute of Electrical Communication, Tohoku University, Katahira 2-1-1, Aoba-ku, Sendai 980-8577, Japan.

††The author is with Faculty of Engineering, Mie University, Kamihama-cho, Tsu-shi 514-8504, Japan.

*A preliminary version of this paper was reported as [11].

In this paper, we introduce a new conditional linearization based on a *left-right separated* conditional term rewriting system R_L. The point of our linearization is that a non-left-linear rule $Dxx \to x$ is translated into a left-linear conditional rule $Dxy \to z \Leftarrow x = z, y = z$ in which the left-hand side and the right-hand side have separate variables. By considering this new system R_L instead of the traditional conditional system R^L we can easily relax the non-overlapping limitation of conditional systems originated from Bergstra and Klop [2] if the original system R is non-duplicating. Here, R is non-duplicating if for any rewrite rule $l \to r$, no variable has more occurrences in r than it has in l.

By developing a new concept of weight decreasing joinability we first present a sufficient condition for the Church-Rosser property of a left-right separated conditional term rewriting system R_L, which may have overlapping rewrite rules. We next apply this result to our conditional linearization, and show a sufficient condition for the unique normal form property of the original non-duplicating system R, which may be non-left-linear and overlapping. Moreover, our result can be naturally applied to proving the Church-Rosser property of some non-duplicating non-left-linear overlapping term rewriting systems such as right-ground systems.

In the next section we give a concise explanation of abstract reduction systems. In section 3 we introduce the notion of weight decreasing joinability, which is the main tool used throughout the paper to prove the Church-Rosser property of conditional term rewriting systems. Section 4 briefly explains the basic notions and definitions concerning term rewriting systems. In section 5 we define the notion of left-right separated conditional term rewriting systems and show a sufficient condition for the Church-Rosser property of the systems. Section 6 introduces a new conditional linearization based on left-right separated conditional term rewriting systems. By using the conditional linearization technique we give a sufficient condition for the unique normal form property of (unconditional) term rewriting systems which are non-duplicating, non-left-linear, and overlapping. In Section 7 we show that the conditional linearization proposed can be used as a useful method for proving the Church-Rosser property of some class of non-duplicating (unconditional) term rewriting systems.

2. Reduction Systems

Assuming that the reader is familiar with the basic concepts and notations concerning reduction systems in [1], [6], we briefly explain notations and definitions.

A reduction system (or an abstract reduction system) is a structure $A = \langle D, \rightarrow \rangle$ consisting of some set D and some binary relation $\rightarrow$ on D (i.e., $\rightarrow \subseteqq D \times D$), called a reduction relation. A reduction (starting with x_0) in A is a finite or infinite sequence $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots$. The identity of elements x, y of D is denoted by $x \equiv y$. $\leftrightarrow$ is the symmetric closure of $\rightarrow$, $\overset{*}{\rightarrow}$ is the transitive reflexive closure of $\rightarrow$, and $\overset{*}{\leftrightarrow}$ is the equivalence relation generated by $\rightarrow$. We write $x \leftarrow y$ if $y \rightarrow x$; likewise $x \overset{*}{\leftarrow} y$. We use $\overset{n}{\rightarrow}$ to denote a reduction of n ($n \geqq 0$) steps.

We say that x is a normal form if $\neg\exists y \in D[x \rightarrow y]$; let NF be the set of normal forms. If $x \overset{*}{\rightarrow} y$ and $y \in NF$ then we say x has a normal form y and y is a normal form of x.

Definition 1: $A = \langle D, \rightarrow \rangle$ is confluent, or Church-Rosser iff
$\forall x, y, z \in D[x \overset{*}{\rightarrow} y \wedge x \overset{*}{\rightarrow} z \Rightarrow \exists w \in D.y \overset{*}{\rightarrow} w \wedge z \overset{*}{\rightarrow} w]$.

Definition 2: $A = \langle D, \rightarrow \rangle$ has unique normal forms iff $\forall x, y \in NF[x \overset{*}{\leftrightarrow} y \Rightarrow x \equiv y]$.

Note that A has unique normal forms if A is Church-Rosser.

The following fact observed by Klop and De Vrijer [5], De Vrijer [4] plays an essential role in our linearization too.

Proposition 1 (Klop and De Vrijer)**:**
Let $A_0 = \langle D, \underset{0}{\rightarrow} \rangle$ and $A_1 = \langle D, \underset{1}{\rightarrow} \rangle$ be two reduction systems with the sets of normal forms NF_0 and NF_1 respectively. Then A_0 has unique normal forms if each of the following conditions holds:
(i) $\underset{1}{\rightarrow}$ extends $\underset{0}{\rightarrow}$,
(ii) A_1 is Church-Rosser,
(iii) NF_1 includes NF_0.

Proof. Let $x, y \in NF_0$ such that $x \overset{*}{\underset{0}{\leftrightarrow}} y$. From (i) and (iii), we have $x, y \in NF_1$ and $x \overset{*}{\underset{1}{\leftrightarrow}} y$. Since A_1 is Church-Rosser, it follows that $x \equiv y$. □

Note. In the above proposition, the condition (ii) can be replaced with the following condition: A_1 has unique normal forms.

3. Weight Decreasing Joinability

This section introduces a new concept of weight decreasing joinability. In the later sections this concept is used for analyzing the Church-Rosser property of conditional term rewriting systems with extra variables occurring in conditional parts of rewrite rules.

Let N^+ be the set of positive integers. $A = \langle D, \rightarrow \rangle$ is a weighted reduction system if $\rightarrow = \cup_{w \in N^+} \rightarrow_w$, that is, positive integers (weights w) are assigned to each reduction step to represent costs.

Definition 3: A proof of $x \overset{*}{\leftrightarrow} y$ is a sequence $\mathcal{P}$: $x_0 \leftrightarrow_{w_1} x_1 \leftrightarrow_{w_2} x_2 \leftrightarrow_{w_3} \cdots \leftrightarrow_{w_n} x_n$ ($n \geqq 0$) such that $x \equiv x_0$ and $y \equiv x_n$. The weight $w(\mathcal{P})$ of the proof $\mathcal{P}$ is $\sum_{i=1}^{n} w_i$. If $\mathcal{P}$ is a 0 step sequence (i.e., $n = 0$), then $w(\mathcal{P}) = 0$.

We usually abbreviate a proof $\mathcal{P}$ of $x \overset{*}{\leftrightarrow} y$ by $\mathcal{P}$: $x \overset{*}{\leftrightarrow} y$. The form of a proof may be indicated by writing, for example, $\mathcal{P}$: $x \overset{*}{\rightarrow} \cdot \overset{*}{\leftarrow} y$, $\mathcal{P}'$: $x \leftarrow \cdot \overset{*}{\rightarrow} \cdot \leftarrow y$, etc. We use the symbols $\mathcal{P}, \mathcal{Q}, \cdots$ for proofs.

Definition 4: A weighted reduction system $A = \langle D, \rightarrow \rangle$ is weight decreasing joinable iff for all $x, y \in D$ and any proof $\mathcal{P}$: $x \overset{*}{\leftrightarrow} y$ there exists some proof $\mathcal{P}'$: $x \overset{*}{\rightarrow} \cdot \overset{*}{\leftarrow} y$ such that $w(\mathcal{P}) \geqq w(\mathcal{P}')$.

It is clear that if a weighted reduction system A is weight decreasing joinable then A is Church-Rosser. We will now show a sufficient condition for the weight decreasing joinability.

Lemma 1: Let A be a weighted reduction system. Then A is weight decreasing joinable if for any $x, y \in D$ and any proof $\mathcal{P}$: $x \leftarrow \cdot \rightarrow y$ one of the following conditions holds:
(i) there exists a proof $\mathcal{P}'$: $x \overset{*}{\leftrightarrow} y$ such that $w(\mathcal{P}) > w(\mathcal{P}')$, or
(ii) there exist proofs $\mathcal{P}'$: $x \rightarrow \cdot \overset{*}{\leftrightarrow} y$ and $\mathcal{P}''$: $x \overset{*}{\leftrightarrow} \cdot \leftarrow y$ such that $w(\mathcal{P}) \geqq w(\mathcal{P}')$ and $w(\mathcal{P}) \geqq w(\mathcal{P}'')$, or
(iii) there exists a proof $\mathcal{P}'$: $x \rightarrow y$ (or $x \leftarrow y$) such that $w(\mathcal{P}) \geqq w(\mathcal{P}')$.

