Strong Sequentiality of Left-Linear Overlapping Term Rewriting Systems

Yoshihito Toyama*
NTT Laboratories
Inuidani, Seika-cho, Soraku-gun, Kyoto 619-02, Japan
email: toyama@ntt-20.ntt.jp

Abstract

Huet and Lévy showed that for every strongly sequential orthogonal (i.e., left-linear and non-overlapping) term rewriting system, index reduction strategy is normalizing. This paper extends their result to overlapping *term rewriting systems. We show that index reduction is normalizing for the class of strongly sequential left-linear term rewriting systems in which every critical pair can be joined with* root balanced reductions*. This class includes all weakly orthogonal left-normal systems, for which leftmost-outermost reduction strategy is normalizing. For example, leftmost-outermost reduction strategy is normalizing for combinatory logic CL with the overlapping rules* $\mathrm{pred} \cdot (\mathrm{succ} \cdot x) \to x$ *and* $\mathrm{succ} \cdot (\mathrm{pred} \cdot x) \to x$.

1 Introduction

The normalizing reduction strategies of reduction systems, such as leftmost-outermost evaluation of lambda calculus, combinatory logic, ordinal recursive program schemata and left-normal term rewriting systems, guarantee a *safe* evaluation which reduces a given expression to its normal form whenever possible [1, 2, 4, 8, 9, 10, 12, 15]. Hence, the normalizing reduction strategies play an important role in the implementation of functional programming languages based on reduction systems.

Strong sequentiality formalized by Huet and Lévy [5] is a well-known practical criterion guaranteeing an efficiently computable normalizing reduction strategy for orthogonal (i.e., left-linear and non-overlapping) term rewriting systems. They showed that for every strongly sequential orthogonal term rewriting system, index reduction is a normalizing strategy, that is, by rewriting a redex called an *index* at each step, every reduction starting with a term having a normal form eventually terminates at the normal from. Here, the index is defined as a *needed redex* concerning an approximation of R which is obtained by analyzing the left-hand sides alone of the rewriting rules of term rewriting systems. Moreover, Huet and Lévy [5] showed the decidability of strong sequentiality.

Extensions of strong sequentiality have been reported by [11, 13, 14]. Thatte [14] introduced left sequentiality and showed the equivalence of strong sequentiality and left sequentiality for the class of orthogonal constructor systems. Oyamaguchi [13] introduced sufficient sequentiality not only based on the analysis of the left-hand sides of the rewriting rules of term rewriting systems but also on the non-variable parts of the right-hand sides. He showed that the notion of sufficient sequentiality properly extends the notion of strong sequentiality. Klop and Middeldorp [11] presented a simple proof of the decidability of strong sequentiality and compared these extensions of strong sequentiality. However, all these results are restricted to orthogonal term rewriting systems; hence, they cannot be applied to term rewriting systems with overlapping rules such as

$$\begin{cases} \mathit{pred}(\mathit{succ}(x)) \to x \\ \mathit{succ}(\mathit{pred}(x)) \to x \end{cases}$$

Concerning overlapping term rewriting systems, Kennway [8] proved the surprising fact that every weakly orthogonal term rewriting system has a computable normalizing strategy. However, the weak orthogonality in [8] is restricted to root overlapping rules such as *parallel-or*, i.e., $\mathit{or}(\mathit{true}, x) \to \mathit{true}$ and $\mathit{or}(x, \mathit{true}) \to \mathit{true}$. Thus, his result cannot be applied to the above non-root overlapping situation. Moreover, the algorithm proposed in [8] is too complicated to find a needed redex effectively.

*Partially supported by grants from NWO, Vrije Universiteit Amsterdam, and Katholieke Universiteit Nijmegen

In this paper, we extend the result by Huet and Lévy to *overlapping* term rewriting systems. The notions of index and strong sequentiality are naturally extended to the overlapping situation. Under these extensions, we show that index reduction is normalizing for the class of strongly sequential *root balanced joinable* term rewriting systems. The root balanced joinable term rewriting system is defined as a left-linear term rewriting system in which every critical pair can be joined with *the root balanced reductions*. We also show that this class includes all weakly orthogonal left-normal systems (not being restricted to root overlapping rules), for which leftmost-outermost reduction strategy is normalizing. For example, leftmost-outermost reduction strategy is normalizing for combinatory logic CL + $\{pred\cdot(succ\cdot x) \to x,\ succ\cdot(pred\cdot x) \to x\}$. Here, combinatory logic CL [1] is the orthogonal term rewriting system having the following rewriting rules:

$$CL \quad \begin{cases} ((S\cdot x)\cdot y)\cdot z \to (x\cdot z)\cdot(y\cdot z) \\ (K\cdot x)\cdot y \to x. \end{cases}$$

Moreover, our result can be applied to term rewriting systems not having the Church-Rosser property too. For example, leftmost-outermost reduction strategy is again normalizing for CL +

$$\begin{cases} (K\cdot A)\cdot y \to (K\cdot B)\cdot y \\ (K\cdot B)\cdot y \to (K\cdot A)\cdot y \\ A \to A \\ B \to B, \end{cases}$$

though the system is not Church-Rosser since $(K\cdot A)\cdot y$ can be reduced into two terms A and B which cannot be joined.

The approach presented here is more direct than that in Huet and Lévy [5]. The key idea behind our approach is the *balanced weak Church-Rosser property* which gives a useful means of analyzing reduction strategies. We first explain this idea in an abstract framework. Section 2 introduces preliminary concepts of abstract reduction systems. In Section 3, we introduce the *balanced weak Church-Rosser property* of abstract reduction systems and explain how this property is related to a normalizing reduction strategy. Our results are carefully partitioned between abstract properties depending solely on reduction relation and properties depending on term structure. In Sections 4 we present preliminary concepts for term rewriting systems and in the next section we extend the notion of strong sequentiality to overlapping term rewriting systems. In Section 6, by using the *balanced weak Church-Rosser property* of index reduction, we prove that index reduction of *root balanced joinable* term rewriting systems is normalizing; our proof does not need the complicated notions and algorithms used in Huet and Lévy [5]. Section 7 extends index reduction to quasi-index reduction. Finally, Section 8 discusses syntactic characterization of strongly sequential overlapping term rewriting systems.

2 Reduction Systems

Assuming that the reader is familiar with the basic concepts and notations concerning reduction systems in [6, 10], we briefly explain notations and definitions.

