Church-Rosser Property and Unique Normal Form Property of Non-Duplicating Term Rewriting Systems

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Abstract. We propose a new type of conditional term rewriting systems: *left-right separated* conditional term rewriting systems, in which the left-hand side and the right-hand side of a rewrite rule have separate variables. By developing a concept of weight decreasing joinability we first present a sufficient condition for the Church-Rosser property of left-right separated conditional term rewriting systems which may have overlapping rewrite rules. We next apply this result to show sufficient conditions for the unique normal form property and the Church-Rosser property of unconditional term rewriting systems which are non-duplicating, non-left-linear, and overlapping.

1 Introduction

The original idea of the conditional linearization of non-left-linear term rewriting systems was introduced by De Vrijer [4], Klop and De Vrijer [6] for giving a simpler proof of Chew's theorem [2, 9]. They developed an interesting method for proving the unique normal form property for some non-Church-Rosser, non-left-linear term rewriting system R. The method is based on the fact that the unique normal form property of the original non-left-linear term rewriting system R follows the Church-Rosser property of an associated left-linear conditional term rewriting system R^L which is obtained from R by *linearizing* a non-left-linear rule, for example $Dxx \to x$, into a left-linear conditional rule $Dxy \to x \Leftarrow x = y$. Klop and Bergstra [1] proved that non-overlapping left-linear semi-equational conditional term rewriting systems are Church-Rosser. Hence, combining these two results, Klop and De Vrijer [4, 5, 6] showed that the term rewriting system R has the unique normal form property if R^L is non-overlapping. However, as their conditional linearization technique is based on the Church-Rosser property for the traditional conditional term rewriting system R^L, its application is restricted in non-overlapping R^L (though this limitation may be slightly relaxed with R^L containing only trivial critical pairs).

In this paper, we introduce a new conditional linearization based on a *left-right separated* conditional term rewriting system R_L. The point of our linearization is that a non-left-linear rule $Dxx \to x$ is translated into a left-linear conditional rule $Dxy \to z \Leftarrow x = z, y = z$ in which the left-hand side and the right-hand side have separate variables. By considering this new system R_L instead of a traditional conditional system R^L we can easily relax the non-overlapping limitation of conditional systems originated from Klop and Bergstra [1] if the original system R is non-duplicating. Here, R is non-duplicating if for any rewrite rule $l \to r$, no variable has more occurrences in r than it has in l.

By developing a new concept of weight decreasing joinability we first present a sufficient condition for the Church-Rosser property of a left-right separated conditional term rewriting system R_L which may have overlapping rewrite rules. We next apply this result to our conditional linearization, and show a sufficient condition for the unique normal form property of the original system R which is non-duplicating, non-left-linear, and overlapping.

Moreover, our result can be naturally applied to proving the Church-Rosser property of some non-duplicating non-left-linear overlapping term rewriting systems such as right-ground systems. More recently, Oyamaguchi and Ohta [7] proved that non-E-overlapping right-ground term rewriting systems are Church-Rosser by using the joinability of E-graphs, and Oyamaguchi extended this result into some overlapping systems [8]. The results by conditional linearization in this paper strengthen some part of Oyamaguchi's results by E-graphs [7, 8], and vice verse.

2 Reduction Systems

Assuming that the reader is familiar with the basic concepts and notations concerning reduction systems in [3, 5], we briefly explain notations and definitions.

A reduction system (or an abstract reduction system) is a structure $A = \langle D, \to \rangle$ consisting of some set D and some binary relation $\to$ on D (i.e., $\to \subseteq D \times D$), called a reduction relation. A reduction (starting with x_0) in A is a finite or infinite sequence $x_0 \to x_1 \to x_2 \to \cdots$. The identity of elements x, y of D is denoted by $x \equiv y$. $\xrightarrow{\equiv}$ is the reflexive closure of $\to$, $\leftrightarrow$ is the symmetric closure of $\to$, $\xrightarrow{*}$ is the transitive reflexive closure of $\to$, and $\xleftrightarrow{*}$ is the equivalence relation generated by $\to$ (i.e., the transitive reflexive symmetric closure of $\to$). We write $x \leftarrow y$ if $y \to x$; likewise $x \xleftarrow{*} y$.

If $x \in D$ is minimal with respect to $\to$, i.e., $\neg\exists y \in D[x \to y]$, then we say that x is a normal form; let NF be the set of normal forms. If $x \xrightarrow{*} y$ and $y \in NF$ then we say x has a normal form y and y is a normal form of x.

Definition 1. $A = \langle D, \to \rangle$ is Church-Rosser (or confluent) iff
$\forall x, y, z \in D[x \xrightarrow{*} y \wedge x \xrightarrow{*} z \Rightarrow \exists w \in D, y \xrightarrow{*} w \wedge z \xrightarrow{*} w]$.

Definition 2. $A = \langle D, \to \rangle$ has unique normal forms iff
$\forall x, y \in NF[x \xleftrightarrow{*} y \Rightarrow x \equiv y]$.

The following fact observed by Klop and De Vrijer [6] plays an essential role in our linearization too.

Proposition 3. *[Klop and De Vrijer] Let* $A_0 = \langle D, \underset{0}{\rightarrow} \rangle$ *and* $A_1 = \langle D, \underset{1}{\rightarrow} \rangle$ *be two reduction systems with the sets of normal forms* NF_0 *and* NF_1 *respectively. Then* A_0 *has unique normal forms if each of the following conditions holds:*

(i) $\underset{1}{\rightarrow}$ *extends* $\underset{0}{\rightarrow}$,
(ii) A_1 *is Church-Rosser,*
(iii) NF_1 *contains* NF_0.

Proof. Easy. □

3 Weight Decreasing Joinability

This section introduces the new concept of weight decreasing joinability. In the later sections this concept is used for analyzing the Church-Rosser property of conditional term rewriting systems with extra variables occurring in conditional parts of rewrite rules.

Let N^+ be the set of positive integers. $A = \langle D, \rightarrow \rangle$ is a weighted reduction system if $\rightarrow = \cup_{w \in N^+} \rightarrow_w$, that is, positive integers (weights w) are assigned to each reduction step to represent costs.

Definition 4. A proof of $x \overset{*}{\leftrightarrow} y$ is a sequence $\mathcal{P}$: $x_0 \leftrightarrow_{w_1} x_1 \leftrightarrow_{w_2} x_2 \cdots \leftrightarrow_{w_n} x_n$ $(n \geq 0)$ such that $x \equiv x_0$ and $y \equiv x_n$. The weight $w(\mathcal{P})$ of the proof $\mathcal{P}$ is $\sum_{i=1}^{n} w_i$. If $\mathcal{P}$ is a 0 step sequence (i.e., $n = 0$), then $w(\mathcal{P}) = 0$.

We usually abbreviate a proof $\mathcal{P}$ of $x \overset{*}{\leftrightarrow} y$ by $\mathcal{P}$: $x \overset{*}{\leftrightarrow} y$. The form of a proof may be indicated by writing, for example, $\mathcal{P}$: $x \overset{*}{\rightarrow} \cdot \overset{*}{\leftarrow} y$, $\mathcal{P}'$: $x \leftarrow \cdot \overset{*}{\rightarrow} \cdot \leftarrow y$, etc. We use the symbols $\mathcal{P}, \mathcal{Q}, \cdots$ for proofs.

Definition 5. A weighted reduction system $A = \langle D, \rightarrow \rangle$ is weight decreasing joinable iff for all $x, y \in D$ and any proof $\mathcal{P}$: $x \overset{*}{\leftrightarrow} y$ there exists some proof $\mathcal{P}'$: $x \overset{*}{\rightarrow} \cdot \overset{*}{\leftarrow} y$ such that $w(\mathcal{P}) \geq w(\mathcal{P}')$.

It is clear that if a weighted reduction system A is weight decreasing joinable then A is Church-Rosser. We will now show a sufficient condition for the weight decreasing joinability.