Proof. By induction on the weight $w(\mathcal{Q})$ of a proof $\mathcal{Q}$: $x \overset{*}{\leftrightarrow} y$, we prove that there exists a proof $\mathcal{Q}'$: $x \overset{*}{\rightarrow} \cdot \overset{*}{\leftarrow} y$ such that $w(\mathcal{Q}) \geqq w(\mathcal{Q}')$. *Base step* ($w(\mathcal{Q}) = 0$) is trivial. *Induction step*: Let $\mathcal{Q} : x \leftrightarrow x' \overset{*}{\leftrightarrow} y$ and let $\mathcal{S} : x' \overset{*}{\leftrightarrow} y$ be the subproof of $\mathcal{Q}$. From induction hypothesis, there exists a proof $\mathcal{S}'$: $x' \overset{*}{\rightarrow} \cdot \overset{*}{\leftarrow} y$ such that $w(\mathcal{S}) \geqq w(\mathcal{S}')$. Thus, if $x \rightarrow x'$ then we have $\mathcal{Q}'$: $x \rightarrow x' \overset{*}{\rightarrow} \cdot \overset{*}{\leftarrow} y$ such that $w(\mathcal{Q}) \geqq w(\mathcal{Q}')$. Otherwise we have a proof $\mathcal{Q}''$: $x \leftarrow x' \overset{n}{\rightarrow} \cdot \overset{*}{\leftarrow} y$ such that $w(\mathcal{Q}) \geqq w(\mathcal{Q}'')$. By induction on n we will prove that $\mathcal{Q}'$ exists. The case $n = 0$ is trivial. Let $\mathcal{Q}''$: $x \leftarrow x' \rightarrow z \overset{n-1}{\longrightarrow} \cdot \overset{*}{\leftarrow} y$ and let $\mathcal{P}$: $x \leftarrow x' \rightarrow z$ be the subproof of $\mathcal{Q}''$. Then $\mathcal{P}$ can be replaced with $\mathcal{P}'$ satisfying one of the above conditions (i), (ii), or (iii).

Case (i). $\mathcal{P}'$: $x \overset{*}{\leftrightarrow} z$ and $w(\mathcal{P}) > w(\mathcal{P}')$. Then we have $\hat{\mathcal{Q}}$: $x \overset{*}{\leftrightarrow} z \overset{n-1}{\longrightarrow} \cdot \overset{*}{\leftarrow} y$ such that $w(\mathcal{Q}'') > w(\hat{\mathcal{Q}})$. Thus, by using induction hypothesis concerning the weight $w(\mathcal{Q})$, we obtain $\mathcal{Q}'$ from $\hat{\mathcal{Q}}$.

Case (ii). $\mathcal{P}'$: $x \rightarrow z' \overset{*}{\leftrightarrow} z$ and $w(\mathcal{P}) \geqq w(\mathcal{P}')$. Then we have $\hat{\mathcal{Q}}$: $x \rightarrow z' \overset{*}{\leftrightarrow} z \overset{n-1}{\longrightarrow} \cdot \overset{*}{\leftarrow} y$ such that $w(\mathcal{Q}'') \geqq w(\hat{\mathcal{Q}})$. Let $\hat{\mathcal{Q}}'$: $z' \overset{*}{\leftrightarrow} z \overset{n-1}{\longrightarrow} \cdot \overset{*}{\leftarrow} y$ be the subproof of $\hat{\mathcal{Q}}$. From induction hypothesis concerning the weight $w(\mathcal{Q})$ there exists a proof $\hat{\mathcal{Q}}''$: $z' \overset{*}{\rightarrow} \cdot \overset{*}{\leftarrow} y$ such that $w(\hat{\mathcal{Q}}') \geqq w(\hat{\mathcal{Q}}'')$. Thus, by replacing $\hat{\mathcal{Q}}'$ of $\hat{\mathcal{Q}}$ with

$\hat{\mathcal{Q}}''$, we have $\mathcal{Q}'$.

Case (iii). $\mathcal{P}' : x \leftarrow z$ and $w(\mathcal{P}) \geq w(\mathcal{P}')$. (If $\mathcal{P}'$:$x \rightarrow z$, the claim trivially holds.) Then we have $\hat{\mathcal{Q}}$: $x \leftarrow z \xrightarrow{n-1} \cdot \xleftarrow{*} y$ such that $w(\mathcal{Q}'') \geq w(\hat{\mathcal{Q}})$. From induction hypothesis concerning the number n of reduction steps, we have $\mathcal{Q}'$. □

The following lemma is used to show the Church-Rosser property of non-left-linear systems in Section 7.

Lemma 2: Let $A_0 = \langle D, \underset{0}{\rightarrow} \rangle$ and $A_1 = \langle D, \underset{1}{\rightarrow} \rangle$. Let $\mathcal{P}_i$: $x_i \overset{*}{\underset{1}{\leftrightarrow}} y$ $(i = 1, \cdots, n)$ and let $\rho = \sum_{i=1}^{n} w(\mathcal{P}_i)$. Assume that for any $a, b \in D$ and any proof $\mathcal{P}$: $a \overset{*}{\underset{1}{\leftrightarrow}} b$ such that $w(\mathcal{P}) \leqq \rho$ there exist proofs $\mathcal{P}'$: $a \overset{*}{\underset{1}{\rightarrow}} c \overset{*}{\underset{1}{\leftarrow}} b$ with $w(\mathcal{P}') \leqq w(\mathcal{P})$ and $a \overset{*}{\underset{0}{\rightarrow}} c \overset{*}{\underset{0}{\leftarrow}} b$ for some $c \in D$. Then, there exist proofs $\mathcal{P}'_i$: $x_i \overset{*}{\underset{0}{\rightarrow}} z$ $(i = 1, \cdots n)$ and $\mathcal{Q}$: $y \overset{*}{\underset{1}{\leftrightarrow}} z$ with $w(\mathcal{Q}) \leqq \rho$ for some z (Figure 1).

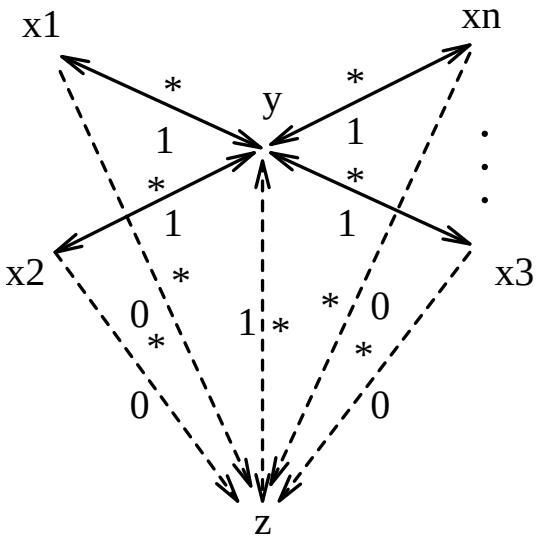

Fig. 1 Reductions in Lemma 2

Proof. By induction on ρ. *Base step* $(\rho = 0)$ is trivial. *Induction step*: Let $w(\mathcal{P}_n) > 0$. From induction hypothesis, we have proofs $\tilde{\mathcal{P}}_i$: $x_i \overset{*}{\underset{0}{\rightarrow}} z'$ $(i = 1, \cdots n-1)$ and $\tilde{\mathcal{Q}}$: $y \overset{*}{\underset{1}{\leftrightarrow}} z'$ for some z' such that $\sum_{i=1}^{n-1} w(\mathcal{P}_i) \geqq w(\tilde{\mathcal{Q}})$. By connecting the proofs $\tilde{\mathcal{Q}}$ and $\mathcal{P}_n$ we have a proof $\hat{\mathcal{P}}$: $z' \overset{*}{\underset{1}{\leftrightarrow}} y \overset{*}{\underset{1}{\leftrightarrow}} x_n$. Since $\sum_{i=1}^{n-1} w(\mathcal{P}_i) \geqq w(\tilde{\mathcal{Q}})$ and $w(\hat{\mathcal{P}}) = w(\tilde{\mathcal{Q}}) + w(\mathcal{P}_n)$, it follows that $\rho \geqq w(\hat{\mathcal{P}})$. By the assumption, we have proofs $\tilde{\mathcal{P}}$: $z' \overset{*}{\underset{1}{\rightarrow}} z \overset{*}{\underset{1}{\leftarrow}} x_n$ with $\rho \geqq w(\hat{\mathcal{P}}) \geqq w(\tilde{\mathcal{P}})$ and $z' \overset{*}{\underset{0}{\rightarrow}} z \overset{*}{\underset{0}{\leftarrow}} x_n$ for some z. Thus we obtain proofs $\mathcal{P}'_i$: $x_i \overset{*}{\underset{0}{\rightarrow}} z$ $(i = 1, \cdots, n)$. By combining subproofs of $\hat{\mathcal{P}}$: $z' \overset{*}{\underset{1}{\leftrightarrow}} y \overset{*}{\underset{1}{\leftrightarrow}} x_n$ and $\tilde{\mathcal{P}}$: $z' \overset{*}{\underset{1}{\rightarrow}} z \overset{*}{\underset{1}{\leftarrow}} x_n$, we can make $\mathcal{Q}'$: $y \overset{*}{\underset{1}{\leftrightarrow}} z' \overset{*}{\underset{1}{\rightarrow}} z$ and $\mathcal{Q}''$: $y \overset{*}{\underset{1}{\leftrightarrow}} x_n \overset{*}{\underset{1}{\rightarrow}} z$. Note that $\rho + \rho \geqq w(\hat{\mathcal{P}}) + w(\tilde{\mathcal{P}}) = w(\mathcal{Q}') + w(\mathcal{Q}'')$. Thus $\rho \geqq w(\mathcal{Q}')$ or $\rho \geqq w(\mathcal{Q}'')$. Take $\mathcal{Q}$ to be $\mathcal{Q}'$ if $\rho \geqq w(\mathcal{Q}')$; otherwise, take $\mathcal{Q}$ to be $\mathcal{Q}''$. □

4. Term Rewriting Systems

In the following sections, we briefly explain the basic notions and definitions concerning term rewriting systems [1], [6].