A reduction system (or an abstract reduction system) is a structure $A = \langle D, \to\rangle$ consisting of some set D and some binary relation $\to$ on D (i.e., $\to \subseteq D\times D$), called a reduction relation. A reduction (starting with x_0) in A is a finite or infinite sequence $x_0 \to x_1 \to x_2 \to \cdots$. The identity of elements x, y of D is denoted by $x \equiv y$. $\xrightarrow{\equiv}$ is the reflexive closure of $\to$, $\leftrightarrow$ is the symmetric closure of $\to$, $\xrightarrow{+}$ is the transitive closure of $\to$, $\xrightarrow{*}$ is the transitive reflexive closure of $\to$, and $=$ is the equivalence relation generated by $\to$ (i.e., the transitive reflexive symmetric closure of $\to$). $\xrightarrow{m}$ denotes a reduction of m ($m \geq 0$) steps. If $x \in D$ is minimal with respect to $\to$, i.e., $\neg\exists y \in D[x \to y]$, then we say that x is a normal form; let NF be the set of normal forms. If $x \xrightarrow{*} y$ and $y \in NF$ then we say x has a normal form y and y is a normal form of x.

Definition 2.1 *$A = \langle D, \to\rangle$ is strongly normalizing (or terminating) iff every reduction in A terminates, i.e., there is no infinite sequence $x_0 \to x_1 \to x_2 \to \cdots$.*

Definition 2.2 *$A = \langle D, \to\rangle$ is Church-Rosser (or confluent) iff*
$\forall x, y, z \in D[x \xrightarrow{*} y \wedge x \xrightarrow{*} z \Rightarrow \exists w \in A, y \xrightarrow{*} w \wedge z \xrightarrow{*} w]$.

Definition 2.3 *$A = \langle D, \to\rangle$ is weakly Church-Rosser (or weakly confluent) iff*
$\forall x, y, z \in D[x \to y \wedge x \to z \Rightarrow \exists w \in A, y \xrightarrow{*} w \wedge z \xrightarrow{*} w]$.

The following propositions are well known [1, 6, 10].

Proposition 2.4 *Let A be Church-Rosser, then,*

(1) $\forall x, y \in D[x = y \Rightarrow \exists w \in D, x \xrightarrow{*} w \wedge y \xrightarrow{*} w]$,

(2) $\forall x, y \in NF[x = y \Rightarrow x \equiv y]$,

(3) $\forall x \in D \forall y \in NF[x = y \Rightarrow x \xrightarrow{*} y]$.

Proposition 2.5 (Newman's Lemma) *If A is weakly Church-Rosser and strongly normalizing, then A is Church-Rosser.*

Definition 2.6 (Reduction Strategy) *Let $A = \langle D, \to \rangle$ and let $\underset{s}{\to}$ be a subrelation of $\xrightarrow{+}$ (i.e., if $x \underset{s}{\to} y$ then $x \xrightarrow{+} y$) such that a normal form concerning $\underset{s}{\to}$ is also a normal form concerning $\to$ (i.e., two binary relations $\underset{s}{\to}$ and $\to$ have the same domain). Then, we say that $\underset{s}{\to}$ is a reduction strategy for A (or for $\to$). If $\underset{s}{\to}$ is a subrelation of $\to$ then we call it a one step reduction strategy; otherwise $\to$ is called a many step reduction strategy.*

Definition 2.7 (Normalizing Strategy) *A reduction strategy $\underset{s}{\to}$ is normalizing iff for each x having a normal form concerning $\to$, there exists no infinite sequence $x \equiv x_0 \underset{s}{\to} x_1 \underset{s}{\to} x_2 \underset{s}{\to} \cdots$ (i.e., every $\underset{s}{\to}$ reduction starting with x must eventually terminate at a normal form of x).*

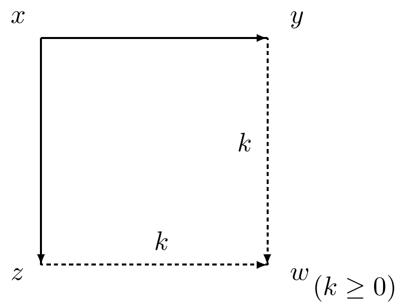
3 Balanced Weak Church-Rosser Property

This section introduces the new concept of balanced weak Church-Rosser property. Though in the later sections this concept will play an important role for analyzing normalizing strategies of term rewriting systems, our results concerning the balanced weak Church-Rosser property can be presented in an abstract framework depending solely on the reduction relation.

Let $A = \langle D, \to \rangle$ be an abstract reduction system.

Definition 3.1 *$A = \langle D, \to \rangle$ (or $\to$) is balanced weakly Church-Rosser (BWCR) iff*

$$\forall x, y, z \in D[x \to y \wedge x \to z \Rightarrow \exists w \in D, \exists k \geq 0, y \xrightarrow{k} w \wedge z \xrightarrow{k} w].$$



We define the *local* Church-Rosser property and the *local* strong normalizing property for an element $x \in D$. x is Church-Rosser if

$\forall y, z \in D[x \xrightarrow{*} y \wedge x \xrightarrow{*} z \Rightarrow \exists w \in A, y \xrightarrow{*} w \wedge z \xrightarrow{*} w]$.

x is strongly normalizing if every reduction starting with x terminates. x is complete if x is Church-Rosser and strongly normalizing.

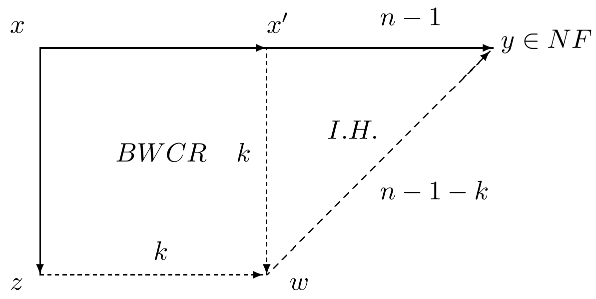
Lemma 3.2 (BWCR Lemma) *Let $A = \langle D, \to \rangle$ be BWCR. Let $x = y$ and $y \in NF$. Then,*

(1) *x is complete,*

(2) *All the reductions from x to y have the same length (i.e., the same number of reduction steps).*

Proof. We first prove the following claim: if $x \xrightarrow{n} y$ and $y \in NF$ ($n \geq 0$) then x satisfies the properties (1) and (2).