Lemma 6. *Let* A *be a weighted reduction system. Then* A *is weight decreasing joinable if for any* $x, y \in D$ *and any proof* $\mathcal{P}$*:* $x \leftarrow \cdot \rightarrow y$ *one of the following conditions holds:*

(i) *there exists a proof* $\mathcal{P}'$: $x \overset{*}{\leftrightarrow} y$ *such that* $w(\mathcal{P}) > w(\mathcal{P}')$*, or*
(ii) *there exist proofs* $\mathcal{P}'$: $x \rightarrow \cdot \overset{*}{\leftrightarrow} y$ *and* $\mathcal{P}''$: $x \overset{*}{\leftrightarrow} \cdot \leftarrow y$ *such that* $w(\mathcal{P}) \geq w(\mathcal{P}')$ *and* $w(\mathcal{P}) \geq w(\mathcal{P}'')$*, or*

(iii) *there exists a proof $\mathcal{P}' : x \to y$ (or $x \leftarrow y$) such that $w(\mathcal{P}) \geq w(\mathcal{P}')$.*

Proof. By induction on the weight $w(\mathcal{Q})$ of a proof $\mathcal{Q} : x \overset{*}{\leftrightarrow} y$, we prove that there exists a proof $\mathcal{Q}'$: $x \overset{*}{\to} \cdot \overset{*}{\leftarrow} y$ such that $w(\mathcal{Q}) \geq w(\mathcal{Q}')$. *Base step* ($w(\mathcal{Q}) = 0$) is trivial. *Induction step*: Let $\mathcal{Q} : x \leftrightarrow x' \overset{*}{\leftrightarrow} y$ and let $\mathcal{S} : x' \overset{*}{\leftrightarrow} y$ be the subproof of $\mathcal{Q}$. From induction hypothesis, there exists a proof $\mathcal{S}'$: $x' \overset{*}{\to} \cdot \overset{*}{\leftarrow} y$ such that $w(\mathcal{S}) \geq w(\mathcal{S}')$. Thus, if $x \to x'$ then we have $\mathcal{Q}'$: $x \to x' \overset{*}{\to} \cdot \overset{*}{\leftarrow} y$ such that $w(\mathcal{Q}) \geq w(\mathcal{Q}')$. Otherwise we have a proof $\mathcal{Q}''$: $x \leftarrow x' \overset{n}{\to} \cdot \overset{*}{\leftarrow} y$ such that $w(\mathcal{Q}) \geq w(\mathcal{Q}'')$, where $\overset{n}{\to}$ denotes a reduction of n ($n \geq 0$) steps. By induction on n we will prove that $\mathcal{Q}'$ exists. The case $n = 0$ is trivial. Let $\mathcal{Q}''$: $x \leftarrow x' \to z \overset{n-1}{\longrightarrow} \cdot \overset{*}{\leftarrow} y$ and let $\mathcal{P}$: $x \leftarrow x' \to z$ be the subproof of $\mathcal{Q}''$. Then $\mathcal{P}$ can be replaced with $\mathcal{P}'$ satisfying one of the above conditions (i), (ii), or (iii).

Case (i). $\mathcal{P}' : x \overset{*}{\leftrightarrow} z$ and $w(\mathcal{P}) > w(\mathcal{P}')$. Then we have $\hat{\mathcal{Q}}$: $x \overset{*}{\leftrightarrow} z \overset{n-1}{\longrightarrow} \cdot \overset{*}{\leftarrow} y$ such that $w(\mathcal{Q}'') > w(\hat{\mathcal{Q}})$. Thus, by using induction hypothesis concerning the weight $w(\mathcal{Q})$, we obtain $\mathcal{Q}'$ from $\hat{\mathcal{Q}}$.

Case (ii). $\mathcal{P}'$: $x \to z' \overset{*}{\leftrightarrow} z$ and $w(\mathcal{P}) \geq w(\mathcal{P}')$. Then we have $\hat{\mathcal{Q}}$: $x \to z' \overset{*}{\leftrightarrow} z \overset{n-1}{\longrightarrow} \cdot \overset{*}{\leftarrow} y$ such that $w(\mathcal{Q}'') \geq w(\hat{\mathcal{Q}})$. Let $\hat{\mathcal{Q}}'$: $z' \overset{*}{\leftrightarrow} z \overset{n-1}{\longrightarrow} \cdot \overset{*}{\leftarrow} y$ be the subproof of $\hat{\mathcal{Q}}$. From induction hypothesis concerning the weight $w(\mathcal{Q})$ there exists a proof $\hat{\mathcal{Q}}''$: $z' \overset{*}{\to} \cdot \overset{*}{\leftarrow} y$ such that $w(\hat{\mathcal{Q}}') \geq w(\hat{\mathcal{Q}}'')$. Thus, by replacing $\hat{\mathcal{Q}}'$ of $\hat{\mathcal{Q}}$ with $\hat{\mathcal{Q}}''$, we have $\mathcal{Q}'$.

Case (iii). $\mathcal{P}' : x \leftarrow z$ and $w(\mathcal{P}) \geq w(\mathcal{P}')$. (If $\mathcal{P}' : x \to z$, the claim trivially holds.) Then we have $\hat{\mathcal{Q}}$: $x \leftarrow z \overset{n-1}{\longrightarrow} \cdot \overset{*}{\leftarrow} y$ such that $w(\mathcal{Q}'') \geq w(\hat{\mathcal{Q}})$. From induction hypothesis concerning the number n of reduction steps, we have $\mathcal{Q}'$. $\square$

The following lemma is used to show the Church-Rosser property of non-left-linear systems in Section 7.

Lemma 7. *Let $A_0 = \langle D, \underset{0}{\to} \rangle$ and $A_1 = \langle D, \underset{1}{\to} \rangle$. Let $\mathcal{P}_i$: $x_i \underset{1}{\overset{*}{\leftrightarrow}} y$ ($i = 1, \cdots, n$) and let $\rho = \sum_{i=1}^{n} w(\mathcal{P}_i)$. Assume that for any $a, b \in D$ and any proof $\mathcal{P}$: $a \underset{1}{\overset{*}{\leftrightarrow}} b$ such that $w(\mathcal{P}) \leq \rho$ there exist proofs $\mathcal{P}'$: $a \underset{1}{\overset{*}{\to}} c \underset{1}{\overset{*}{\leftarrow}} b$ with $w(\mathcal{P}') \leq w(\mathcal{P})$ and $a \underset{0}{\overset{*}{\to}} c \underset{0}{\overset{*}{\leftarrow}} b$ for some $c \in D$. Then, there exist proofs $\mathcal{P}'_i$: $x_i \underset{0}{\overset{*}{\to}} z$ ($i = 1, \cdots n$) and $\mathcal{Q}$: $y \underset{1}{\overset{*}{\leftrightarrow}} z$ with $w(\mathcal{Q}) \leq \rho$ for some z (Figure 3.1).*

Proof. By induction on ρ. *Base step* ($\rho = 0$) is trivial. *Induction step*: From induction hypothesis, we have proofs $\tilde{\mathcal{P}}_i$: $x_i \underset{0}{\overset{*}{\to}} z'$ ($i = 1, \cdots n-1$) and $\tilde{\mathcal{Q}}$: $y \underset{1}{\overset{*}{\leftrightarrow}} z'$ for some z' such that $\sum_{i=1}^{n-1} w(\mathcal{P}_i) \geq w(\tilde{\mathcal{Q}})$. By connecting the proofs $\tilde{\mathcal{Q}}$ and $\mathcal{P}_n$ we have a proof $\hat{\mathcal{P}}$: $z' \underset{1}{\overset{*}{\leftrightarrow}} y \underset{1}{\overset{*}{\leftrightarrow}} x_n$. Since $\sum_{i=1}^{n-1} w(\mathcal{P}_i) \geq w(\tilde{\mathcal{Q}})$ and $w(\hat{\mathcal{P}}) = w(\tilde{\mathcal{Q}}) + w(\mathcal{P}_n)$, it follows that $\rho \geq w(\hat{\mathcal{P}})$. By the assumption, we have proofs $\tilde{\mathcal{P}}$: $z' \underset{1}{\overset{*}{\to}} z \underset{1}{\overset{*}{\leftarrow}} x_n$ with $\rho \geq w(\hat{\mathcal{P}}) \geq w(\tilde{\mathcal{P}})$ and $z' \underset{0}{\overset{*}{\to}} z \underset{0}{\overset{*}{\leftarrow}} x_n$ for some z. Thus we obtain proofs $\mathcal{P}'_i$: $x_i \underset{0}{\overset{*}{\to}} z$ ($i = 1, \cdots, n$).