Let $\mathcal{F}$ be an enumerable set of function symbols denoted by $f, g, h, \cdots$, and let $\mathcal{V}$ be an enumerable set of variable symbols denoted by $x, y, z, \cdots$ where $\mathcal{F} \cap \mathcal{V} = \phi$. By $T(\mathcal{F}, \mathcal{V})$, we denote the set of terms constructed from $\mathcal{F}$ and $\mathcal{V}$. $V(t)$ denotes the set of variables occurring in a term t.

A substitution θ is a mapping from a term set $T(\mathcal{F}, \mathcal{V})$ to $T(\mathcal{F}, \mathcal{V})$ such that $\theta(f(t_1, \ldots, t_n)) \equiv f(\theta(t_1), \ldots, \theta(t_n))$ if $f(t_1, \ldots, t_n) \in T(\mathcal{F}, \mathcal{V})$ $(n \geqq 0)$. Thus, for a term t, $\theta(t)$ is completely determined by its values on the variable symbols occurring in t. Following common usage, we write this as $t\theta$ instead of $\theta(t)$.

Consider an extra constant □ called a hole and the set $T(\mathcal{F} \cup \{\Box\}, \mathcal{V})$. Then $C \in T(\mathcal{F} \cup \{\Box\}, \mathcal{V})$ is called a context on $\mathcal{F}$. We use the notation $C[\ , \ldots, \ ]$ for the context containing n holes $(n \geqq 0)$, and if $t_1, \ldots, t_n \in T(\mathcal{F}, \mathcal{V})$, then $C[t_1, \ldots, t_n]$ denotes the result of placing $t_1, \ldots, t_n$ in the holes of $C[\ , \ldots, \ ]$ from left to right. In particular, $C[\ ]$ denotes a context containing precisely one hole. s is called a subterm of t if $t \equiv C[s]$.

A rewrite rule is a pair $\langle l, r \rangle$ of terms such that $l \notin \mathcal{V}$ and any variable in r also occurs in l. We write $l \rightarrow r$ for $\langle l, r \rangle$. A redex is a term $l\theta$, where $l \rightarrow r$. The set of rewrite rules defines a reduction relation $\rightarrow$ on $T(\mathcal{F}, \mathcal{V})$ as follows:

$t \rightarrow s$ iff $t \equiv C[l\theta]$, $s \equiv C[r\theta]$ for some rule $l \rightarrow r$, and some $C[\ ]$, θ.

When we want to specify the redex occurrence $\Delta \equiv l\theta$ of t in this reduction, we write $t \overset{\Delta}{\rightarrow} s$.

Definition 5: A (finite) term rewriting system R is a reduction system $R = \langle T(\mathcal{F}, \mathcal{V}), \rightarrow \rangle$ such that the reduction relation $\rightarrow$ on $T(\mathcal{F}, \mathcal{V})$ is defined by a finite set of rewrite rules. When we want to specify the term rewriting system R in the reduction relation $\rightarrow$, we write $\underset{R}{\rightarrow}$. If R has $l \rightarrow r$ as a rewrite rule, we write $l \rightarrow r \in R$.

We say that R is left-linear if for any $l \rightarrow r \in R$, l is linear (i.e., every variable in l occurs only once). A rewrite rule $l \rightarrow r$ is duplicating if r contains more occurrences of some variable than l; otherwise, $l \rightarrow r$ is non-duplicating. We say that R is non-duplicating if every $l \rightarrow r \in R$ is non-duplicating.

Let $l \rightarrow r$ and $l' \rightarrow r'$ be two rules in R. Assume that we have renamed the variables appropriately, so that l and l' share no variables. Assume that $s \notin \mathcal{V}$ is a

subterm occurrence in t, i.e., $t \equiv C[s]$, such that s and l' are unifiable, i.e., $s\theta \equiv l'\theta$, with a minimal unifier θ. Then we say that $l \to r$ and $l' \to r'$ are overlapping, and that the pair $\langle C[r']\theta, r\theta\rangle$ of terms is critical in R. We may choose $l \to r$ and $l' \to r'$ to be the same rule, but in this case we shall not consider the case $s \equiv l$, which gives the trivial pair $\langle r, r\rangle$. If R has overlapping rewrite rules then we say that R is overlapping; otherwise, non-overlapping [1], [6].

5. Left-Right Separated Conditional Systems

In this section we introduce a new conditional term rewriting system R in which l and r of any rewrite rule $l \to r \Leftarrow x_1 = y_1, \cdots, x_n = y_n$ do not share the same variable; every variable y in r is connected to some variable x in l through some equation $x = y$ occurring in the equational condition $x_1 = y_1, \cdots, x_n = y_n$. A decidable sufficient condition for the Church-Rosser property of R is presented.

Definition 6: A left-right separated conditional term rewriting system is a conditional term rewriting system with extra variables in which every conditional rewrite rule has the form $l \to r \Leftarrow x_1 = y_1, \cdots, x_n = y_n$ with $l, r \in T(\mathcal{F}, \mathcal{V})$, $V(l) = \{x_1, \cdots, x_n\}$ and $V(r) \subseteqq \{y_1, \cdots, y_n\}$ $(n \geqq 0)$ such that:
(i) $l \notin \mathcal{V}$ and l is linear,
(ii) $\{x_1, \cdots, x_n\} \cap \{y_1, \cdots, y_n\} = \phi$,
(iii) $x_i \not\equiv x_j$ if $i \neq j$ (but, $y_i \equiv y_j$ may hold even if $i \neq j$).
We say that a conditional rewrite rule $l \to r \Leftarrow x_1 = y_1, \cdots, x_n = y_n$ satisfying the above conditions is duplicating if some variable has more occurrences in r than it has in the conditional part "$x_1 = y_1, \cdots, x_n = y_n$"; otherwise, it is non-duplicating. A left-right separated conditional term rewriting system is non-duplicating if every conditional rewrite rule is non-duplicating.

Note. In the above conditional rewrite rule, the left-hand side l and the right-hand side r have separate variables, i.e., $V(l) \cap V(r) = \phi$, because of (ii). Since every variable y in r is connected to some variable x in l through the equational condition, it holds that $V(r\theta) \subseteqq V(l\theta)$ for the substitution $\theta = [x_1 := y_1, \cdots, x_n := y_n]$. Thus, $l\theta \to r\theta$ makes an unconditional rewrite rule, and it is non-duplicating if the conditional rewrite rule is non-duplicating.

Example 1: The following R is a non-duplicating left-right separated conditional term rewriting system:

$$R \left\{ \begin{array}{l} f(x, x') \to g(y, y) \Leftarrow x = y, x' = y \\ h(x, x', x'') \to c \Leftarrow x = y, x' = y, x'' = y \end{array} \right.$$

The following R' is however not a non-duplicating left-right separated conditional term rewriting system since the rewrite rule is duplicating:

$$R' \left\{ f(x, x') \to h(y, y, y) \Leftarrow x = y, x' = y \right.$$

Definition 7: Let R be a non-duplicating left-right separated conditional term rewriting system. We inductively define reduction relations $\underset{R_i}{\longrightarrow}$ for $i \geqq 0$ as follows:
(i) $\underset{R_0}{\longrightarrow} = \phi$,
(ii) $\underset{R_{i+1}}{\longrightarrow} = \{\langle C[l\theta], C[r\theta]\rangle \mid l \to r \Leftarrow x_1 = y_1, \cdots, x_n = y_n \in R \text{ and } x_j\theta \overset{*}{\underset{R_i}{\longleftrightarrow}} y_j\theta \ (j = 1, \cdots, n)\}$.
In the reduction $t \equiv C[l\theta] \underset{R_{i+1}}{\longrightarrow} s \equiv C[r\theta]$, the redex occurrence $\Delta \equiv l\theta$ is specified by writing $t \overset{\Delta}{\to} s$, if necessary.
Note that $\underset{R_i}{\longrightarrow} \subseteqq \underset{R_{i+1}}{\longrightarrow}$ for all $i \geqq 0$. The reduction relation $s \to t$ (i.e., $\underset{R}{\to}$) is defined as $s \underset{R_i}{\longrightarrow} t$ for some i.

The weight $w(\mathcal{P})$ of a proof $\mathcal{P}$ of a non-duplicating left-right separated conditional term rewriting system R is defined as the total number of reduction steps appearing in the recursive structure of $\mathcal{P}$.