Proof of the claim. We show the claim by induction on n. The case $n = 0$ is trivial. Let $x \to x' \xrightarrow{n-1} y \in NF$. Take any one step reduction $x \to z$ starting with x. By the balanced weak Church-Rosser property, there exists some w and $k \geq 0$ such that $z \xrightarrow{k} w$ and $x' \xrightarrow{k} w$. By induction hypothesis, the properties (1) and (2) hold at x'; hence $x' \xrightarrow{k} w \xrightarrow{*} y$ must have $n-1$ steps in length. Thus, $w \xrightarrow{n-1-k} y$. Since $z \xrightarrow{k} w$, we obtain $z \xrightarrow{n-1} y$. By induction hypothesis, z satisfies the properties (1) and (2). Therefore, the claim follows.



We next show that if $x \overset{n}{\leftrightarrow} y$ and $y \in NF$ ($n \geq 0$) then $x \xrightarrow{*} y$. Here, $\overset{n}{\leftrightarrow}$ denotes the n-step connection of $\leftrightarrow$. The proof is by induction on n. The case $n = 0$ is trivial. Let $x \leftrightarrow x' \overset{n-1}{\leftrightarrow} y$. By induction hypothesis, we have $x' \xrightarrow{*} y$. The case $x \to x'$ is trivial. Let $x \leftarrow x'$. By applying the claim to $x' \xrightarrow{*} y \in NF$, it is obtained that x' is complete. Thus, $x \xrightarrow{*} y$.

Therefore, from the claim it follows that if $x = y$ and $y \in NF$ then x satisfies the properties (1) and (2). □

We say that A (or $\to$) has the normal form property if $\forall x \in D, \forall y \in NF[x = y \Rightarrow x \xrightarrow{*} y]$. A (or $\to$) has the unique normal form property if $\forall x, y \in NF[x = y \Rightarrow x \equiv y]$. It is well-known that the normal form property implies the unique normal form property [9, 10].

Corollary 3.3 *If an abstract reduction system A is BWCR then A has the normal form property.*

Proof. From the BWCR lemma, it is trivial. $\Box$

Next we will explain how the blanced weak Church-Rosser property is related to a normalizing reduction strategy.

$d(x)$ denotes the length of a reduction from x to a normal form if it exists. Note that if $\to$ is balanced weakly Church-Rosser and x has a normal form then $d(x)$ is well-defined according to the BWCR lemma. We write $x \overset{m}{\leftarrow} y$ if $y \overset{m}{\to} x$.

Lemma 3.4 *Let $\to$ be balanced weakly Church-Rosser. Let $x \xrightarrow{m_1} \cdot \xleftarrow{n_1} \cdot \xrightarrow{m_2} \cdot \xleftarrow{n_2} \cdots \xrightarrow{m_p} \cdot \xleftarrow{n_p} y$ for some p, $m_1, \cdots, m_p, n_1, \cdots, n_p \geq 0$ and let x have a normal form. Then y has a normal and $d(x) - d(y) = \sum m_i - \sum n_i$.*

Proof. By the BWCR lemma it is clear that y has a normal form. We prove $d(x) - d(y) = \sum m_i - \sum n_i$ by induction on p. The case $p = 0$ is trivial. Let $x \xrightarrow{m_1} \cdot \xleftarrow{n_1} \cdot \xrightarrow{m_2} \cdot \xleftarrow{n_2} \cdots \xrightarrow{m_{p-1}} \cdot \xleftarrow{n_{p-1}} y' \xrightarrow{m_p} z \xleftarrow{n_p} y$. By the BWCR lemma, $d(y')$ and $d(z)$ are well-defined and $d(y') - m_p = d(z) = d(y) - n_p$. Thus, we have $d(y') - d(y) = m_p - n_p$. From induction hypothesis, $d(x) - d(y') = \sum_{i=1}^{p-1} m_i - \sum_{i=1}^{p-1} n_i$. Therefore, $d(x) - d(y) = \sum m_i - \sum n_i$. $\Box$

We write $x \longleftrightarrow\!\!\!\!\rightarrow y$ if there exists a connection $x \xrightarrow{m_1} \cdot \xleftarrow{n_1} \cdot \xrightarrow{m_2} \cdot \xleftarrow{n_2} \cdots \xrightarrow{m_p} \cdot \xleftarrow{n_p} y$ such that $\sum m_i > \sum n_i$. We sometimes write $x \leftarrow\!\!\!\!\longleftrightarrow y$ instead of $y \longleftrightarrow\!\!\!\!\rightarrow x$.

Lemma 3.5 *Let $\to$ be balanced weakly Church-Rosser. Let $x \longleftrightarrow\!\!\!\!\rightarrow y$ and let x have a normal form. Then y has a normal form and $d(x) > d(y)$.*

Proof. It is trivial from Lemma 3.4. $\Box$

The following lemma and corollary explain how the BWCR lemma implies the normalizing property of a reduction strategy $\underset{s}{\to} \subseteq \to$.

Lemma 3.6 *Let $\underset{s}{\to}$ be a reduction strategy for $\to$ such that:*

(1) *$\underset{s}{\to}$ is balanced weakly Church-Rosser,*

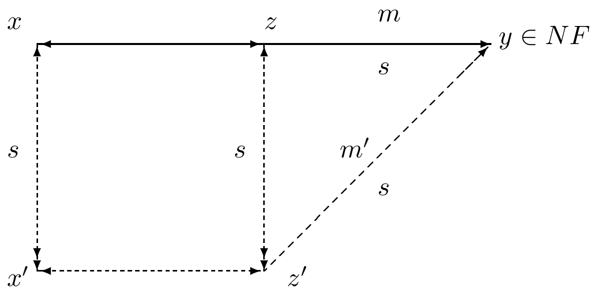
(2) *If $x \to y$ then;*

- **(i)** *$x \underset{s}{=} y$ or,*
- **(ii)** *$x \underset{s}{\longleftrightarrow\!\!\!\!\rightarrow} \cdot \leftrightarrow \cdot \underset{s}{\leftarrow\!\!\!\!\longleftrightarrow} y$.*

If $x = y$ and $y \in NF$ then we have $x \underset{s}{\overset{*}{\to}} y$.

Proof. We first show the claim: If $x \leftrightarrow \cdot \underset{s}{\overset{m}{\to}} y$ and $y \in NF$ $(m \geq 0)$, then we have $x \underset{s}{=} y$.