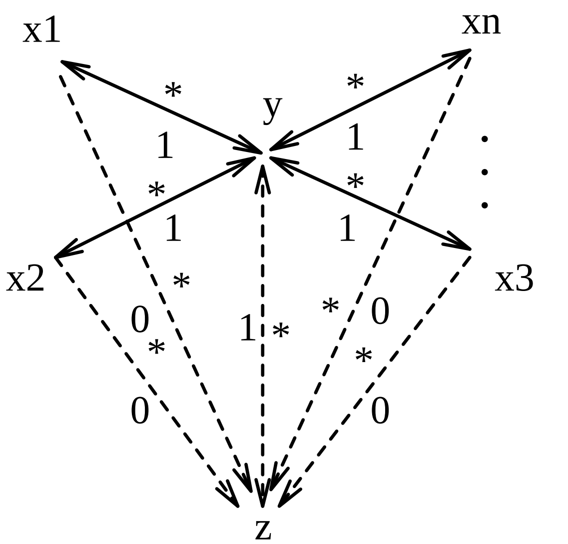

Figure 3.1

By combining subproofs of $\hat{\mathcal{P}}$: $z' \overset{*}{\underset{1}{\leftrightarrow}} y \overset{*}{\underset{1}{\leftrightarrow}} x_n$ and $\tilde{\mathcal{P}}$: $z' \overset{*}{\underset{1}{\rightarrow}} z \overset{*}{\underset{1}{\leftarrow}} x_n$, we can make $\mathcal{Q}'$: $y \overset{*}{\underset{1}{\leftrightarrow}} z' \overset{*}{\underset{1}{\rightarrow}} z$ and $\mathcal{Q}''$: $y \overset{*}{\underset{1}{\leftrightarrow}} x_n \overset{*}{\underset{1}{\rightarrow}} z$. Note that $\rho + \rho \geq w(\hat{\mathcal{P}}) + w(\tilde{\mathcal{P}}) = w(\mathcal{Q}') + w(\mathcal{Q}'')$. Thus $\rho \geq w(\mathcal{Q}')$ or $\rho \geq w(\mathcal{Q}'')$. Take $\mathcal{Q}'$ as Q if $\rho \geq w(\mathcal{Q}')$; otherwise, take $\mathcal{Q}''$ as Q. $\square$

4 Term Rewriting Systems

In the following sections, we briefly explain the basic notions and definitions concerning term rewriting systems [3, 5].

Let $\mathcal{F}$ be an enumerable set of function symbols denoted by $f, g, h, \cdots$, and let $\mathcal{V}$ be an enumerable set of variable symbols denoted by $x, y, z, \cdots$ where $\mathcal{F} \cap \mathcal{V} = \phi$. By $T(\mathcal{F}, \mathcal{V})$, we denote the set of terms constructed from $\mathcal{F}$ and $\mathcal{V}$. The term set $T(\mathcal{F}, \mathcal{V})$ is sometimes denoted by T. $V(t)$ denotes the set of variables occurring in a term t.

A substitution θ is a mapping from a term set $T(\mathcal{F}, \mathcal{V})$ to $T(\mathcal{F}, \mathcal{V})$ such that for a term t, $\theta(t)$ is completely determined by its values on the variable symbols occurring in t. Following common usage, we write this as $t\theta$ instead of $\theta(t)$.

Consider an extra constant $\square$ called a hole and the set $T(\mathcal{F} \cup \{\square\}, \mathcal{V})$. Then $C \in T(\mathcal{F} \cup \{\square\}, \mathcal{V})$ is called a context on $\mathcal{F}$. We use the notation $C[\ ,\ldots,\ ]$ for the context containing n holes ($n \geq 0$), and if $t_1, \ldots, t_n \in T(\mathcal{F}, \mathcal{V})$, then $C[t_1, \ldots, t_n]$ denotes the result of placing $t_1, \ldots, t_n$ in the holes of $C[\ ,\ldots,\ ]$ from left to right. In particular, $C[\ ]$ denotes a context containing precisely one hole. s is called a subterm of t if $t \equiv C[s]$. If s is a subterm occurrence of t, then we write $s \subseteq t$. If a term t has an occurrence of some (function or variable) symbol e, we write $e \in t$. The variable occurrences $z_1, \cdots, z_n$ of $C[z_1, \cdots, z_n]$ are fresh if $z_1, \cdots, z_n \notin C[\ , \cdots, \ ]$ and $z_i \not\equiv z_j$ $(i \neq j)$.

A rewrite rule is a pair $\langle l, r \rangle$ of terms such that $l \notin \mathcal{V}$ and any variable in r also occurs in l. We write $l \to r$ for $\langle l, r \rangle$. A redex is a term $l\theta$, where $l \to r$. In this case $r\theta$ is called a contractum of $l\theta$. The set of rewrite rules defines a reduction relation $\to$ on T as follows:

$t \to s$ iff $t \equiv C[l\theta]$, $s \equiv C[r\theta]$ for some rule $l \to r$, and some $C[\ ]$, θ.

When we want to specify the redex occurrence $\Delta \equiv l\theta$ of t in this reduction, we write $t \stackrel{\Delta}{\to} s$.

Definition 8. A term rewriting system R is a reduction system $R = \langle T(\mathcal{F}, \mathcal{V}), \to \rangle$ such that the reduction relation $\to$ on $T(\mathcal{F}, \mathcal{V})$ is defined by a set of rewrite rules. When we want to specify the term rewriting system R in the reduction relation $\to$, we write $\underset{R}{\to}$. If R has $l \to r$ as a rewrite rule, we write $l \to r \in R$.

We say that R is left-linear if for any $l \to r \in R$, l is linear (i.e., every variable in l occurs only once). If R has a critical pair then we say that R is overlapping: otherwise non-overlapping [3, 5]. A rewrite rule $l \to r$ is duplicating if r contains more occurrences of some variable than l; otherwise, $l \to r$ is non-duplicating. We say that R is non-duplicating if every $l \to r \in R$ is non-duplicating [5].

5 Left-Right Separated Conditional Systems

In this section we introduce a new conditional term rewriting system R in which l and r of any rewrite rule $l \to r \Leftarrow x_1 = y_1, \cdots, x_n = y_n$ do not share the same variable; every variable y_i in r is connected to some variable x_i in l through the equational conditions $x_1 = y_1, \cdots, x_n = y_n$. A decidable sufficient condition for the Church-Rosser property of R is presented.

Definition 9. A left-right separated conditional term rewriting system is a conditional term rewriting system with extra variables in which every conditional rewrite rule has the form:

$l \to r \ \Leftarrow \ x_1 = y_1, \cdots, x_n = y_n$

with $l, r \in T(\mathcal{F}, \mathcal{V})$, $V(l) = \{x_1, \cdots, x_n\}$ and $V(r) \subseteq \{y_1, \cdots, y_n\}$ $(n \geq 0)$ such that:

(i) $l \notin V$ is linear,
(ii) $\{x_1, \cdots, x_n\} \cap \{y_1, \cdots, y_n\} = \phi$,
(iii) $x_i \not\equiv x_j$ if $i \neq j$,
(iv) no variable has more occurrences in r than it has in the conditional part "$x_1 = y_1, \cdots, x_n = y_n$".

Note. In the above conditional rewrite rule, the left-hand side l and the right-hand side r have separate variables, i.e., $V(l) \cap V(r) = \phi$, because of (ii). Since every variable y_i in r is connected to some variable x_i in l through the equational condition, it holds that $V(r\theta) \subseteq V(l\theta)$ for the substitution $\theta = [x_1 := y_1, \cdots, x_m := y_m]$. Thus, $l\theta \to r\theta$ is an unconditional rewrite rule, and it is non-duplicating due to (iv).

Example 1. The following R is a left-right separated conditional term rewriting system:

$$R \quad \begin{cases} f(x, x') \to g(y, y) \Leftarrow x = y, x' = y \\ h(x, x', x'') \to c \Leftarrow x = y, x' = y, x'' = y \end{cases}$$

The following R' is however not a left-right separated conditional term rewriting system since the condition (iv) does not hold:

$$R' \quad \left\{ f(x, x') \to h(y, y, y) \Leftarrow x = y, x' = y \right.$$

□

Definition 10. Let R be a left-right separated conditional term rewriting system. We inductively define reduction relations $\xrightarrow[R_i]{}$ for $i \geq 0$ as follows: $\xrightarrow[R_0]{} = \phi$, $\xrightarrow[R_{i+1}]{} = \{\langle C[l\theta], C[r\theta] \rangle \mid l \to r \Leftarrow x_1 = y_1, \cdots, x_n = y_n \in R$ and $x_j\theta \xleftrightarrow[R_i]{*} y_j\theta \ (j = 1, \cdots, n)\}$. Note that $\xrightarrow[R_i]{} \subseteq \xrightarrow[R_{i+1}]{}$ for all $i \geq 0$. $s \to t$ iff $s \xrightarrow[R_i]{} t$ for some i.