Definition 8: A proof $\mathcal{P}$ and its weight $w(\mathcal{P})$ are inductively defined as follows:
(i) The empty sequence λ is a (0 step) proof of $t \overset{*}{\underset{R_n}{\longleftrightarrow}} t$ $(n \geqq 0)$ and $w(\lambda) = 0$.
(ii) An expression $\mathcal{P} : s \underset{[\mathbf{r}, C[\,], \theta, \mathcal{P}_1, \cdots, \mathcal{P}_m]}{\longrightarrow} t$ (resp. $t \underset{[\mathbf{r}, C[\,], \theta, \mathcal{P}_1, \cdots, \mathcal{P}_m]}{\longleftarrow} s$) is a proof of $s \underset{R_n}{\longrightarrow} t$ (resp. $t \underset{R_n}{\longleftarrow} s$) $(n \geqq 1)$, where $\mathbf{r}$ is a rewrite rule $l \to r \Leftarrow x_1 = y_1, \cdots, x_m = y_m \in R$, $C[\,]$ is a context, and θ is a substitution such that $t \equiv C[l\theta]$, $s \equiv C[r\theta]$, and $\mathcal{P}_i$ is a proof of $x_i\theta \overset{*}{\underset{R_{n-1}}{\longrightarrow}} y_i\theta$ $(i = 1, \cdots, m)$. $w(\mathcal{P}) = 1 + \sum_{i=1}^{m} w(\mathcal{P}_i)$. $\mathcal{P}_1, \cdots, \mathcal{P}_m$ are referred to as subproofs associated with the proof $\mathcal{P}$.
(iii) A finite sequence $\mathcal{P} : \mathcal{P}_1 \cdots \mathcal{P}_m$ $(m \geqq 1)$ of proofs is a proof of $t_0 \overset{*}{\underset{R_n}{\longleftrightarrow}} t_m$ $(n \geqq 1)$, where $\mathcal{P}_i$ $(i = 1, \cdots, m)$ is a proof of $t_{i-1} \underset{R_n}{\longrightarrow} t_i$ or $t_{i-1} \underset{R_n}{\longleftarrow} t_i$. $w(\mathcal{P}) = \sum_{i=1}^{m} w(\mathcal{P}_i)$.

We say that $\mathcal{P}$ is a proof of $s \overset{*}{\leftrightarrow} t$ if it is a proof of $s \overset{*}{\underset{R_n}{\longleftrightarrow}} t$ for some n. For convenience, we often abbreviate a proof $\mathcal{P}$ of $s \overset{*}{\leftrightarrow} t$ by $\mathcal{P}$: $s \overset{*}{\leftrightarrow} t$, and the form of a proof is indicated by writing, for example, $\mathcal{P}$: $s \overset{*}{\to} \cdot \overset{*}{\leftarrow} t$, $\mathcal{P}'$: $s \leftarrow \cdot \overset{*}{\to} \cdot \leftarrow t$, $\mathcal{P}''$: $s \overset{\Delta}{\leftarrow} \cdot \overset{\Delta'}{\to} t$, etc.

The concept of critical pairs can be naturally extended to left-right separated conditional term rewriting systems. Let $l \to r \Leftarrow x_1 = y_1, \cdots, x_m = y_m$ and $l' \to r' \Leftarrow x'_1 = y'_1, \cdots, x'_n = y'_n$ be two rules in a left-right separated conditional term rewriting system R. Assume that two rules share no variables. Assume that $s \notin \mathcal{V}$ is a subterm occurrence in l, i.e., $l \equiv C[s]$, such that s and l' are unifiable, i.e., $s\theta \equiv l'\theta$, with the most general unifier θ. Note that $r\theta \equiv r$, $r'\theta \equiv r'$, $y_i\theta \equiv y_i$ $(i = 1, \cdots, m)$ and $y'_j\theta \equiv y'_j$

$(j = 1, \cdots, n)$ as $\{x_1, \cdots, x_m\} \cap \{y_1, \cdots, y_m\} = \phi$ and $\{x'_1, \cdots, x'_n\} \cap \{y'_1, \cdots, y'_n\} = \phi$. Since $l \equiv C[s]$ is linear and the domain of θ is contained in $V(s)$, $C[s]\theta \equiv C[s\theta]$. Thus, from $l\theta \equiv C[s]\theta \equiv C[l'\theta]$, two reductions starting with $l\theta$, i.e., $l\theta \to C[r']$ and $l\theta \to r$, can be obtained by using $l \to r \Leftarrow x_1 = y_1, \cdots, x_m = y_m$ and $l' \to r' \Leftarrow x'_1 = y'_1, \cdots, x'_n = y'_n$ if we assume the equations $x_1\theta = y_1, \cdots, x_m\theta = y_m$ and $x'_1\theta = y'_1, \cdots, x'_n\theta = y'_n$. Then we say that $l \to r \Leftarrow x_1 = y_1, \cdots, x_m = y_m$ and $l' \to r' \Leftarrow x'_1 = y'_1, \cdots, x'_n = y'_n$ are overlapping, and $E \vdash \langle C[r'], r\rangle$ is a conditional critical pair associated with the multiset of equations $E = [x_1\theta = y_1, \cdots, x_m\theta = y_m, x'_1\theta = y'_1, \cdots, x'_n\theta = y'_n]$ in R. We may choose $l \to r \Leftarrow x_1 = y_1, \cdots, x_m = y_m$ and $l' \to r' \Leftarrow x'_1 = y'_1, \cdots, x'_n = y'_n$ to be the same rule, but in this case we shall not consider the case $s \equiv l$. If R has overlapping rewrite rules then we say that R is overlapping; otherwise, non-overlapping

Example 2: Let R be the non-duplicating left-right separated conditional term rewriting system with the following rewrite rules:

$$R \begin{cases} f(x', x'') \to g(x) \Leftarrow x' = x, x'' = x \\ f(y', h(y'')) \to g(y) \Leftarrow y' = y, y'' = y \end{cases}$$

Let $\theta = [x' := y', x'' := h(y'')]$ be the most general unifier of $f(x', x'')$ and $f(y', h(y''))$. By applying the substitution θ to the conditional parts "$x' = x, x'' = x$" and "$y' = y, y'' = y$" we have the multiset of equations $E = [y' = x, h(y'') = x, y' = y, y'' = y]$. Then, assuming the equations in E, $g(x) \equiv g(x)\theta \leftarrow f(x', x'')\theta \equiv f(y', h(y'')) \to g(y)$. Thus, we have a conditional critical pair $E \vdash \langle g(x), g(y)\rangle$.

Note that in a left-right separated conditional term rewriting system the application of the same rule at the same position does not imply the same result as the variables occurring in the left-hand side of the rule do not cover those in the right-hand side: See the following example.

Example 3: Let R be the (non-duplicating) left-right separated conditional term rewriting system with the following rewrite rules:

$$R \begin{cases} f(x) \to g(y) \Leftarrow x = y \\ a \to c \\ b \to c \end{cases}$$

It is obvious that R is non-overlapping. We have however two reductions $f(c) \to g(a)$ and $f(c) \to g(b)$, as $c \overset{*}{\leftarrow} a$ and $c \overset{*}{\leftarrow} b$. Thus the application of the first rule at the root position of $f(c)$ does not guarantee the unique result.

We next discuss how to compare the weights of proofs by rewrite rules in R and assumed equations in E. $E \sqcup E'$ denotes the union of multisets E and E'. We write $E \sqsubseteq E'$ if no elements in E occur more than in E'.

Definition 9: Let R be a non-duplicating left-right separated conditional term rewriting system. Let E be a multiset of equations $t' = s'$ and a fresh constant $\bullet$. Then relations $t \underset{E}{\sim} s$ and $t \underset{E}{\sim\triangleright} s$ on terms are inductively defined as follows:
(i) $t \underset{[t=s]}{\sim} s$.
(ii) If $t \underset{E}{\sim} s$ then $s \underset{E}{\sim} t$.
(iii) If $t \underset{E}{\sim} r$ and $r \underset{E'}{\sim} s$ then $t \underset{E \sqcup E'}{\sim} s$.
(iv) If $t \underset{E}{\sim} s$ then $C[t] \underset{E}{\sim} C[s]$.
(v) If $l \to r \Leftarrow x_1 = y_1, \cdots, x_n = y_n \in R$ and $x_i\theta \underset{E_i}{\sim} y_i\theta$ $(i = 1, \cdots, n)$ then $C[l\theta] \underset{E}{\sim\triangleright} C[r\theta]$ where $E = E_1 \sqcup \cdots \sqcup E_n$.
(vi) If $t \underset{E}{\sim\triangleright} s$ then $t \underset{E \sqcup [\bullet]}{\sim} s$.

In the above definition the fresh constant $\bullet$ keeps in E the number of concrete rewriting steps appearing in an *abstract* proof. We write $t \underset{E}{\triangleleft\!\sim} s$ if $s \underset{E}{\sim\triangleright} t$.

Lemma 3: Let $E = [p_1 = q_1, \cdots, p_m = q_m, \bullet, \cdots, \bullet]$ be a multiset in which $\bullet$ occurs k times $(k \geqq 0)$, and let $\mathcal{P}_i$: $p_i\theta \overset{*}{\leftrightarrow} q_i\theta$ $(i = 1, \cdots, m)$.
(1) If $t \underset{E}{\sim} s$ then there exists a proof $\mathcal{Q}$: $t\theta \overset{*}{\leftrightarrow} s\theta$ with $w(\mathcal{Q}) \leqq \sum_{i=1}^{m} w(\mathcal{P}_i) + k$.
(2) If $t \underset{E}{\sim\triangleright} s$ then there exists a proof $\mathcal{Q}'$: $t\theta \to s\theta$ with $w(\mathcal{Q}') \leqq \sum_{i=1}^{m} w(\mathcal{P}_i) + k + 1$.