The proof is by induction on m. The case $m = 0$ is trivial from $y \in NF$ and (i) (since (ii) does not hold form Lemma 3.5). Let $x \leftrightarrow z \underset{s}{\overset{m}{\to}} y \in NF$ $(m > 0)$. Then $x \to z$ must satisfy (i) or (ii). If $x \underset{s}{=} z$ then $x \underset{s}{=} y$ is trivial. Assume that $x \underset{s}{\longleftrightarrow\!\!\!\!\rightarrow} x' \leftrightarrow z' \underset{s}{\leftarrow\!\!\!\!\longleftrightarrow} z$. By applying Lemma 3.5 to $z \underset{s}{\longleftrightarrow\!\!\!\!\rightarrow} z'$, we have $z' \underset{s}{\overset{m'}{\to}} y$ with $m' < m$. Applying induction hypothesis of the claim to x', we obtain that $x \underset{s}{\longleftrightarrow\!\!\!\!\rightarrow} x' \underset{s}{=} y$.



We next prove that if $x \overset{n}{\leftrightarrow} y$ and $y \in NF$ $(n \geq 0)$, then $x \underset{s}{\overset{*}{\to}} y$. The proof is by induction on n. The case $n = 0$ is trivial. Let $x \leftrightarrow x' \overset{n-1}{\leftrightarrow} y \in NF$. From induction hypothesis, we have $x' \underset{s}{\overset{*}{\to}} y$. Thus, form the claim, $x \underset{s}{=} y$. From the BWCR lemma, it follows that $x \underset{s}{\overset{*}{\to}} y$. $\Box$

Corollary 3.7 *Let $\underset{s}{\to}$ be a reduction strategy for $\to$ such that:*

(1) *$\underset{s}{\to}$ is balanced weakly Church-Rosser,*

(2) *If $x \to y$ then;*

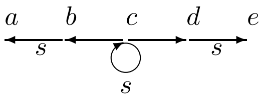
- **(i)** *$x \underset{s}{=} y$ or,*

(ii) $x \underset{s}{\leftarrow\!\!\!\twoheadrightarrow} \cdot \leftrightarrow \cdot \underset{s}{\twoheadleftarrow\!\!\!\rightarrow} y$.

Then $\rightarrow$ has the normal form property and $\underset{s}{\rightarrow}$ is a normalizing strategy.

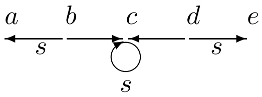
Proof. It is trivial form the BWCR lemma and Lemma 3.6. □

Remark. In Lemma 3.6 and Corollary 3.7 we cannot relax (ii) $x \underset{s}{\leftarrow\!\!\!\twoheadrightarrow} \cdot \leftrightarrow \cdot \underset{s}{\twoheadleftarrow\!\!\!\rightarrow} y$ to $x \underset{s}{\leftarrow\!\!\!\twoheadrightarrow} \cdot \overset{+}{\leftrightarrow} \cdot \underset{s}{\twoheadleftarrow\!\!\!\rightarrow} y$. A counterexample is:



Note that $c \underset{s}{\leftarrow\!\!\!\twoheadrightarrow} \cdot \overset{+}{\leftrightarrow} \cdot \underset{s}{\twoheadleftarrow\!\!\!\rightarrow} b$ and $c \underset{s}{\leftarrow\!\!\!\twoheadrightarrow} \cdot \overset{+}{\leftrightarrow} \cdot \underset{s}{\twoheadleftarrow\!\!\!\rightarrow} d$ since $c \underset{s}{\rightarrow} c \rightarrow b \rightarrow a \underset{s}{\leftarrow} b$ and $c \underset{s}{\rightarrow} c \rightarrow d \rightarrow e \underset{s}{\leftarrow} d$.

Remark. In Corollary 3.7 if we need only to show that $\underset{s}{\rightarrow}$ is a normalizing strategy for $\rightarrow$, we may replace (ii) $x \underset{s}{\leftarrow\!\!\!\twoheadrightarrow} \cdot \leftrightarrow \cdot \underset{s}{\twoheadleftarrow\!\!\!\rightarrow} y$ with $x \underset{s}{=} \cdot \rightarrow \cdot \underset{s}{\twoheadleftarrow\!\!\!\rightarrow} y$. However, in this case the normal form property of $\rightarrow$ does not hold. An example in which $\underset{s}{\rightarrow}$ is a normalizing strategy is:



Remark. One might think that by taking index reduction $\underset{I}{\rightarrow}$ (which is defined later) as $\underset{s}{\rightarrow}$ in Corollary 3.7, it might be shown directly that index reduction is normalizing for strongly sequential term rewriting systems. However, this is not the case since the property (2)-(ii) in the lemma does not hold.

4 Term Rewriting Systems

In the following sections, we will explain how to apply the BWCR lemma to term rewriting systems. We briefly explain the basic notions and definitions concerning term rewriting systems [3, 6, 7, 10].

Let $\mathcal{F}$ be an enumerable set of function symbols denoted by $f, g, h, \cdots$, and let $\mathcal{V}$ be an enumerable set of variable symbols denoted by $x, y, z, \cdots$ where $\mathcal{F} \cap \mathcal{V} = \phi$. By $T(\mathcal{F}, \mathcal{V})$, we denote the set of terms constructed from $\mathcal{F}$ and $\mathcal{V}$. The term set $T(\mathcal{F}, \mathcal{V})$ is sometimes denoted by T.

A substitution θ is a mapping from a term set $T(\mathcal{F}, \mathcal{V})$ to $T(\mathcal{F}, \mathcal{V})$ such that for a term t, $\theta(t)$ is completely determined by its values on the variable symbols occurring in t. Following common usage, we write this as $t\theta$ instead of $\theta(t)$.

Consider an extra constant $\square$ called a hole and the set $T(\mathcal{F} \cup \{\square\}, \mathcal{V})$. Then $C \in T(\mathcal{F} \cup \{\square\}, \mathcal{V})$ is called a context on $\mathcal{F}$. We use the notation $C[\ ,\ldots,\ ]$ for the context containing n holes ($n \geq 0$), and if $t_1, \ldots, t_n \in T(\mathcal{F}, \mathcal{V})$, then $C[t_1, \ldots, t_n]$ denotes the result of placing $t_1, \ldots, t_n$ in the holes of $C[\ ,\ldots,\ ]$ from left to right. In particular, $C[\ ]$ denotes a context containing precisely one hole. s is called a subterm of $t \equiv C[s]$. If s is a subterm occurrence of t, then we write $s \subseteq t$. If a term t has an occurrence of some (function or variable) symbol e, we write $e \in t$. The variable occurrences $z_1, \cdots, z_n$ of $C[z_1, \cdots, z_n]$ are fresh if $z_1, \cdots, z_n \notin C[\ ,\cdots,\ ]$ and $z_i \not\equiv z_j$ $(i \neq j)$.