The weight $w(\mathcal{P})$ of a proof $\mathcal{P}$ of a left-right separated conditional term rewriting system R is defined as the total redctuton steps appearing in the recursive structure of $\mathcal{P}$.

Definition 11. A proof $\mathcal{P}$ and its weight $w(\mathcal{P})$ are inductively defined as follows:

(i) The empty sequence λ is a (0 step) proof of $t \xleftrightarrow[R_n]{*} t \quad (n \geq 0)$ and $w(\lambda) = 0$.

(ii) An expression $\mathcal{P} : s \xrightarrow[{[\mathbf{r}, C[\ ], \mathcal{P}_1, \cdots, \mathcal{P}_m]}]{} t$ (resp. $t \xleftarrow[{[\mathbf{r}, C[\ ], \mathcal{P}_1, \cdots, \mathcal{P}_m]}]{} s$) is a proof of $s \xrightarrow[R_n]{} t$ (resp. $t \xleftarrow[R_n]{} s$) $\quad (n \geq 1)$, where $\mathbf{r}$ is a rewrite rule $l \to r \Leftarrow x_1 = y_1, \cdots, x_m = y_m \in R$ and $C[\ ]$ is a context such that for some substitution θ, $t \equiv C[l\theta]$, $s \equiv C[r\theta]$, and $\mathcal{P}_i$ is a proof of $x_i\theta \xleftrightarrow[R_{n-1}]{*} y_i\theta \quad (i = 1, \cdots, m)$. $w(\mathcal{P}) = 1 + \sum_{i=1}^{m} w(\mathcal{P}_i)$. $\mathcal{P}_1, \cdots, \mathcal{P}_m$ are subproofs associated with the proof $\mathcal{P}$.

(iii) A finite sequence $\mathcal{P} : \mathcal{P}_1 \cdots \mathcal{P}_m \quad (m \geq 1)$ of proofs is a proof of $t_0 \xleftrightarrow[R_n]{*} t_m$ $(n \geq 1)$, where $\mathcal{P}_i \ (i = 1, \cdots, m)$ is a proof of $t_{i-1} \xrightarrow[R_n]{} t_i$ or $t_{i-1} \xleftarrow[R_n]{} t_i$. $w(\mathcal{P}) = \sum_{i=1}^{m} w(\mathcal{P}_i)$.

$\mathcal{P}$ is a proof of $s \overset{*}{\leftrightarrow} t$ if it is a proof of $s \overset{*}{\underset{R_n}{\longleftrightarrow}} t$ for some n. For convenience, we often use the abbreviations introduced in Section 3; i.e., we abbreviate a proof $\mathcal{P}$ of $s \overset{*}{\leftrightarrow} t$ by $\mathcal{P}$: $s \overset{*}{\leftrightarrow} t$, and the form of a proof is indicated by writing, for example, $\mathcal{P}$: $s \overset{*}{\rightarrow} \cdot \overset{*}{\leftarrow} t$, $\mathcal{P}'$: $s \leftarrow \cdot \overset{*}{\rightarrow} \cdot \leftarrow t$, etc. We use the symbols $\mathcal{P}, \mathcal{Q}, \cdots$ for proofs.

Let $l \rightarrow r \Leftarrow\ x_1 = y_1, \cdots, x_m = y_m$ and $l' \rightarrow r' \Leftarrow\ x'_1 = y'_1, \cdots, x'_n = y'_n$ be two rules in a left-right separated conditional term rewriting system R. Assume that we have renamed the variables appropriately, so that two rules share no variables. Assume that $s \notin V$ is a subterm occurrence in l, i.e., $l \equiv C[s]$, such that s and l' are unifiable, i.e., $s\theta \equiv l'\theta$, with the most general unifier θ. Note that $r\theta \equiv r$, $r'\theta \equiv r'$, $y_i\theta \equiv y_i$ $(i = 1, \cdots, m)$ and $y'_j\theta \equiv y'_j$ $(j = 1, \cdots, n)$ as $\{x_1, \cdots, x_m\} \cap \{y_1, \cdots, y_m\} = \phi$ and $\{x'_1, \cdots, x'_n\} \cap \{y'_1, \cdots, y'_n\} = \phi$. Since $l \equiv C[s]$ is linear and the domain of θ is contained in $V(s)$, $C[s]\theta \equiv C[s\theta]$. Thus, from $l\theta \equiv C[s]\theta \equiv C[l'\theta]$, two reductions starting with $l\theta$, i.e., $l\theta \rightarrow C[r']$ and $l\theta \rightarrow r$, can be obtained by using $l \rightarrow r \Leftarrow\ x_1 = y_1, \cdots, x_m = y_m$ and $l' \rightarrow r' \Leftarrow\ x'_1 = y'_1, \cdots, x'_n = y'_n$ if we assume the equations $x_1\theta = y_1, \cdots, x_m\theta = y_m$ and $x'_1\theta = y'_1, \cdots, x'_n\theta = y'_n$. Then we say that $l \rightarrow r \Leftarrow\ x_1 = y_1, \cdots, x_m = y_m$ and $l' \rightarrow r' \Leftarrow\ x'_1 = y'_1, \cdots, x'_n = y'_n$ are overlapping, and $E\ \vdash\ \langle C[r'], r\rangle$ is a conditional critical pair associated with the multiset of equations $E = [x_1\theta = y_1, \cdots, x_m\theta = y_m, x'_1\theta = y'_1, \cdots, x'_n\theta = y'_n]$ in R. We may choose $l \rightarrow r \Leftarrow\ x_1 = y_1, \cdots, x_m = y_m$ and $l' \rightarrow r' \Leftarrow\ x'_1 = y'_1, \cdots, x'_n = y'_n$ to be the same rule, but in this case we shall not consider the case $s \equiv l$. If R has no critical pair, then we say that R is non-overlapping.

Example 2. Let R be the left-right separated conditional term rewriting system with the following rewrite rules:

$$R \quad \begin{cases} f(x', x'') \rightarrow g(x) \Leftarrow x' = x, x'' = x \\ f(y', h(y'')) \rightarrow g(y) \Leftarrow y' = y, y'' = y \end{cases}$$

Let $\theta = [x' := y', x'' := h(y'')]$ be the most general unifier of $f(x', x'')$ and $f(y', h(y''))$. By applying the substitution θ to the conditional parts "$x' = x, x'' = x$" and "$y' = y, y'' = y$" we have the multiset of equations $E = [y' = x, h(y'') = x, y' = y, y'' = y]$. Then, assuming the equations in E, $g(x) \leftarrow g(x)\theta \leftarrow f(x', x'')\theta \equiv f(y', h(y'')) \rightarrow g(y)$. Thus, we have a condtional critecal pair $E\ \vdash\ \langle g(x), g(y)\rangle$. $\Box$

Note that in a left-right separated conditional term rewriting system the application of the same rule at the same position does not imply the same result as the variables occurring in the left-hand side of a rule do not cover that in the right-hand side: See the following example.

Example 3. Let R be the left-right separated conditional term rewriting system with the following rewrite rules:

$$R \quad \begin{cases} f(x) \to g(y) \Leftarrow x = y \\ a \to c \\ b \to c \end{cases}$$

It is obvious that R is non-overlapping. We have however two reductions $f(c) \to g(a)$ and $f(c) \to g(b)$, as $c \overset{*}{\leftrightarrow} a$ and $c \overset{*}{\leftrightarrow} b$. Thus the application of the first rule at the root position of $f(c)$ does not guarantee a unique result. $\square$

We next discuss how to compare the weights of *abstract* proofs including the assumed equations of E. $E \sqcup E'$ denotes the union of multisets E and E'. We write $E \sqsubseteq E'$ if no elements in E occur more than E'.

Definition 12. Let E be a multiset of equations $t' = s'$ and a fresh constant $\bullet$. Then relations $t \underset{E}{\sim} s$ and $t \underset{E}{\sim\!\triangleright} s$ on terms is inductively defined as follows:

(i) $t \underset{[t=s]}{\sim} s$.

(ii) If $t \underset{E}{\sim} s$ then $s \underset{E}{\sim} t$.

(iii) If $t \underset{E}{\sim} r$ and $r \underset{E'}{\sim} s$ then $t \underset{E \sqcup E'}{\sim} s$.