Proof. By induction on the construction of $t \underset{E}{\sim} s$ and $t \underset{E}{\sim\triangleright} s$ in Definition 9, we prove (1) and (2) simultaneously. *Base Step*: Trivial as (i) $t \underset{[t=s]}{\sim} s$ of Definition 9. *Induction Step*: If we have $t \underset{E}{\sim} s$ by (ii) (iii) (iv) and $t \underset{E}{\sim\triangleright} s$ by (vi) of Definition 9, then from the induction hypothesis (1) and (2) clearly follow. Assume that $t \underset{E}{\sim\triangleright} s$ by (v) of Definition 9. Then we have a rule $l \to r \Leftarrow x_1 = y_1, \cdots, x_n = y_n$ such that $t \equiv C[l\theta']$, $s \equiv C[r\theta']$, $x_i\theta' \underset{E_i}{\sim} y_i\theta'$ $(i = 1, \cdots, n)$ for some θ' and $E = E_1 \sqcup \cdots \sqcup E_n$. From the induction hypothesis and $E = E_1 \sqcup \cdots \sqcup E_n$, it can be easily shown that there exist proofs $\mathcal{Q}_i$: $x_i\theta'\theta \overset{*}{\leftrightarrow} x_i\theta'\theta$ $(i = 1, \cdots, n)$ and $\sum_{i=1}^{n} w(\mathcal{Q}_i) \leqq \sum_{i=1}^{m} w(\mathcal{P}_i) + k$. Therefore we have a proof $\mathcal{Q}'$: $t\theta \to s\theta$ with $w(\mathcal{Q}') \leqq \sum_{i=1}^{m} w(\mathcal{P}_i) + k + 1$. □

Theorem 1: Let R be a non-duplicating left-right separated conditional term rewriting system. Then R is weight decreasing joinable if for any conditional critical pair $E \vdash \langle q, q'\rangle$ one of the following conditions holds:
(i) $q \underset{E'}{\sim} q'$ for some E' such that $E' \sqsubseteq E \sqcup [\bullet]$, or
(ii) $q \underset{E_1}{\sim\triangleright} \cdot \underset{E_2}{\sim} q'$ and $q \underset{E'_1}{\sim} \cdot \underset{E'_2}{\triangleleft\!\sim} q'$ for some E_1, E_2, E'_1, and E'_2 such that $E_1 \sqcup E_2 \sqsubseteq E \sqcup [\bullet]$ and $E'_1 \sqcup E'_2 \sqsubseteq E \sqcup [\bullet]$, or
(iii) $q \underset{E'}{\sim\triangleright} q'$ (or $q \underset{E'}{\triangleleft\!\sim} q'$) for some E' such that $E' \sqsubseteq$

$E \sqcup [\bullet]$.

Note. If R has finitely many rewrite rules then R has finitely many conditional critical pairs. For each $E \vdash \langle q, q' \rangle$, it is decidable whether one of the above conditions (i), (ii), or (iii) holds since each relation between q and q' is restricted by an upper bound $E \sqcup [\bullet]$. Thus, the theorem presents a decidable sufficient condition for guaranteeing the Church-Rosser property of R having a finite number of rewrite rules.

Proof. The theorem follows from Lemma 1 if for any $\mathcal{P}$: $t \leftarrow p \rightarrow s$ ($t \not\equiv s$) one of the following conditions holds: (i) there exists a proof $\mathcal{Q}$: $t \overset{*}{\leftrightarrow} s$ such that $w(\mathcal{P}) > w(\mathcal{Q})$, or (ii) there exist proofs $\mathcal{Q}_1$: $t \rightarrow \cdot \overset{*}{\leftrightarrow} s$ and $\mathcal{Q}_2$: $t \overset{*}{\leftrightarrow} \cdot \leftarrow s$ such that $w(\mathcal{P}) \geq w(\mathcal{Q}_1)$ and $w(\mathcal{P}) \geq w(\mathcal{Q}_2)$, or (iii) there exists a proof $\mathcal{Q} : t \rightarrow s$ (or $t \leftarrow s$) such that $w(\mathcal{P}) \geq w(\mathcal{Q})$. Hence we will show that one of (i), (ii), or (iii) holds for a given proof $\mathcal{P}$: $t \leftarrow p \rightarrow s$.

Let $\mathcal{P}$: $t \overset{\Delta}{\leftarrow} p \overset{\Delta'}{\rightarrow} s$ where two redexes $\Delta \equiv l\theta$ and $\Delta' \equiv l'\theta'$ are associated with two rules $\mathbf{r}_1$: $l \rightarrow r \Leftarrow x_1 = y_1, \cdots, x_m = y_m$ and $\mathbf{r}_2$: $l' \rightarrow r' \Leftarrow x'_1 = y'_1, \cdots, x'_{m'} = y'_{m'}$ respectively.

Case 1. Δ and Δ' are disjoint. Then $p \equiv C[\Delta, \Delta']$ for some context $C[\ ,\ ]$ and $\mathcal{P}$: $t \equiv C[t', \Delta'] \overset{\Delta}{\leftarrow} C[\Delta, \Delta'] \overset{\Delta'}{\rightarrow} C[\Delta, s'] \equiv s$ for some t' and s'. Since we can take $\mathcal{Q}_1 = \mathcal{Q}_2$: $t \equiv C[t', \Delta'] \overset{\Delta'}{\rightarrow} C[t', s'] \overset{\Delta}{\leftarrow} C[\Delta, s'] \equiv s$ with $w(\mathcal{Q}_1) = w(\mathcal{Q}_2) = w(\mathcal{P})$, (ii) holds.

Case 2. Δ' occurs in θ of $\Delta \equiv l\theta$ (i.e., Δ' occurs below the pattern l). Without loss of generality we may assume that $\mathbf{r}_1$: $C_L[x_1, \cdots, x_m] \rightarrow C_R[y_1, \cdots, y_n] \Leftarrow x_1 = y_1, \cdots, x_m = y_m$ (all the variable occurrences of l and r are displayed), $\mathcal{P}'$: $p \equiv C[C_L[p_1, \cdots, p_m]] \overset{\Delta}{\rightarrow} t \equiv C[C_R[t_1, \cdots, t_n]]$ with subproofs $\mathcal{P}_i$: $p_i \overset{*}{\leftrightarrow} t_i$ ($i = 1, \cdots, m$), and $\mathcal{P}''$: $p \equiv C[C_L[p_1, p_2, \cdots, p_m]] \overset{\Delta'}{\rightarrow} s \equiv C[C_L[p'_1, p_2, \cdots, p_m]]$ by $p_1 \overset{\Delta'}{\rightarrow} p'_1$. Thus $w(\mathcal{P}) = w(\mathcal{P}') + w(\mathcal{P}'')$ and $w(\mathcal{P}') = 1 + \sum_{i=1}^{m} w(\mathcal{P}_i)$. Since we have a proof $\mathcal{Q}'$: $p'_1 \overset{\Delta'}{\leftarrow} p_1 \overset{*}{\leftrightarrow} t_1$ with $w(\mathcal{Q}') = w(\mathcal{P}'') + w(\mathcal{P}_1)$, we can apply $\mathbf{r}_1$ to $s \equiv C[C_L[p'_1, p_2, \cdots, p_m]]$ too. Then, we have a proof $\mathcal{Q}$: $s \equiv C[C_L[p'_1, \cdots, p_m]] \rightarrow t \equiv C[C_R[t_1, \cdots, t_n]]$ with $w(\mathcal{Q}) = 1 + w(\mathcal{Q}') + \sum_{i=2}^{m} w(\mathcal{P}_i) = w(\mathcal{P})$. Thus, (iii) follows.

Case 3. Δ and Δ' coincide by the application of the same rule, i.e., $\mathbf{r} = \mathbf{r}_1 = \mathbf{r}_2$. (We mentioned in Example 3 that in a non-duplicating left-right separated conditional term rewriting system the application of the same rule at the same position does not imply the same result as the variables occurring in the left-hand side of a rule do not cover that in the right-hand side. Thus this case is necessary even if the system is non-overlapping.) Let the rule applied to Δ and Δ' be $\mathbf{r}$: $C_L[x_1, \cdots, x_m] \rightarrow C_R[y_1, \cdots, y_n] \Leftarrow x_1 = y_1, \cdots, x_m = y_m$ (all the variable occurrences of l and r are displayed, and $m \geq n$ by the condition (iv) of Definition 6), and let $\mathcal{P}'$: $p \equiv C[C_L[p_1, \cdots, p_m]] \overset{\Delta}{\rightarrow} t \equiv C[C_R[t_1, \cdots, t_n]]$ with subproofs $\mathcal{P}'_i$: $p_i \overset{*}{\leftrightarrow} t_i$ ($i = 1, \cdots, m$) and $\mathcal{P}''$: $p \equiv C[C_L[p_1, \cdots, p_m]] \overset{\Delta'}{\rightarrow} s \equiv C[C_R[s_1, \cdots, s_n]]$ with subproofs $\mathcal{P}''_i$: $p_i \overset{*}{\leftrightarrow} s_i$ ($i = 1, \cdots, m$). Here $w(\mathcal{P}) = w(\mathcal{P}') + w(\mathcal{P}'') = 2 + \sum_{i=1}^{m} w(\mathcal{P}'_i) + \sum_{i=1}^{m} w(\mathcal{P}''_i)$. Then, we have a proof $\mathcal{Q}$: $t \equiv C[C_R[t_1, \cdots, t_n]] \overset{*}{\leftrightarrow} C[C_R[p_1, \cdots, p_n]] \overset{*}{\leftrightarrow} C[C_R[s_1, \cdots, s_n]] \equiv s$ with $w(\mathcal{Q}) = \sum_{i=1}^{n} w(\mathcal{P}'_i) + \sum_{i=1}^{n} w(\mathcal{P}''_i) < 2 + \sum_{i=1}^{m} w(\mathcal{P}'_i) + \sum_{i=1}^{m} w(\mathcal{P}''_i) = w(\mathcal{P})$. (Note that $m \geq n$ is necessary to guarantee $w(\mathcal{Q}) < w(\mathcal{P})$.) Hence (i) holds.