A rewriting rule is a pair $\langle l, r\rangle$ of terms such that $l \notin \mathcal{V}$ and any variable in r also occurs in l. We write $l \rightarrow r$ for $\langle l, r\rangle$. A redex is a term $l\theta$, where $l \rightarrow r$. In this case $r\theta$ is called a contractum of $l\theta$. The set of rewriting rules defines a reduction relation $\rightarrow$ on T as follows:

$$t \rightarrow s \text{ iff } t \equiv C[l\theta],\ s \equiv C[r\theta] \quad \text{for some rule } l \rightarrow r, \text{ and some } C[\ ],\ \theta.$$

When we want to specify the redex occurrence $\Delta \equiv l\theta$ of t in this reduction, we write $t \overset{\Delta}{\rightarrow} s$.

Definition 4.1 *A term rewriting system R is a reduction system $R = \langle T(\mathcal{F}, \mathcal{V}), \rightarrow\rangle$ such that the reduction relation $\rightarrow$ on $T(\mathcal{F}, \mathcal{V})$ is defined by a set of rewriting rules. If R has $l \rightarrow r$ as a rewriting rule, we write $l \rightarrow r \in R$.*

We say that R is left-linear if for any $l \rightarrow r \in R$, l is linear (i.e., every variable in l occurs only once).

Let $l \rightarrow r$ and $l' \rightarrow r'$ be two rules in R. Assume that we have renamed the variables appropriately, so that l and l' share no variables. Assume that $s \notin V$ is a subterm occurrence in l, i.e., $l \equiv C[s]$, such that s and l' are unifiable, i.e., $s\theta \equiv l'\theta$, with a minimal unifier θ [6, 10]. Since $l\theta \equiv C[s]\theta \equiv C\theta[l'\theta]$, two reductions starting with $l\theta$, i.e., $l\theta \rightarrow C\theta[r'\theta] \equiv C[r']\theta$ and $l\theta \rightarrow r\theta$, can be obtained by using $l' \rightarrow r'$ and $l \rightarrow r$. Then we say that $l \rightarrow r$ and $l' \rightarrow r'$ are overlapping, and that the pair $\langle C[r']\theta, r\theta\rangle$ of terms is critical in R [6, 7] . We may choose $l \rightarrow r$ and $l' \rightarrow r'$ to be the same rule, but in this case we shall not consider the case $s \equiv l$, which gives the trivial pair $\langle r, r\rangle$. If R has no critical pair, then we say that R is non-overlapping. If every critical pair $\langle s, t\rangle$ is trivial, i.e., $s \equiv t$, then R is weakly overlapping [6, 7, 10].

R is orthogonal if R is left-linear and non-overlapping. R is weakly orthogonal if R is left-linear and weakly overlapping. The following result is well known [6, 10].

Proposition 4.2 *Let R be orthogonal (or weakly orthogonal). Then R is Church-Rosser.*

5 Strong Sequentiality

The fundamental concept of strong sequentiality for orthogonal term rewriting systems was introduced by Huet and Lévy [5]. In this section we explain the basic notions and properties related to strong sequentiality, according to Huet and Lévy [5], and Klop and Middeldorp [11]. Instead of the orthogonality, we assume only left-linearity for term rewriting systems. Thus, strong sequentiality defined here is an extension of the original one in [5].

Note. From here on we assume that R is a left-linear term rewriting system which may have *overlapping* rules.

Consider an extra constant Ω and the set $T(\mathcal{F} \cup \{\Omega\}, \mathcal{V})$, denoted by T_Ω. The element of T_Ω is called a Ω-term.

Definition 5.1 *The preordering $\geq$ on T_Ω is defined as follows:*

$$t \geq \Omega \text{ for all } t \in T_\Omega,$$
$$f(t_1, \cdots, t_n) \geq f(s_1, \cdots, s_n) \quad (n \geq 0) \quad \text{if } t_i \geq s_i \text{ for } i = 1, \cdots, n.$$

We write $t > s$ if $t \geq s$ and $t \not\equiv s$.

Definition 5.2 (Compatibility) *Two Ω-terms t and s are compatible, denoted by $t \uparrow s$, if there exists some Ω-term r such that $r \geq t$ and $r \geq s$; otherwise, t and s are incompatible, denoted by $t \# s$. Let S be a set of Ω-term. Then $t \uparrow S$ if there exists some $s \in S$ such that $t \uparrow s$; otherwise, $t \# S$.*

Let t_Ω denote the Ω-term obtained from a term t by replacing each variable in t with Ω. The set of redex schemata of R is $Red = \{\, l_\Omega \mid l \to r \in R\}$. The Ω-reduction $\to_\Omega$ is defined on T_Ω as $C[s] \to_\Omega C[\Omega]$ where $s \uparrow Red$ and $s \not\equiv \Omega$. Since $\to_\Omega$ is strongly normalizing, the following lemma can be easily proven [11].

Lemma 5.3 *$\to_\Omega$ is complete (i.e., Church-Rosser and strongly normalizing).*

Proof. It is clear that $\to_\Omega$ is weakly Church-Rosser. Apply Newman's Lemma. □

$\omega(t)$ denotes the normal form of t concerning $\to_\Omega$. Note that $\omega(t)$ is well-defined according to the completeness of $\to_\Omega$.

Lemma 5.4 *If $t \geq s$ then $\omega(t) \geq \omega(s)$.*

Proof. By induction on the size of t, it can be easily proven. □

Example 5.5 *Let R be the (left-linear) term rewriting system with the following rewriting rules:*

$$R \quad \begin{cases} f(c, h(x)) \to c \\ g(h(x), y) \to f(x, y) \\ d \to c. \end{cases}$$

Then $Red = \{f(c, h(\Omega)), g(h(\Omega), \Omega), d\}$ and $\omega(f(g(d, c), c)) \equiv f(\Omega, c)$.

Definition 5.6 (Index) *Let Δ be a redex occurrence in $C[\Delta]$ such that $z \in \omega(C[z])$ where z is a fresh variable. Then the redex occurrence Δ is called an index of t. If Δ is an index of $C[\Delta]$ then we write $C[\Delta_I]$; otherwise $C[\Delta_{NI}]$.*

The original definition of index in Huet and Lévy [5] is restricted to orthogonal term rewriting systems; hence, any two indexes occurring in a term must be disjoint. On the other hand we assume only left-linearity for term rewriting systems. Hence, if a term rewriting system is overlapping then two indexes may be overlapping as follows.