(iv) If $t \underset{E}{\sim} s$ then $C[t] \underset{E}{\sim} C[s]$.

(v) If $l \to r \;\Leftarrow\; x_1 = y_1, \cdots, x_n = y_n \;\in R$ and $x_i\theta \underset{E_i}{\sim} y_i\theta$ $(i = 1, \cdots, n)$ then $C[l\theta] \underset{E}{\sim\!\triangleright} C[r\theta]$ where $E = E_1 \sqcup \cdots \sqcup E_n$.

(vi) If $t \underset{E}{\sim\!\triangleright} s$ then $t \underset{E \sqcup [\bullet]}{\sim} s$.

In the above definition the fresh constant $\bullet$ keeps in E the number of concrete rewriting steps appearing in an *abstract* proof. We write $t \underset{E}{\triangleleft\!\sim} s$ if $s \underset{E}{\sim\!\triangleright} t$.

Lemma 13. *Let $E = [p_1 = q_1, \cdots, p_m = q_m, \bullet, \cdots, \bullet]$ be a multiset in which $\bullet$ occurs k times $(k \geq 0)$, and let $\mathcal{P}_i$: $p_i\theta \overset{*}{\leftrightarrow} q_i\theta$ $(i = 1, \cdots, m)$.*

(1) *If $t \underset{E}{\sim} s$ then there exists a proof $\mathcal{Q}$: $t\theta \overset{*}{\leftrightarrow} s\theta$ with $w(\mathcal{Q}) \leq \sum_{i=1}^m w(\mathcal{P}_i) + k$.*

(2) *If $t \underset{E}{\sim\!\triangleright} s$ then there exists a proof $\mathcal{Q}'$: $t\theta \to s\theta$ with $w(\mathcal{Q}') \leq \sum_{i=1}^m w(\mathcal{P}_i) + k + 1$.*

Proof. By induction on the construction of $t \underset{E}{\sim} s$ and $t \underset{E}{\sim\!\triangleright} s$ in Definition 12, we prove (1) and (2) simultaneously. *Base Step*: Trivial as (i) $t \underset{[t=s]}{\sim} s$ of Definition 12. *Induction Step*: If we have $t \underset{E}{\sim} s$ by (ii) (iii) (iv) and $t \underset{E}{\sim\!\triangleright} s$ by (vi) of Definition 12, then from the induction hypothesis (1) and (2) clearly follow. Assume that $t \underset{E}{\sim\!\triangleright} s$ by (v) of Definition 12. Then we have a rule $l \to r \;\Leftarrow\; x_1 = y_1, \cdots, x_n = y_n$ such that $t \equiv C[l\theta']$, $s \equiv C[r\theta']$, $x_i\theta' \underset{E_i}{\sim} y_i\theta'$ $(i = 1, \cdots, n)$ for some θ' and $E = E_1 \sqcup \cdots \sqcup E_n$. From the induction hypothesis and $E = E_1 \sqcup \cdots \sqcup E_n$, it can be easily shown that $\mathcal{Q}_i$: $x_i\theta'\theta \overset{*}{\leftrightarrow} x_i\theta'\theta$ $(i = 1, \cdots, n)$ and $\sum_{i=1}^n w(\mathcal{Q}_i) \leq \sum_{i=1}^m w(\mathcal{P}_i) + k$. Therefore we have a proof $\mathcal{Q}'$: $t\theta \to s\theta$ with $w(\mathcal{Q}') \leq \sum_{i=1}^m w(\mathcal{P}_i) + k + 1$. $\square$

Theorem 14. *Let R be a left-right separated conditional term rewriting system. Then R is weight decreasing joinable if for any conditional critical pair $E \vdash \langle q, q' \rangle$ one of the following conditions holds:*

(i) $q \underset{E'}{\sim} q'$ *for some E' such that $E' \sqsubseteq E \sqcup [\bullet]$, or*

(ii) $q \underset{E_1}{\sim\!\triangleright} \cdot \underset{E_2}{\sim} q'$ *and* $q \underset{E_1'}{\sim} \cdot \underset{E_2'}{\triangleleft\!\sim} q'$ *for some E_1, E_2, E_1', and E_2' such that $E_1 \sqcup E_2 \sqsubseteq E \sqcup [\bullet]$ and $E_1' \sqcup E_2' \sqsubseteq E \sqcup [\bullet]$, or*

(iii) $q \underset{E'}{\sim\!\triangleright} q'$ *(or* $q \underset{E'}{\triangleleft\!\sim} q'$*) for some E' such that $E' \sqsubseteq E \sqcup [\bullet]$.*

Note. If R has finitely many rewrite rules then R has finitely many conditional critical pairs. For each $E \vdash \langle q, q' \rangle$, it is decidable whether one of the above conditions (i), (ii), or (iii) holds since each relation between q and q' is restricted by an upper bound $E \sqcup [\bullet]$. Thus, the theorem presents a decidable sufficient condition for guaranteeing the Church-Rosser property of R having finte rewrite rules.

Proof. The theorem follows from Lemma 6 if for any $\mathcal{P}$: $t \leftarrow p \rightarrow s$ ($t \not\equiv s$) one of the following conditions holds: (i) there exists a proof $\mathcal{Q}$: $t \overset{*}{\leftrightarrow} s$ $w(\mathcal{P}) > w(\mathcal{Q})$, or (ii) there exist proofs $\mathcal{Q}_1$: $t \rightarrow \cdot \overset{*}{\leftrightarrow} s$ and $\mathcal{Q}_2$: $t \overset{*}{\leftrightarrow} \cdot \leftarrow s$ such that $w(\mathcal{P}) \geq w(\mathcal{Q}_1)$ and $w(\mathcal{P}) \geq w(\mathcal{Q}_2)$, or (iii) there exists a proof $\mathcal{Q} : t \rightarrow s$ (or $t \leftarrow s$) such that $w(\mathcal{P}) \geq w(\mathcal{Q})$. Hence we will show that one of (i), (ii), or (iii) holds for a given proof $\mathcal{P}$: $t \leftarrow p \rightarrow s$.

Let $\mathcal{P}$: $t \overset{\Delta}{\leftarrow} p \overset{\Delta'}{\rightarrow} s$ where two redexes $\Delta \equiv l\theta$ and $\Delta' \equiv l'\theta'$ are associated with two rules $\mathbf{r}_1$: $l \rightarrow r \Leftarrow \ x_1 = y_1, \cdots, x_m = y_m$ and $\mathbf{r}_2$: $l' \rightarrow r' \Leftarrow \ x_1' = y_1', \cdots, x_{m'}' = y_{m'}'$ respectively.

Case 1. Δ and Δ' are disjoint. Then $p \equiv C[\Delta, \Delta']$ for some context $C[\ ,\ ]$ and $\mathcal{P}$: $t \equiv C[t', \Delta'] \overset{\Delta}{\leftarrow} C[\Delta, \Delta'] \overset{\Delta'}{\rightarrow} C[\Delta, s'] \equiv s$ for some t' and s'. Since we can take $\mathcal{Q}_1 = \mathcal{Q}_2$: $t \equiv C[t', \Delta'] \overset{\Delta'}{\rightarrow} C[t', s'] \overset{\Delta}{\leftarrow} C[\Delta, s'] \equiv s$ with $w(\mathcal{Q}_1) = w(\mathcal{Q}_2) = w(\mathcal{P})$, (ii) holds.