Case 4. Δ' occurs in Δ but neither Case 2 nor Case 3 (i.e., Δ' overlaps with the pattern l of $\Delta \equiv l\theta$). Then, there exists a conditional critical pair $[p_1 = q_1, \cdots, p_m = q_m] \vdash \langle q, q' \rangle$ between $\mathbf{r}_1$ and $\mathbf{r}_2$, and we can write $\mathcal{P}$: $t \equiv C[q\theta] \overset{\Delta}{\leftarrow} p \equiv C[\Delta] \overset{\Delta'}{\rightarrow} s \equiv C[q'\theta]$ with subproofs $\mathcal{P}_i$: $p_i\theta \overset{*}{\leftrightarrow} q_i\theta$ ($i = 1, \cdots, m$). Thus $w(\mathcal{P}) = \sum_{i=1}^{m} w(\mathcal{P}_i) + 2$. From the assumption about critical pairs the possible relations between q and q' are given in the following subcases.

Subcase 4.1. $q \underset{E'}{\sim} q'$ for some E' such that $E' \sqsubseteq E \sqcup [\bullet]$. By Lemma 3 and $E' \sqsubseteq E \sqcup [\bullet]$, we have a proof $\mathcal{Q}'$: $q\theta \overset{*}{\leftrightarrow} q'\theta$ with $w(\mathcal{Q}') \leq \sum_{i=1}^{m} w(\mathcal{P}_i) + 1 < w(\mathcal{P})$. Hence it is obtained that $\mathcal{Q}$: $t \equiv C[q\theta] \overset{*}{\leftrightarrow} s \equiv C[q'\theta]$ with $w(\mathcal{Q}) < w(\mathcal{P})$. Thus, (i) holds.

Subcase 4.2. $q \underset{E_1}{\sim} \triangleright \cdot \underset{E_2}{\sim} q'$ and $q \underset{E'_1}{\sim} \cdot \underset{E'_2}{\lessdot\sim} q'$ for some E_1, E_2, E'_1, and E'_2 such that $E_1 \sqcup E_2 \sqsubseteq E \sqcup [\bullet]$ and $E'_1 \sqcup E'_2 \sqsubseteq E \sqcup [\bullet]$. By Lemma 3 and $E_1 \sqcup E_2 \sqsubseteq E \sqcup [\bullet]$, we have a proof $\mathcal{Q}'$: $q\theta \rightarrow \cdot \overset{*}{\leftrightarrow} q'\theta$ with $w(\mathcal{Q}') \leq \sum_{i=1}^{m} w(\mathcal{P}_i) + 2 = w(\mathcal{P})$. Hence we can take $\mathcal{Q}_1$: $t \equiv C[q\theta] \rightarrow \cdot \overset{*}{\leftrightarrow} s \equiv C[q'\theta]$ with $w(\mathcal{Q}_1) \leq w(\mathcal{P})$. Similarly we have $\mathcal{Q}_2$: $t \equiv C[q\theta] \overset{*}{\leftrightarrow} \cdot \leftarrow s \equiv C[q'\theta]$ with $w(\mathcal{Q}_2) \leq w(\mathcal{P})$. Thus, (ii) follows.

Subcase 4.3. $q \underset{E'}{\sim} \triangleright q'$ (or $q \underset{E'}{\lessdot\sim} q'$) and $E' \sqsubseteq E \sqcup [\bullet]$. By Lemma 3 and $E' \sqsubseteq E \sqcup [\bullet]$, we have a proof $\mathcal{Q}'$: $q\theta \rightarrow q'\theta$ with $w(\mathcal{Q}') \leq \sum_{i=1}^{m} w(\mathcal{P}_i) + 2 = w(\mathcal{P})$. Hence we obtain $\mathcal{Q}$: $t \equiv C[q\theta] \rightarrow s \equiv C[q'\theta]$ with $w(\mathcal{Q}) \leq w(\mathcal{P})$. For the case of $q \underset{E'}{\lessdot\sim} q'$ we can obtain $\mathcal{Q}$: $s \leftarrow t$ with $w(\mathcal{Q}) \leq w(\mathcal{P})$ similarly. Thus, (iii) holds. □

Corollary 1: Let R be a non-duplicating left-right separated conditional term rewriting system. Then R is weight decreasing joinable if R is non-overlapping.

Example 4: Let R be the non-duplicating left-right separated conditional term rewriting system with the following rewrite rules:

$$R \left\{ \begin{array}{l} f(x', x'') \rightarrow h(x, f(x, b)) \Leftarrow x' = x, x'' = x \\ f(g(y'), y'') \rightarrow h(y, f(g(y), a)) \Leftarrow y' = y, y'' = y \\ a \rightarrow b \end{array} \right.$$

Here, R has the conditional critical pair $[g(y') = x, y'' = x, y' = y, y'' = y] \vdash \langle h(x, f(x, b)), h(y, f(g(y), a)) \rangle$.

Since $h(x,f(x,b)) \underset{[y''=x]}{\sim} h(y'',f(x,b)) \underset{[g(y')=x]}{\sim} h(y'',f(g(y'),b)) \underset{[y''=y,y'=y]}{\sim} h(y,f(g(y),b)) \underset{[\bullet]}{\sim} h(y,f(g(y),a))$, we have $h(x,f(x,b)) \; assume \underset{E'}{\sim} h(y,f(g(y),a))$ where $E' = [g(y') = x, y'' = x, y'' = y, y' = y, \bullet]$. Thus, from Theorem 1 it follows that R is weight decreasing joinable.

We say that $E = [p_1 = q_1, \cdots, p_m = q_m]$ is satisfiable (in R) if there exist proofs $\mathcal{P}_i$: $p_i\theta \overset{*}{\leftrightarrow} q_i\theta$ $(i = 1, \cdots, m)$ for some θ; otherwise, E is unsatisfiable. Note that the satisfiability of E is generally undecidable. Theorem 1 requests that every conditional critical pair $E \vdash \langle q, q' \rangle$ satisfies (i), (ii) or (iii). However, it is clear that we can ignore conditional critical pairs having unsatisfiable E. Thus, we can strengthen Theorem 1 as follows.

Corollary 2: Let R be a non-duplicating left-right separated conditional term rewriting system. Then R is weight decreasing joinable if any conditional critical pair $E \vdash \langle q, q' \rangle$ such that E is satisfiable in R satisfies (i), (ii) or (iii) in Theorem 1.

6. Conditional Linearization

The original idea of the conditional linearization of non-left-linear term rewriting systems was introduced by Klop and De Vrijer [5], De Vrijer [4]. In this section, we introduce a new conditional linearization based on non-duplicating left-right separated conditional term rewriting systems. The point of our linearization is that by replacing traditional conditional systems with non-duplicating left-right separated conditional systems we can easily relax the non-overlapping limitation.

Now we explain a new linearization of non-left-linear rules. For instance, consider a non-duplicating non-left-linear rule $f(x,x,x,y,y,z) \to g(x,x,x,z)$. Then, by replacing all the variable occurrences x,x,x,y,y,z from left to right in the left-hand side with distinct fresh variable occurrences $x', x'', x''', y', y'', z'$ respectively and connecting every fresh variable to corresponding original one with equation, we can make a non-duplicating left-right separated conditional rule $f(x', x'', x''', y', y'', z') \to g(x,x,x,z) \Leftarrow x' = x, x'' = x, x''' = x, y' = y, y' = y, z' = z$. More formally we have the following definition, the framework of which originates essentially from Klop and De Vrijer [5], De Vrijer [4].

Definition 10: (i) If $\mathbf{r}$ is a non-duplicating rewrite rule $l \to r$ and $l \equiv C[y_1, \cdots, y_m]$ (all the variable occurrences of l are displayed), then the (non-duplicating left-right separated) conditional linearization of $\mathbf{r}$ is a non-duplicating left-right separated conditional rewrite rule $\mathbf{r}_L$: $l' \to r \Leftarrow x_1 = y_1, \cdots, x_m = y_m$ where $l' \equiv C[x_1, \cdots, x_m]$ and $x_1, \cdots, x_m$ are distinct fresh variables. Note that $l'\theta \equiv l$ for the substitution $\theta = [x_1 := y_1, \cdots, x_m := y_m]$.