Example 5.7 *Let $Red = \{p(s(\Omega)), s(p(\Omega))\}$. Then we have the overlapping indexes $f(s(p(s(x))_I)_I)$ since $\omega(f(z)) \equiv f(z)$ and $\omega(f(s(z))) \equiv f(s(z))$.*

One might think that overlapping redex occurrences always make overlapping indexes if one of them is an index, but this is not the case from the following example.

Example 5.8 *Let $Red = \{0, f(0), g(f(\Omega), 1)\}$. Then we have $g(f(0_{NI})_I, 0_I)$. Note that two redex occurrences $f(0)$ and 0 are overlapping but 0 occurring in $f(0)$ is not an index.*

$t \xrightarrow{\Delta} s$ is an index reduction if Δ is an index of t. We write $t \underset{I}{\rightarrow} s$ if there exists an index reduction $t \xrightarrow{\Delta} s$ for some index Δ; otherwise $t \underset{NI}{\rightarrow} s$.

We say that R is strongly sequential if for each term $t \notin NF$, t has an index [5, 11, 10]. Note that index reduction of a strongly sequential system R is a reduction strategy because we can always apply an index reduction to a term being not a normal form.

The decidability of strong sequentiality for orthogonal term rewriting systems was first proven by Huet and Lévy [5], through a complicated decision procedure. A simple proof by Klop and Middeldorp can be found in [11]. This result can be immediately generalized to left-linear term rewriting systems.

Theorem 5.9 *Strong sequentiality of left-linear term rewriting systems (which may have overlapping rules) is decidable.*

Proof. The proof of the decidability by Klop and Middeldorp [11] does not use the non-overlapping limitation. Thus, their proof can be applied to the above theorem without modification. □

6 Index Reduction of Overlapping Systems

We will now explain how to prove the normalizing property of index reduction for *overlapping* term rewriting systems by using the BWCR lemma. We first define *root balanced joinable* term rewriting systems.

Let R be a term rewriting system. The root reduction $t \underset{r}{\rightarrow} s$ is defined as $t \xrightarrow{\Delta} s$ and $\Delta \equiv t$.

Lemma 6.1 *Let $C[\tilde{\Delta}_I]$ for some $\tilde{\Delta}$ and let $t \underset{r}{\rightarrow} s$. Then $C[t] \underset{I}{\rightarrow} C[s]$.*

Proof. It is trivial form the definition of the root reduction. □

Definition 6.2 *A critical pair $\langle s, t\rangle$ is root balanced joinable if $s \underset{r}{\overset{k}{\rightarrow}} t'$ and $t \underset{r}{\overset{k}{\rightarrow}} t'$ for some t' and $k \geq 0$. A term rewriting system R is root balanced joinable if every critical pair is root balanced joinable.*

$$\begin{tikzcd} & t \arrow[d, dotted, "k"', "r"] \\ s \arrow[r, dotted, "k", "r"'] & t' \end{tikzcd} \quad (k \geq 0)$$

Note that every weakly orthogonal term rewriting system is trivially root balanced joinable since every critical pair is root balanced joinable with $k = 0$.

Definition 6.3 *Let Δ and Δ' be two redex occurrences in a term t, and let $\Delta \equiv C[s_1, \cdots, s_n]$ and $C[\Omega, \cdots, \Omega] \in Red$. Then Δ and Δ' (or Δ' and Δ) are overlapping if $\Delta' \subseteq \Delta$ and $\Delta' \not\subseteq s_i$ for any i.*

Lemma 6.4 *Let R be root balanced joinable. Let $t \underset{I}{\overset{\Delta}{\rightarrow}} t'$ and $t \xrightarrow{\Delta'} t''$, where $\Delta' \subseteq \Delta$ and Δ and Δ' are overlapping. Then, we have $t' \underset{I}{\overset{k}{\rightarrow}} s$ and $t'' \underset{I}{\overset{k}{\rightarrow}} s$ for some s and $k \geq 0$.*

$$\begin{tikzcd} t \arrow[r, "\Delta'"] \arrow[d, "\Delta"', "I"] & t'' \arrow[d, dotted, "k"', "I"] \\ t' \arrow[r, dotted, "k", "I"'] & s \end{tikzcd} \quad (k \geq 0)$$

Proof. Let $t \equiv C[\Delta] \equiv C[C'[\Delta']]$, $t' \equiv C[p]$, $t'' \equiv C[q]$. From the root balanced joinability of the critical pair concerning Δ and Δ', we have $p \underset{r}{\overset{k}{\rightarrow}} s'$ and $q \underset{r}{\overset{k}{\rightarrow}} s'$ for some s' and $k \geq 0$. Thus, from $C[\Delta_I]$ and Lemma 6.1, it follows that $C[p] \underset{I}{\overset{k}{\rightarrow}} C[s']$ and $C[q] \underset{I}{\overset{k}{\rightarrow}} C[s']$. □

Lemma 6.5 *Let $C[\Delta_I, \Delta']$. Then $C[\Delta_I, t]$ for any t.*

Proof. $z \in \omega(C[z, \Delta']) \equiv \omega(C[z, \Omega]) \leq \omega(C[z, t])$. Thus, we have $z \in \omega(C[z, t])$. □

.

Lemma 6.6 *Let R be root balanced joinable. Let $t \xrightarrow[I]{\Delta} t'$, $t \xrightarrow[I]{\Delta'} t''$. Then, we have $t' \xrightarrow[I]{k} s$ and $t'' \xrightarrow[I]{k} s$ for some s and $k \geq 0$ (i.e., $\underset{I}{\rightarrow}$ is BWCR).*

$$\begin{tikzcd}
t \arrow[r, "\Delta'", "I"'] \arrow[d, "\Delta"', "I"] & t'' \arrow[d, dashed, "k"', "I"] \\
t' \arrow[r, dashed, "k", "I"'] & s \quad (k \geq 0)
\end{tikzcd}$$

Proof. If Δ and Δ' are disjoint, then from Lemma 6.5 the theorem clearly holds with $k = 1$. Assume that Δ and Δ' are not disjoint, say $\Delta' \subseteq \Delta$. Then, Δ and Δ' must be overlapping as they both are the indexes. Apply Lemma 6.4. □

The parallel reduction $t \mathrel{-\!\!\!\Vert\!\!\!\twoheadrightarrow} s$ is defined with $t \equiv C[\Delta_1, \cdots, \Delta_n] \xrightarrow{\Delta_1} \cdots \xrightarrow{\Delta_n} s$ $(n \geq 0)$.