Case 2. Δ' occurs in θ of $\Delta \equiv l\theta$ (i.e., Δ' occurs below the pattern l). Without loss of generality we may assume that $\mathbf{r}_1$: $C_L[x_1, \cdots, x_m] \rightarrow C_R[y_1, \cdots, y_n] \Leftarrow x_1 = y_1, \cdots, x_m = y_m$ (all the variable occurrences are displayed), $\mathcal{P}'$: $p \equiv C[C_L[p_1, \cdots, p_m]] \overset{\Delta}{\rightarrow} t \equiv C[C_R[t_1, \cdots, t_n]]$ with subproofs $\mathcal{P}_i$: $p_i \overset{*}{\leftrightarrow} t_i$ ($i = 1, \cdots, m$), and $\mathcal{P}''$: $p \equiv C[C_L[p_1, p_2, \cdots, p_m]] \overset{\Delta'}{\rightarrow} s \equiv C[C_L[p_1', p_2, \cdots, p_m]]$ by $p_1 \overset{\Delta'}{\rightarrow} p_1'$. Thus $w(\mathcal{P}) = w(\mathcal{P}') + w(\mathcal{P}'')$ and $w(\mathcal{P}') = 1 + \sum_{i=1}^{m} w(\mathcal{P}_i)$. Since we have a proof $\mathcal{Q}'$: $p_1' \overset{\Delta'}{\leftarrow} p_1 \overset{*}{\leftrightarrow} t_1$ with $w(\mathcal{Q}') = w(\mathcal{P}'') + w(\mathcal{P}_1)$, we can apply $\mathbf{r}_1$ to $s \equiv C[C_L[p_1', p_2, \cdots, p_m]]$ too. Then, we have a proof $\mathcal{Q}$: $s \equiv C[C_L[p_1', \cdots, p_m]] \rightarrow t \equiv C[C_R[t_1, \cdots, t_n]]$ with $w(\mathcal{Q}) = 1 + w(\mathcal{Q}') + \sum_{i=2}^{m} w(\mathcal{P}_i) = w(\mathcal{P})$. Thus, (iii) follows.

Case 3. Δ and Δ' coincide by the application of the same rule, i.e., $\mathbf{r} = \mathbf{r}_1 = \mathbf{r}_2$. (We mentioned in Example 3 that in a left-right separated conditional term rewriting system the application of the same rule at the same position does not imply the same result as the variables occurring in the left-hand side of a rule do not cover that in the right-hand side. Thus this case is necessary even if the system is non-overlapping.) Let the rule applied to Δ and Δ' be $\mathbf{r}$: $C_L[x_1, \cdots, x_m] \to C_R[y_1, \cdots, y_n] \Leftarrow x_1 = y_1, \cdots, x_m = y_m$ (all the variable occurrences are displayed, and $m \geq n$ by the condition (iv) of Definition 9), and let $\mathcal{P}'$: $p \equiv C[C_L[p_1, \cdots, p_m]] \overset{\Delta}{\to} t \equiv C[C_R[t_1, \cdots, t_n]]$ with subproofs $\mathcal{P}'_i$: $p_i \overset{*}{\leftrightarrow} t_i$ $(i = 1, \cdots, m)$ and $\mathcal{P}''$: $p \equiv C[C_L[p_1, \cdots, p_m]] \overset{\Delta'}{\to} s \equiv C[C_R[s_1, \cdots, s_n]]$ with subproofs $\mathcal{P}''_i$: $p_i \overset{*}{\leftrightarrow} s_i$ $(i = 1, \cdots, m)$. Here $w(\mathcal{P}) = w(\mathcal{P}') + w(\mathcal{P}'') = 2 + \sum_{i=1}^m w(\mathcal{P}'_i) + \sum_{i=1}^m w(\mathcal{P}''_i)$. Then, we have a proof $\mathcal{Q}$: $t \equiv C[C_R[t_1, \cdots, t_n]] \overset{*}{\leftrightarrow} C[C_R[p_1, \cdots, p_n]] \overset{*}{\leftrightarrow} C[C_R[s_1, \cdots, s_n]] \equiv s$ with $w(\mathcal{Q}) = \sum_{i=1}^n w(\mathcal{P}'_i) + \sum_{i=1}^n w(\mathcal{P}''_i) < 2 + \sum_{i=1}^m w(\mathcal{P}'_i) + \sum_{i=1}^m w(\mathcal{P}''_i) = w(\mathcal{P})$. (Note that $m \geq n$ is necessary to guarantee $w(\mathcal{Q}) < w(\mathcal{P})$.) Hence (i) holds.

Case 4. Δ' occurs in Δ but neither Case 2 nor Case 3 (i.e., Δ' overlaps with the pattern l of $\Delta \equiv l\theta$). Then, there exists a conditional critical pair $[p_1 = q_1, \cdots, p_m = q_m] \vdash \langle q, q' \rangle$ between $\mathbf{r}_1$ and $\mathbf{r}_2$, and we can write $\mathcal{P}$: $t \equiv C[q\theta] \overset{\Delta}{\leftarrow} p \equiv C[\Delta] \overset{\Delta'}{\to} s \equiv C[q'\theta]$ with subproofs $\mathcal{P}_i$: $p_i\theta \overset{*}{\leftrightarrow} q_i\theta$ $(i = 1, \cdots, m)$. Thus $w(\mathcal{P}) = \sum_{i=1}^m w(\mathcal{P}_i) + 2$. From the assumption about critical pairs the possible relations between q and q' are give in the following subcases.

Subcase 4.1. $q \underset{E'}{\sim} q'$ for some E' such that $E' \sqsubseteq E \sqcup [\bullet]$. By Lemma 13 and $E' \sqsubseteq E \sqcup [\bullet]$, we have a proof $\mathcal{Q}'$: $q\theta \overset{*}{\leftrightarrow} q'\theta$ with $w(\mathcal{Q}') \leq \sum_{i=1}^m w(\mathcal{P}_i) + 1 < w(\mathcal{P})$. Hence it is obtained that $\mathcal{Q}$: $t \equiv C[q\theta] \overset{*}{\leftrightarrow} s \equiv C[q'\theta]$ with $w(\mathcal{Q}) < w(\mathcal{P})$. Thus, (i) holds.

Subcase 4.2. $q \underset{E_1}{\sim\!\triangleright} \cdot \underset{E_2}{\sim} q'$ and $q \underset{E_1'}{\sim} \cdot \underset{E_2'}{\triangleleft\!\sim} q'$ for some E_1, E_2, E_1', and E_2' such that $E_1 \sqcup E_2 \sqsubseteq E \sqcup [\bullet]$ and $E_1' \sqcup E_2' \sqsubseteq E \sqcup [\bullet]$. By Lemma 13 and $E_1 \sqcup E_2 \sqsubseteq E \sqcup [\bullet]$, we have a proof $\mathcal{Q}'$: $q\theta \to \cdot \overset{*}{\leftrightarrow} q'\theta$ with $w(\mathcal{Q}') \leq \sum_{i=1}^m w(\mathcal{P}_i) + 2 = w(\mathcal{P})$. Hence we can take $\mathcal{Q}_1$: $t \equiv C[q\theta] \to \cdot \overset{*}{\leftrightarrow} s \equiv C[q'\theta]$ with $w(\mathcal{Q}_1) \leq w(\mathcal{P})$. Simiraly we have $\mathcal{Q}_2$: $t \equiv C[q\theta] \overset{*}{\leftrightarrow} \cdot \leftarrow s \equiv C[q'\theta]$ with $w(\mathcal{Q}_2) \leq w(\mathcal{P})$. Thus, (ii) follows.

Subcase 4.3. $q \underset{E'}{\sim\!\triangleright} q'$ (or $q \underset{E'}{\triangleleft\!\sim} q'$) and $E' \sqsubseteq E \sqcup [\bullet]$. By Lemma 13 and $E' \sqsubseteq E \sqcup [\bullet]$, we have a proof $\mathcal{Q}'$: $q\theta \to q'\theta$ with $w(\mathcal{Q}') \leq \sum_{i=1}^m w(\mathcal{P}_i) + 2 = w(\mathcal{P})$. Hence we obtain $\mathcal{Q}$: $t \equiv C[q\theta] \to s \equiv C[q'\theta]$ with $w(\mathcal{Q}) \leq w(\mathcal{P})$. For the case of $q \underset{E'}{\triangleleft\!\sim} q'$ we can obtain $\mathcal{Q}$: $s \leftarrow t$ with $w(\mathcal{Q}) \leq w(\mathcal{P})$ similarly. Thus, (iii) holds. □

Corollary 15. *Let R be a left-right separated conditional term rewriting system. Then R is weight decreasing joinable if R is non-overlapping.*

Example 4. Let R be the left-right separated conditional term rewriting system with the following rewrite rules:

$$R \quad \begin{cases} f(x', x'') \to h(x, f(x, b)) \Leftarrow x' = x, x'' = x \\ f(g(y'), y'') \to h(y, f(g(y), a)) \Leftarrow y' = y, y'' = y \\ a \to b \end{cases}$$

Here, R has the conditional critical pair

$$[g(y') = x, y'' = x, y' = y, y'' = y] \vdash \langle h(x, f(x, b)), h(y, f(g(y), a)) \rangle.$$

Since $h(x, f(x, b)) \underset{[y''=x]}{\sim} h(y'', f(x, b)) \underset{[g(y')=x]}{\sim} h(y'', f(g(y'), b)) \underset{[y''=y,y'=y]}{\sim} h(y, f(g(y), b)) \underset{[\bullet]}{\sim} h(y, f(g(y), a))$, we have $h(x, f(x, b)) \underset{E'}{\sim} h(y, f(g(y), a))$ where $E' = [g(y') = x, y'' = x, y'' = y, y' = y, \bullet]$. Thus, from Theorem 14 it follows that R is weight decreasing joinable. □

We say that $E = [p_1 = q_1, \cdots, p_m = q_m]$ is satisfiable (in R) if there exist proofs $\mathcal{P}_i$: $p_i\theta \overset{*}{\leftrightarrow} q_i\theta$ $(i = 1, \cdots, m)$ for some θ; otherewise E is unsatisfiable. Note that the satisfiability of E is generally undecidable. Theorem 14 requests that every conditional critical pair $E \vdash \langle q, q' \rangle$ satisfies (i), (ii) or (iii). However, it is clear that we can ignore conditional critical pairs having unsatisfiable E. Thus, we can strengthen Theorem 14 as follows.