(ii) If R is a non-duplicating term rewriting system, then R_L, the conditional linearization of R, is defined as the set of the rewrite rules $\{\mathbf{r}_L | \mathbf{r} \in R\}$.

Note. The non-duplicating limitation of R in the above definition is necessary to guarantee that R_L is a non-duplicating left-right separated conditional term rewriting system. Otherwise R_L does not satisfy the non-duplicating condition of Definition 6 in general.

The above conditional linearization is different from the original one by Klop and De Vrijer [5], De Vrijer [4] in which the left-linear version of a rewrite rule $\mathbf{r}$ is a traditional conditional rewrite rule without extra variables in the right-hand side and the conditional part. Hence, in the case $\mathbf{r}$ is already left-linear, the linearization in [4], [5] can take $\mathbf{r}$ itself as its conditional linearization. On the other hand, in our definition we cannot take $\mathbf{r}$ itself as its conditional linearization since $\mathbf{r}$ is not a non-duplicating left-right separated rewrite rule.

Theorem 2: If a conditional linearization R_L of a non-duplicating term rewriting system R is Church-Rosser, then R has unique normal forms.

Proof. By Proposition 1, similar to Klop and De Vrijer [5], De Vrijer [4]. □

Example 5: Let R be the non-duplicating term rewriting system with the following rewrite rules:

$$R \begin{cases} f(x,x) \to h(x,f(x,b)) \\ f(g(y),y) \to h(y,f(g(y),a)) \\ a \to b \end{cases}$$

Note that R is non-left-linear and non-terminating. Then we have the following R_L as the linearization of R:

$$R_L \begin{cases} f(x',x'') \to h(x,f(x,b)) \Leftarrow x' = x, x'' = x \\ f(g(y'),y'') \to h(y,f(g(y),a)) \Leftarrow y' = y, y'' = y \\ a \to b \end{cases}$$

In Example 4 the Church-Rosser property of R_L has already been shown. Thus, from Theorem 2 it follows that R has unique normal forms.

7. Church-Rosser Property of Non-Duplicating Systems

In the previous section we have shown a general method based on the conditional linearization technique to prove the unique normal form property of non-left-linear overlapping non-duplicating term rewriting systems. In this section we show that the same conditional linearization technique can be used as a general method for proving the Church-Rosser property of some class of non-duplicating term rewriting systems.

Theorem 3: Let R be a right-ground (i.e., no variables occur in the right-hand side of rewrite rules) term rewriting system. If the conditional linearization R_L of R is weight decreasing joinable then R is Church-Rosser.

Proof. Let R and R_L have reduction relations $\rightarrow$ and $\underset{L}{\rightarrow}$ respectively. Since $\underset{L}{\rightarrow}$ extends $\rightarrow$ and R_L is weight decreasing joinable, the theorem clearly holds if we show the claim: For any t, s and $\mathcal{P}$: $t \overset{*}{\underset{L}{\leftrightarrow}} s$ there exist proofs $\mathcal{Q}$: $t \overset{*}{\underset{L}{\rightarrow}} r \overset{*}{\underset{L}{\leftarrow}} s$ with $w(\mathcal{P}) \geq w(\mathcal{Q})$ and $t \overset{*}{\rightarrow} r \overset{*}{\leftarrow} s$ for some term r. We will prove this claim by induction on $w(\mathcal{P})$. *Base Step* ($w(\mathcal{P}) = 0$) is trivial. *Induction Step*: Let $w(\mathcal{P}) = \rho > 0$. From the weight decreasing joinability of R_L, we have a proof $\mathcal{P}'$: $t \overset{*}{\underset{L}{\rightarrow}} \cdot \overset{*}{\underset{L}{\leftarrow}} s$ with $\rho \geq w(\mathcal{P}')$. Let $\mathcal{P}'$ have the form $t \underset{L}{\rightarrow} s' \overset{*}{\underset{L}{\rightarrow}} \cdot \overset{*}{\underset{L}{\leftarrow}} s$. Without loss of generality we may assume that $C_L[x_1, \cdots, x_m, y_1, \cdots, y_n, \cdots, z_1, \cdots, z_p] \rightarrow C_R \Leftarrow x_1 = x, \cdots, x_m = x, y_1 = y, \cdots, y_n = y, \cdots, z_1 = z, \cdots, z_p = z$ (all the variable occurrences of l and r are displayed) is a linearization of $C_L[x, \cdots, x, y, \cdots, y, \cdots, z, \cdots, z] \rightarrow C_R$ and $\mathcal{P}''$: $t \equiv C[C_L[t_1^x, \cdots, t_m^x, t_1^y, \cdots, t_n^y, \cdots, t_1^z, \cdots, t_p^z]] \underset{L}{\rightarrow} s' \equiv C[C_R]$ with subproofs $\mathcal{P}_i^x$: $t_i^x \overset{*}{\underset{L}{\leftrightarrow}} t^x$ $(i = 1, \cdots, m)$, $\mathcal{P}_j^y$: $t_j^y \overset{*}{\underset{L}{\leftrightarrow}} t^y$ $(j = 1, \cdots, n)$, $\cdots$, $\mathcal{P}_k^z$: $t_k^z \overset{*}{\underset{L}{\leftrightarrow}} t^z$ $(k = 1, \cdots, p)$ for some t^x, t^y, $\cdots$, t^z. Then, from Lemma 2 and the induction hypothesis we have proofs $t_i^x \overset{*}{\rightarrow} \tilde{t_x}$ $(i = 1, \cdots, m)$, $t_j^y \overset{*}{\rightarrow} \tilde{t_y}$ $(j = 1, \cdots, n)$, $\cdots$, $t_k^z \overset{*}{\rightarrow} \tilde{t_z}$ $(k = 1, \cdots, p)$. Hence we can take the reduction $t \equiv C[C_L[t_1^x, \cdots, t_m^x, t_1^y, \cdots, t_n^y, \cdots, t_1^z, \cdots, t_p^z]] \overset{*}{\rightarrow} C[C_L[\tilde{t_x}, \cdots, \tilde{t_x}, \tilde{t_y}, \cdots, \tilde{t_y}, \cdots, \tilde{t_z}, \cdots, \tilde{t_z}]] \rightarrow s' \equiv C[C_R]$. Let $\hat{\mathcal{P}}$: $s' \overset{*}{\underset{L}{\rightarrow}} \cdot \overset{*}{\underset{L}{\leftarrow}} s$ be the subproof of $\mathcal{P}'$. From $\rho > w(\hat{\mathcal{P}})$ and induction hypothesis, we have $\hat{\mathcal{Q}}$: $s' \overset{*}{\underset{L}{\rightarrow}} r \overset{*}{\underset{L}{\leftarrow}} s$ with $w(\hat{\mathcal{P}}) \geq w(\hat{\mathcal{Q}})$ and $s' \overset{*}{\rightarrow} r \overset{*}{\leftarrow} s$ for some r. Thus, the theorem follows. □

The following corollary was originally proven by Oyamaguchi and Ohta [8].

Corollary 3: [Oyamaguchi and Ohta] Let R be a right-ground term rewriting system having a non-overlapping conditional linearization R_L. Then R is Church-Rosser.

Proof. By Theorem 3 and Corollary 1. □

We next relax the right-ground limitation of R in Theorem 3.

Theorem 4: Let R be a term rewriting system in which every rewrite rule $l \rightarrow r$ is right-linear (i.e., r is linear) and non-linear variables in l do not occur in r. If the conditional linearization R_L of R is weight decreasing joinable then R is Church-Rosser.