Lemma 6.7 *Let R be strongly sequential and root balanced joinable, and let $t \mathrel{-\!\!\!\Vert\!\!\!\twoheadrightarrow} s$. Then $t \underset{I}{=} s$ or $t \underset{I}{\longleftrightarrow\!\!\!\!\!\twoheadrightarrow} \cdot \mathrel{-\!\!\!\Vert\!\!\!\twoheadrightarrow} \cdot \underset{I}{\twoheadleftarrow\!\!\!\!\!\longleftrightarrow} s$.*

Proof. Let $t \overset{\Delta_1 \cdots \Delta_n}{\mathrel{-\!\!\!\Vert\!\!\!\twoheadrightarrow}} s$ $(n \geq 0)$. The proof is by induction on n. The case $n = 0$ is trivial. Induction Step:

Case 1. Some Δ_i, say Δ_1, is an index.

We have $t \xrightarrow[I]{\Delta_1} t' \overset{\Delta_2 \cdots \Delta_n}{\mathrel{-\!\!\!\Vert\!\!\!\twoheadrightarrow}} s$. By applying induction hypothesis to $t' \overset{\Delta_2 \cdots \Delta_n}{\mathrel{-\!\!\!\Vert\!\!\!\twoheadrightarrow}} s$, we obtain the lemma.

Case 2. No Δ_i is an index.

From the strong sequentiality there must exist an index, say Δ, in t. Let $t \xrightarrow[I]{\Delta} t''$ and consider the following two cases.

Case 2-1. Δ and Δ_i $(i = 1 \cdots n)$ are non-overlapping.

By using the left-linearity of R, we can easily show that $t'' \mathrel{-\!\!\!\Vert\!\!\!\twoheadrightarrow} s'$ and $s \underset{I}{\rightarrow} s'$ for some s'.

$$\begin{tikzcd}
t \arrow[r, "{\Delta_1, \cdots, \Delta_n}", "\Vert" marking] \arrow[d, "I"', "\Delta"] & s \arrow[d, dashed, "I"'] \\
t'' \arrow[r, dashed, "\Vert" marking] & s'
\end{tikzcd}$$

Case 2-2. Δ and some Δ_i, say Δ_1, are overlapping.

Let $t \xrightarrow[NI]{\Delta_1} t' \overset{\Delta_2 \cdots \Delta_n}{\mathrel{-\!\!\!\Vert\!\!\!\twoheadrightarrow}} s$. Note that $\Delta_1 \subseteq \Delta$. From Lemma 6.4, it follows that $t'' \xrightarrow[I]{k} s'$ and $t' \xrightarrow[I]{k} s'$ for some $k \geq 0$. Thus, we can obtain $t \underset{I}{\longleftrightarrow\!\!\!\!\!\twoheadrightarrow} t'$. Apply induction hypothesis to $t' \overset{\Delta_2 \cdots \Delta_n}{\mathrel{-\!\!\!\Vert\!\!\!\twoheadrightarrow}} s$. □

$$\begin{tikzcd}
t \arrow[r, "\Delta_1", "NI"'] \arrow[d, "\Delta"', "I"] & t' \arrow[r, "{\Delta_2, \cdots, \Delta_n}", "\Vert" marking] \arrow[d, dashed, "k"', "I"] & s \\
t'' \arrow[r, dashed, "k", "I"'] & s'
\end{tikzcd}$$

Theorem 6.8 *Let R be strongly sequential and root balanced joinable. Then, R has the normal form property, and index reduction $\underset{I}{\rightarrow}$ is a normalizing strategy.*

Proof. Note that from the strong sequentiality, we have $NF = NF_I$ (NF_I denotes the set of the normal forms concerning $\underset{I}{\rightarrow}$). Thus, $\underset{I}{\rightarrow}$ is a reduction strategy for $\rightarrow$. Let $t \mathrel{-\!\!\!\Vert\!\!\!\twoheadrightarrow}' s$ be $t \equiv C[\Delta_1, \cdots, \Delta_n] \xrightarrow{\Delta_1} \cdots \xrightarrow{\Delta_n} s$ $(n > 0)$. Then it is clear that $\underset{I}{\rightarrow}$ is a reduction strategy for $\mathrel{-\!\!\!\Vert\!\!\!\twoheadrightarrow}'$ too. Taking $\underset{I}{\rightarrow}$ as $\underset{s}{\rightarrow}$ and $\mathrel{-\!\!\!\Vert\!\!\!\twoheadrightarrow}'$ as $\rightarrow$ respectively, from Lemmas 6.6 and 6.7, we can apply Corollary 3.7. From $\overset{*}{\rightarrow} = \overset{*}{\mathrel{-\!\!\!\Vert\!\!\!\twoheadrightarrow}}'$, the theorem follows. □

7 Quasi-Index Reduction

Quasi-index reduction (or hyper-index reduction) is defined as $\xrightarrow[NI]{*} \cdot \xrightarrow[I]{}$. We show that in Theorem 6.8 index reduction can be relaxed into quasi-index reduction. We first prove the next lemma.

Lemma 7.1 *Let R be strongly sequential and root balanced joinable. Let $t \xrightarrow[I]{n} s \in NF$ for some $n \geq 0$ and $t \xrightarrow{*} t'$. Then, we have $t' \xrightarrow[I]{m} s$ for some $m \leq n$.*

$$\begin{tikzcd} t \arrow[rr, "n", "I"'] \arrow[dd, "*"'] & & s \in NF \\ & & \\ t' \arrow[uurr, dashed, "m", "I"'] & & \end{tikzcd}$$

Proof. The proof is by induction on n. The case $n = 0$ is trivial. Induction Step: To prove the lemma, we only have to show that if $t \xrightarrow[I]{\Delta} t'' \xrightarrow[I]{n-1} s \in NF$ and $t \xrightarrow{\Delta'} t'$, then $t' \xrightarrow[I]{m} s \in NF$ for some $m \leq n$.

$$\begin{tikzcd} t \arrow[r, "\Delta", "I"'] \arrow[dd, "\Delta'"'] & t'' \arrow[r, "n-1", "I"'] & s \in NF \\ & & \\ t' \arrow[uurr, dashed, "m", "I"'] & & \end{tikzcd}$$

Case 1. Δ and Δ' are non-overlapping.

By the left-linearity of R, we can easily show that $t' \xrightarrow[I]{} s'$ and $t'' \xrightarrow{*} s'$ for some s'. From induction hypothesis, it follows that $s' \xrightarrow[I]{m'} s$ for some $m' \leq n-1$. Thus, we obtain $t' \xrightarrow[I]{m'+1} s$.