Corollary 16. *Let R be a left-right separated conditional term rewriting system. Then R is weight decreasing joinable if any conditional critical pair $E \vdash \langle q, q' \rangle$ such that E is satisfiable in R satisfies (i), (ii) or (iii) in Theorem 14.*

6 Conditional Linearization

The original idea of the conditional linearization of non-left-linear term rewriting systems was introduced by De Vrijer [4], Klop and De Vrijer [6] for giving a simpler proof of Chew's theorem [2, 9]. In this section, we introduce a new conditional linearization based on left-right separated conditional term rewriting systems. The point of our linearization is that by replacing traditional conditional systems with left-right separated conditional systems we can easily relax the non-overlapping limitation.

Now we explain a new linearization of non-left-linear rules. For instance, let consider a non-duplicating non-left-linear rule $f(x, x, x, y, y, z) \to g(x, x, x, z)$. Then, by replacing all the variable occurrences x, x, x, y, y, z from left to right in the left handside with distinct fresh variable occurrences $x', x'', x''', y', y'', z'$ respectively and connecting every fresh variable to corresponding original one with equation, we can make a left-right separated conditional rule $f(x', x'', x''', y', y'', z') \to g(x, x, x, z) \Leftarrow x' = x, x'' = x, x''' = x, y' = y, y' = y, z' = z$. More formally we have the following definition, the framework of which originates essentially from De Vrijer [4], Klop and De Vrijer [6].

Definition 17. **(i)** If $\mathbf{r}$ is a non-duplicating rewrite rule $l \to r$ and $l \equiv C[y_1, \cdots, y_m]$ (all the variable occurences of l are displayed), then the (left-right separated) conditional linearization of $\mathbf{r}$ is a left-right separated conditional rewrite rule $\mathbf{r}_L$: $l' \to r \Leftarrow x_1 = y_1, \cdots, x_m = y_m$ where $l' \equiv C[x_1, \cdots, x_m]$ and $x_1, \cdots, x_m$ are distinct fresh variables. Note that $l'\theta \equiv l$ for the substitution $\theta = [x_1 := y_1, \cdots, x_m := y_m]$.

(ii) If R is a non-duplicating term rewriting system, then R_L, the conditional linearization of R, is defined as the set of the rewrite rules $\{\mathbf{r}_L | \mathbf{r} \in R\}$.

Note. The non-duplicating limitation of R in the above definition is necessary to guarantee that R_L is a left-right separated conditional term rewriting system. Otherwise R_L does not satisfy the condition (iv) of Definition 9 in general.

The above conditional linearization is different from the original one by Klop and De Vrijer [4, 6] in which the left-linear version of a rewrite rule **r** is a traditional conditional rewrite rule without extra variables in the right handside and the conditional part. Hence, in the case **r** is already left-linear, Klop and De Vrijer [4, 6] can take **r** itself as its conditional linearization. On the other hand, in our definition we cannot take **r** itself as its conditional linearization since **r** is not a left-right separated rewrite rule.

Theorem 18. *If a conditional linearization R_L of a non-duplicating term rewriting system R is Church-Rosser, then R has unique normal forms.*

Proof. By Propsiton 3, similar to Klop and De Vrijer [4, 6]. □

Example 5. Let R be the non-duplicating term rewriting system with the following rewrite rules:

$$R \quad \begin{cases} f(x,x) \to h(x, f(x,b)) \\ f(g(y),y) \to h(y, f(g(y),a)) \\ a \to b \end{cases}$$

Note that R is non-left-linear and non-terminating. Then we have the following R_L as the linearization of R:

$$R_L \quad \begin{cases} f(x',x'') \to h(x, f(x,b)) \Leftarrow x' = x, x'' = x \\ f(g(y'),y'') \to h(y, f(g(y),a)) \Leftarrow y' = y, y'' = y \\ a \to b \end{cases}$$

In Example 4 the Church-Rosser property of R_L has already been shown. Thus, from Theorem 18 it follows that R has unique normal forms. □

7 Church-Rosser Property of Non-Duplicating Systems

In the previous section we have shown a general method based on the conditional linearization technique to prove the unique normal form property of non-left-linear overlapping non-duplicating term rewriting systems. In this section we show that the same conditional linearization technique can be used as a general method for proving the Church-Rosser property of some class of non-duplicating term rewriting systems.

Theorem 19. *Let R be a right-ground (i.e., no variables occur in the right hand-side of rewrite rules) term rewriting system. If the conditional linearization R_L of R is weight decreasing joinable then R is Church-Rosser.*

Proof. Let R and R_L have reduction relations $\rightarrow$ and $\underset{L}{\rightarrow}$ respectively. Since $\underset{L}{\rightarrow}$ extends $\rightarrow$ and R_L is weight decreasing joinable, the theorem clearly holds if we show the claim: For any t, s and $\mathcal{P}$: $t \underset{L}{\overset{*}{\leftrightarrow}} s$ there exist proofs $\mathcal{Q}$: $t \underset{L}{\overset{*}{\rightarrow}} r \underset{L}{\overset{*}{\leftarrow}} s$ with $w(\mathcal{P}) \geq w(\mathcal{Q})$ and $t \overset{*}{\rightarrow} r \overset{*}{\leftarrow} s$ for some term r. We will prove this claim by induction on $w(\mathcal{P})$. *Base Step* ($w(\mathcal{P}) = 0$) is trivial. *Induction Step*: Let $w(\mathcal{P}) = \rho > 0$. Form the weight decreasing joinability of R_L, we have a proof $\mathcal{P}'$: $t \underset{L}{\overset{*}{\rightarrow}} \cdot \underset{L}{\overset{*}{\leftarrow}} s$ with $\rho \geq w(\mathcal{P}')$. Let $\mathcal{P}'$ have the form $t \underset{L}{\rightarrow} s' \underset{L}{\overset{*}{\rightarrow}} \cdot \underset{L}{\overset{*}{\leftarrow}} s$. Without loss of generality we may assume that $C_L[x_1, \cdots, x_m, y_1, \cdots, y_n, \cdots, z_1, \cdots, z_p] \rightarrow C_R \Leftarrow x_1 = x, \cdots, x_m = x, y_1 = y, \cdots, y_n = y, \cdots, z_1 = z, \cdots, z_p = z$ (all the variable occurrences are displayed) is a linearization of $C_L[x, \cdots, x, y, \cdots, y, \cdots, z, \cdots, z] \rightarrow C_R$ and $\mathcal{P}''$: $t \equiv C[C_L[t_1^x, \cdots, t_m^x, t_1^y, \cdots, t_n^y, \cdots, t_1^z, \cdots, t_p^z]] \underset{L}{\rightarrow} s' \equiv C[C_R]$ with subproofs $\mathcal{P}_i^x$: $t_i^x \underset{L}{\overset{*}{\leftrightarrow}} t'_x$ $(i = 1, \cdots, m)$, $\mathcal{P}_j^y$: $t_j^y \underset{L}{\overset{*}{\leftrightarrow}} t'_y$ $(j = 1, \cdots, n)$, $\cdots$, $\mathcal{P}_k^z$: $t_k^z \underset{L}{\overset{*}{\leftrightarrow}} t'_z$ $(k = 1, \cdots, p)$ for some t'_x, t'_y, $\cdots$, t'_z. Then, from Lemma 7 and the induction hypothesis we have proofs $t_i^x \overset{*}{\rightarrow} t''_x$ $(i = 1, \cdots, m)$, $t_j^y \overset{*}{\rightarrow} t''_y$ $(j = 1, \cdots, n)$, $\cdots$, $t_k^z \overset{*}{\rightarrow} t''_z$ $(k = 1, \cdots, p)$. Hence we can take the reduction $t \equiv C[C_L[t_1^x, \cdots, t_m^x, t_1^y, \cdots, t_n^y, \cdots, t_1^z, \cdots, t_p^z]] \overset{*}{\rightarrow} C[C_L[t''_x, \cdots, t''_x, t''_y, \cdots, t''_y, \cdots, t''_z, \cdots, t''_z]] \rightarrow s' \equiv C[C_R]$. Let $\hat{\mathcal{P}}$: $s' \underset{L}{\overset{*}{\rightarrow}} \cdot \underset{L}{\overset{*}{\leftarrow}} s$ be the subproof of P'. From $\rho > w(\hat{\mathcal{P}})$ and induction hypothesis, we have $\hat{\mathcal{Q}}$: $s' \underset{L}{\overset{*}{\rightarrow}} r \underset{L}{\overset{*}{\leftarrow}} s$ with $w(\hat{\mathcal{P}}) \geq w(\hat{\mathcal{Q}})$ and $s' \overset{*}{\rightarrow} r \overset{*}{\leftarrow} s$ for some r. Thus, the theorem follows. □