Proof. The proof is similar to that of Theorem 3. Let R and R_L have reduction relations $\rightarrow$ and $\underset{L}{\rightarrow}$ respectively. Since $\underset{L}{\rightarrow}$ extends $\rightarrow$ and R_L is weight decreasing joinable, the theorem clearly holds if we show the claim: for any t, s and $\mathcal{P}$: $t \overset{*}{\underset{L}{\leftrightarrow}} s$ there exist proofs $\mathcal{Q}$: $t \overset{*}{\underset{L}{\rightarrow}} r \overset{*}{\underset{L}{\leftarrow}} s$ with $w(\mathcal{P}) \geq w(\mathcal{Q})$ and $t \overset{*}{\rightarrow} r \overset{*}{\leftarrow} s$ for some term r. We will prove this claim by induction on $w(\mathcal{P})$. *Base Step* ($w(\mathcal{P}) = 0$) is trivial. *Induction Step*: Let $w(\mathcal{P}) = \rho > 0$. From the weight decreasing joinability of R_L, we have a proof $\mathcal{P}'$: $t \overset{*}{\underset{L}{\rightarrow}} \cdot \overset{*}{\underset{L}{\leftarrow}} s$ with $\rho \geq w(\mathcal{P}')$. Let $\mathcal{P}'$ have the form $t \underset{L}{\rightarrow} \hat{s} \overset{*}{\underset{L}{\rightarrow}} \cdot \overset{*}{\underset{L}{\leftarrow}} s$. Without loss of generality we may assume that $C_L[x_1, \cdots, x_m, y_1, \cdots, y_n, \cdots, z_1, \cdots, z_p, v_1, \cdots, w_1] \rightarrow C_R[v, \cdots, w] \Leftarrow x_1 = x, \cdots, x_m = x, y_1 = y, \cdots, y_n = y, \cdots, z_1 = z, \cdots, z_p = z, v_1 = v, \cdots, w_1 = w$ (all the variable occurrences l and r are displayed) is a linearization of a right-linear rewrite rule $C_L[x, \cdots, x, y, \cdots, y, \cdots, z, \cdots, z, v, \cdots, w] \rightarrow C_R[v, \cdots, w]$ and $t \equiv C[C_L[t_1^x, \cdots, t_m^x, t_1^y, \cdots, t_n^y, \cdots, t_1^z, \cdots, t_p^z, t_1^v, \cdots, t_1^w]] \underset{L}{\rightarrow} \hat{s} \equiv C[C_R[t^v, \cdots, t^w]]$ with subproofs $\mathcal{P}_i^x$: $t_i^x \overset{*}{\underset{L}{\leftrightarrow}} t^x$ $(i = 1, \cdots, m)$, $\mathcal{P}_j^y$: $t_j^y \overset{*}{\underset{L}{\leftrightarrow}} t^y$ $(j = 1, \cdots, n)$, $\cdots$, $\mathcal{P}_k^z$: $t_k^z \overset{*}{\underset{L}{\leftrightarrow}} t^z$ $(k = 1, \cdots, p)$ for some t^x, t^y, $\cdots$, t^z, and $\mathcal{P}^v$: $t_1^v \overset{*}{\underset{L}{\leftrightarrow}} t^v$, $\cdots$, $\mathcal{P}^w$: $t_1^w \overset{*}{\underset{L}{\leftrightarrow}} t^w$. Then, we can take $t \equiv C[C_L[t_1^x, \cdots, t_m^x, t_1^y, \cdots, t_n^y, \cdots, t_1^z, \cdots, t_p^z, t_1^v, \cdots, t_1^w]] \underset{L}{\rightarrow} s' \equiv C[C_R[t_1^v, \cdots, t_1^w]] \overset{*}{\underset{L}{\leftrightarrow}} \hat{s} \equiv C[C_R[t^v, \cdots, t^w]] \overset{*}{\underset{L}{\rightarrow}} \cdot \overset{*}{\underset{L}{\leftarrow}} s$ with the weight $w(\mathcal{P}')$. Let $\mathcal{P}''$: $t \equiv C[C_L[t_1^x, \cdots, t_m^x, t_1^y, \cdots, t_n^y, \cdots, t_1^z, \cdots, t_p^z, t_1^v, \cdots, t_1^w]] \underset{L}{\rightarrow} s' \equiv C[C_R[t_1^v, \cdots, t_1^w]]$. Then, from Lemma 2 and the induction hypothesis we have proofs $t_i^x \overset{*}{\rightarrow} \tilde{t_x}$ $(i = 1, \cdots, m)$, $t_j^y \overset{*}{\rightarrow} \tilde{t_y}$ $(j = 1, \cdots, n)$, $\cdots$, $t_k^z \overset{*}{\rightarrow} \tilde{t_z}$ $(k = 1, \cdots, p)$. Hence we can take the reduction $t \equiv C[C_L[t_1^x, \cdots, t_m^x, t_1^y, \cdots, t_n^y, \cdots, t_1^z, \cdots, t_p^z, t_1^v, \cdots, t_1^w]] \overset{*}{\rightarrow} C[C_L[\tilde{t_x}, \cdots, \tilde{t_x}, \tilde{t_y}, \cdots, \tilde{t_y}, \cdots, \tilde{t_z}, \cdots, \tilde{t_z}, t_1^v, \cdots, t_1^w]] \rightarrow s' \equiv C[C_R[t_1^v \cdots, t_1^w]]$. Let $\hat{\mathcal{P}}$: $s' \overset{*}{\underset{L}{\leftrightarrow}} \hat{s} \overset{*}{\underset{L}{\rightarrow}} \cdot \overset{*}{\underset{L}{\leftarrow}} s$. From $\rho > w(\hat{\mathcal{P}})$ and induction hypothesis, we have $\hat{\mathcal{Q}}$: $s' \overset{*}{\underset{L}{\rightarrow}} r \overset{*}{\underset{L}{\leftarrow}} s$ with $w(\hat{\mathcal{P}}) \geq w(\hat{\mathcal{Q}})$ and $s' \overset{*}{\rightarrow} r \overset{*}{\leftarrow} s$ for some r. Thus, the theorem follows. □

The following corollary was proven also in [9] by a different method.

Corollary 4: Let R be a term rewriting system in which every rewrite rule $l \rightarrow r$ is right-linear and non-linear variables in l do not occur in r. If the conditional linearization R_L of R is non-overlapping then R is Church-Rosser.

Proof. By Theorem 4 and Corollary 1. □

Example 6: Let R be the term rewriting system with

the following rewrite rules:

$$R \left\{ \begin{array}{l} f(x,x,y) \to h(y,c) \\ g(x) \to f(x,c,g(c)) \\ c \to h(c,c) \end{array} \right.$$

Note that R is non-left-linear and non-terminating. Then we have the following R_L as the linearization of R:

$$R_L \left\{ \begin{array}{l} f(x',x'',y') \to h(y,c) \Leftarrow x' = x, x'' = x, y' = y \\ g(x') \to f(x,c,g(c)) \Leftarrow x' = x \\ c \to h(c,c) \end{array} \right.$$

From Corollary 1, R_L is Church-Rosser. Thus, from Corollary 4 it follows that R is Church-Rosser.

8. Conclusion

In this paper we have proposed a new conditional linearization based on non-duplicating left-right separated conditional term rewriting systems. Our conditional linearization technique has been effectively applied to proving the unique normal form property and the Church-Rosser property of non-duplicating term rewriting systems, which may be non-left-linear and overlapping.

Oyamaguchi and Ohta [8] proved that non-E-overlapping right-ground term rewriting systems are Church-Rosser by using the joinability of E-graphs, which is a different technique from the conditional linearization proposed here, and Ohta, Oyamaguchi, Toyama [9] extended this result into some overlapping systems. The results by the conditional linearization in this paper strengthen some part of the results by E-graphs in [8], [9], and vice versa. To unify the both results into a more general framework is an interesting future work.

Acknowledgments

This work is partially supported by Grants from Ministry of Education, Science and Culture of Japan, 10680346, 12680344 and 09245102.

References

[1] F. Baader and T. Nipkow, "Term rewriting and all that", Cambridge University Press, 1998.

[2] J. A. Bergstra and J. W. Klop, "Conditional rewrite rules: Confluence and termination", J. Comput. and Syst. Sci., Vol.32, pp.323-362, 1986.

[3] P. Chew, "Unique normal forms in term rewriting systems with repeated variables", Proc. 13th ACM Symposium on the theory of computing, pp.7-18, 1981.

[4] R. C. de Vrijer, "Conditional linearization", Indagationes Mathematicae, N.S., 10(1), pp.145-159, 1999.

[5] J. W. Klop and R. C. de Vrijer, "Extended term rewriting systems", Lecture Notes in Comput. Sci., Vol.516, pp.26-50, 1990.

[6] J. W. Klop, "Term rewriting systems", in: S. Abramsky, D. Gabbay and T. Maibaum, eds., Handbook of Logic in Computer Science, Vol.2, pp.1-116, Oxford University Press, 1992.

[7] K. Mano and M. Ogawa, "Unique normal form property of compatible term rewriting systems: a new proof of Chew's theorem", *to appear in Theoretical Computer Science.*

[8] M. Oyamaguchi and Y. Ohta, "On the confluent property of right-ground term rewriting systems", Trans. IEICE Japan, Vol.J76-D-I, pp.39-45, 1993, *in Japanese.*

[9] Y. Ohta, M. Oyamaguchi, Y. Toyama, "On the Church-Rosser property of simple-right-linear TRS's", Trans. IECE Japan, Vol.J78-D-I, pp.263-268, 1995, *in Japanese.*

[10] M. Ogawa, "Chew's theorem revisited: uniquely normalizing property of nonlinear term rewriting systems", Lecture Notes in Comput. Sci., Vol.650, pp.309-318, 1992.

[11] Y. Toyama and M. Oyamguchi, "Church-Rosser property and unique normal form property of non-duplicating term rewriting systems", Lecture Notes in Comput. Sci., Vol.968, pp.316-331, 1995.

Yoshihito Toyama received B.E. from Niigata University in 1975, M.E. and D.E. from Tohoku University in 1977 and 1990. He worked at NTT Laboratories from 1977 to 1993, and at Japan Advanced Institute of Science and Technology (JAIST) from 1993 to 2000. Since 2000, he is a professor of Research Institute of Electrical Communication (RIEC), Tohoku University. His research interests include term rewriting systems, program theory, and automated theorem proving. He is a member of IEICE, IPSJ, JSSST, ACM, EATCS.

Michio Oyamaguchi received D.E. in electrical and communication engineering from Tohoku University in 1977. He is now a professor of the Department of Information Engineering of Mie University. He has been engaged in education and research in the fields of theoretical computer science and software. During 1985-1986, he worked at Passau University, F.R.G. as a research fellow of the AvH Foundation.