The following corollary was originally proven by Oyamaguchi and Ohta [7].

Corollary 20. *[Oyamaguchi] Let R be a right-ground term rewriting system having a non-overlapping conditional linearization R_L. Then R is Church-Rosser.*

We next relax the right-ground limitation of R in Theorem 19.

Theorem 21. *Let R be a term rewriting system in which every rewrite rule $l \rightarrow r$ is right-linear (i.e., r is linear) and no non-linear variables in l occur in r. If the conditional linearization R_L of R is weight decreasing joinable then R is Church-Rosser.*

Proof. The proof is similar to that of Theorem 19. Let R and R_L have reduction relations $\rightarrow$ and $\underset{L}{\rightarrow}$ respectively. Since $\underset{L}{\rightarrow}$ extends $\rightarrow$ and R_L is weight decreasing joinable, the theorem clearly holds if we show the claim: for any t, s and $\mathcal{P}$: $t \underset{L}{\overset{*}{\leftrightarrow}} s$ there exist proofs $\mathcal{Q}$: $t \underset{L}{\overset{*}{\rightarrow}} r \underset{L}{\overset{*}{\leftarrow}} s$ with $w(\mathcal{P}) \geq w(\mathcal{Q})$ and $t \overset{*}{\rightarrow} r \overset{*}{\leftarrow} s$

for some term r. We will prove this claim by induction on $w(\mathcal{P})$. *Base Step* ($w(\mathcal{P}) = 0$) is trivial. *Induction Step*: Let $w(\mathcal{P}) = \rho > 0$. Form the weight decreasing joinability of R_L, we have a proof $\mathcal{P}'$: $t \xrightarrow[L]{*} \cdot \xleftarrow[L]{*} s$ with $\rho \geq w(\mathcal{P}')$. Let $\mathcal{P}'$ have the form $t \xrightarrow[L]{} \hat{s} \xrightarrow[L]{*} \cdot \xleftarrow[L]{*} s$. Without loss of generality we may assume that $C_L[x_1, \cdots, x_m, y_1, \cdots, y_n, \cdots, z_1, \cdots, z_p, v_1, \cdots, w_1] \to C_R[v, \cdots, w] \Leftarrow\ x_1 = x, \cdots, x_m = x,\ y_1 = y, \cdots, y_n = y,\ \cdots,\ z_1 = z, \cdots, z_p = z, v_1 = v, \cdots, w_1 = w$ (all the variable occurrences are displayed) is a linearization of a right-linear rewrite rule $C_L[x, \cdots, x, y, \cdots, y, \cdots, z, \cdots, z, v, \cdots, w] \to C_R[v, \cdots, w]$ and $t \equiv C[C_L[t^x_1, \cdots, t^x_m, t^y_1, \cdots, t^y_n, \cdots, t^z_1, \cdots, t^z_p, p_1, \cdots, q_1]] \xrightarrow[L]{} \hat{s} \equiv C[C_R[p, \cdots, q]]$ with subproofs $\mathcal{P}^x_i$: $t^x_i \overset{*}{\underset{L}{\leftrightarrow}} t'_x$ $(i = 1, \cdots, m)$, $\mathcal{P}^y_j$: $t^y_j \overset{*}{\underset{L}{\leftrightarrow}} t'_y$ $(j = 1, \cdots, n)$, $\cdots$, $\mathcal{P}^z_k$: $t^z_k \overset{*}{\underset{L}{\leftrightarrow}} t'_z$ $(k = 1, \cdots, p)$ for some t'_x, t'_y, $\cdots$, t'_z, and $\mathcal{P}^v$: $p_1 \overset{*}{\underset{L}{\leftrightarrow}} p$, $\cdots$, $\mathcal{P}^w$: $q_1 \overset{*}{\underset{L}{\leftrightarrow}} q$. Then, we can take $t \equiv C[C_L[t^x_1, \cdots, t^x_m, t^y_1, \cdots, t^y_n, \cdots, t^z_1, \cdots, t^z_p, p_1, \cdots, q_1]] \xrightarrow[L]{} s' \equiv C[C_R[p_1, \cdots, q_1]] \overset{*}{\underset{L}{\leftrightarrow}} \hat{s} \equiv C[C_R[p, \cdots, q]] \xrightarrow[L]{*} \cdot \xleftarrow[L]{*} s$ with the weight $w(\mathcal{P}')$. Let $\mathcal{P}''$: $t \equiv C[C_L[t^x_1, \cdots, t^x_m, t^y_1, \cdots, t^y_n, \cdots, t^z_1, \cdots, t^z_p, p_1, \cdots, q_1]] \xrightarrow[L]{} s' \equiv C[C_R[p_1, \cdots, q_1]]$. Then, from Lemma 7 and the induction hypothesis we have proofs $t^x_i \xrightarrow{*} t''_x$ $(i = 1, \cdots, m)$, $t^y_j \xrightarrow{*} t''_y$ $(j = 1, \cdots, n)$, $\cdots$, $t^z_k \xrightarrow{*} t''_z$ $(k = 1, \cdots, p)$. Hence we can take the reduction $t \equiv C[C_L[t^x_1, \cdots, t^x_m, t^y_1, \cdots, t^y_n, \cdots, t^z_1, \cdots, t^z_p, p_1, \cdots, q_1]] \xrightarrow{*} C[C_L[t''_x, \cdots, t''_x, t''_y, \cdots, t''_y, \cdots, t''_z, \cdots, t''_z, p_1, \cdots, q_1]] \to s' \equiv C[C_R[p_1 \cdots, q_1]]$. Let $\hat{\mathcal{P}}$: $s' \overset{*}{\underset{L}{\leftrightarrow}} \hat{s} \xrightarrow[L]{*} \cdot \xleftarrow[L]{*} s$. From $\rho > w(\hat{\mathcal{P}})$ and induction hypothesis, we have $\hat{\mathcal{Q}}$: $s' \xrightarrow[L]{*} r \xleftarrow[L]{*} s$ with $w(\hat{\mathcal{P}}) \geq w(\hat{\mathcal{Q}})$ and $s' \xrightarrow{*} r \xleftarrow{*} s$ for some r. Thus, the theorem follows. □

Corollary 22. *Let R be a term rewriting system in which every rewrite rule $l \to r$ is right-linear and no non-linear variables in l occur in r. If the conditional linearization R_L of R is non-overlapping then R is Church-Rosser.*

Example 6. Let R be the term rewriting system with the following rewrite rules:

$$R \quad \begin{cases} f(x, x, y) \to h(y, c) \\ g(x) \to f(x, c, g(c)) \\ c \to h(c, c) \end{cases}$$

Note that R is non-left-linear and non-terminating. Then we have the following R_L as the linearization of R: w

$$R_L \quad \begin{cases} f(x', x'', y') \to h(y, c) \Leftarrow x' = x, x'' = x, y' = y \\ g(x') \to f(x, c, g(c)) \Leftarrow x' = x \\ c \to h(c, c) \end{cases}$$

From Corollary 15, R_L is Church-Rosser. Thus, from Corollary 22 it follows that R is Church-Rosser. □

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