Case 2. Δ and Δ' are overlapping.

By using Lemma 6.4, we have $t' \xrightarrow[I]{k} s'$ and $t'' \xrightarrow[I]{k} s'$ for some s' and $k \geq 0$. Applying the BWCR lemma to t'', we have $s' \xrightarrow[I]{n-1-k} s$. Thus, it follows that $t' \xrightarrow[I]{n-1} s$. □

Theorem 7.2 *Let R be strongly sequential and root balanced joinable. Then quasi-index reduction $\xrightarrow[NI]{*} \cdot \xrightarrow[I]{}$ is a normalizing strategy.*

Proof. Let t have a normal form s. Then by Theorem 6.8 we have $t \xrightarrow[I]{n} s$ for some n. By using induction on n we prove that every quasi-index reduction starting with t is normalizing. The case $n = 0$ is trivial as t is the normal from. Let $t \xrightarrow[I]{n} s$ $(n > 0)$. Take any one-step quasi-index reduction starting with t, say $t \xrightarrow[NI]{*} t' \xrightarrow[I]{} t''$. From Lemma 7.1 we have $t' \xrightarrow[I]{m} s$ for some $m \leq n$. Thus, by applying the BWCR lemma to t' we obtain $t'' \xrightarrow[I]{n'} s$ for some $n' < n$ as $m = n'+1 \leq n$. From induction hypothesis it follows that every quasi-index reduction starting with t'' is normalizing. Therefore the theorem holds. □

8 Root Balanced Joinable Left-Normal Systems

In this section we discuss syntactical characterization of strongly sequential overlapping term rewriting systems. An answer concerning orthogonal term rewriting systems was found by O'Donnell [12]. He proved that if an orthogonal term rewriting system R is left-normal then R is strongly sequential and leftmost-outermost reduction is normalizing. We show that his result can be naturally extended to root balanced joinable term rewriting systems.

Definition 8.1 *The set T_L of the left-normal terms is inductively defined as follows:*

1. *$x \in T_L$ if x is a variable,*
2. *$f(t_1, \cdots, t_{p-1}, t_p, t_{p+1} \cdots, t_n) \in T_L$ $\quad (0 \leq p \leq n)$ if $t_1, \cdots, t_{p-1}$ are ground terms (i.e., no variable occurs in $t_1, \cdots, t_{p-1}$), $t_p \in T_L$, and $t_{p+1}, \cdots, t_n$ are variables.*

The set of the left-normal schemata is $T_{L\Omega} = \{t_\Omega \mid t \in T_L\}$. We say that R is left-normal [12, 5, 10] iff for any rule $l \to r$ in R, l is a left-normal term, i.e., $Red \subseteq T_{L\Omega}$.

Let Δ be the leftmost-outermost redex in $t \equiv C[\Delta]$. Then we write $t^z \equiv C[z]$, where z is a fresh variable.

Lemma 8.2 *Let R be left-linear and left-normal (note that R may be overlapping). If a term t is not a normal form, then:*

(1) $z \in \omega(t^z)$ *(i.e., the leftmost-outermost redex of t is an index),*

(2) *For any $s \in T_{L\Omega}$, if $\omega(t^z) \uparrow s$ then $t \uparrow s$.*

Proof. Let Δ be the leftmost-outermost redex in $t \equiv C[\Delta]$. By induction on the size of $C[\ ]$, we will prove the lemma.

Basic step: $C[\ ] \equiv \Box$. Then $t \equiv \Delta$ and $t^z \equiv z$. Thus (1) is trivial. If $\omega(t^z) \uparrow s$ then $s \equiv \Omega$. Hence, (2) is also trivial.

Induction step: Since $C[\ ] \not\equiv \Box$, we can write $t \equiv f(t_1, \cdots, t_{p-1}, t_p, t_{p+1}, \cdots, t_n)$, where $t_1, \cdots, t_{p-1}$ are normal forms and the leftmost-outermost redex Δ occurs in t_p. Note that Δ is the leftmost-outermost redex not only of t but also of t_p. Thus we can write $t^z \equiv f(t_1, \cdots, t_{p-1}, t_p^z, t_{p+1}, \cdots, t_n)$. Let us consider $\tilde{t}^z \equiv f(\omega(t_1), \cdots, \omega(t_{p-1}), \omega(t_p^z), \omega(t_{p+1}), \cdots, \omega(t_n))$. Then $\tilde{t}^z \equiv f(t_1, \cdots, t_{p-1}, \omega(t_p^z), \omega(t_{p+1}), \cdots, \omega(t_n))$ as $t_1, \cdots, t_{p-1}$ are normal forms. From induction hypothesis, $z \in \omega(t_p^z)$.

We will show the claim: For any $s \in T_{L\Omega}$, if $\tilde{t}^z \uparrow s$ then $t \uparrow s$.

Proof of the claim. Let $\tilde{t}^z \uparrow s$ and $s \equiv f(s_1, \cdots, s_p, \cdots, s_n)$. Then $\omega(t_p^z) \uparrow s_p$. By induction hypothesis, we have $t_p \uparrow s_p$; hence, $\Omega \in s_p$ as z occurs in $\omega(t_p^z)$. From $s \in T_{L\Omega}$ it follows that $s_{p+1} \equiv, \cdots, \equiv s_n \equiv \Omega$. Hence, it is clear that $t \uparrow s$.

Assume that $\tilde{t}^z \uparrow Red$. Then, from the claim, we have $t \uparrow Red$; contradiction to $\Delta \not\equiv t$. Thus, $\tilde{t}^z \# Red$. Hence $\omega(t^z) \equiv \tilde{t}^z$. Therefore, $z \in \omega(t^z)$. From the claim, it is clear that (2) also holds. $\Box$

Theorem 8.3 *Let R be root balanced joinable and left-normal. Then, R has the normal form property, and leftmost-outermost reduction is a normalizing strategy.*

Proof. This follows from Theorem 6.8 and Lemma 8.2. $\Box$

Note that every weakly orthogonal left-normal term rewriting system is root balanced joinable. Thus the following corollary holds.

Corollary 8.4 *Let R be weakly orthogonal left-normal. Then, R has the normal form property, and leftmost-outermost reduction is normalizing.*

Example 8.5 *Combinatory logic CL [1] is the orthogonal term rewriting system having the following rewriting rules:*

$$CL \quad \begin{cases} ((S \cdot x) \cdot y) \cdot z \to (x \cdot z) \cdot (y \cdot z) \\ (K \cdot x) \cdot y \to x. \end{cases}$$

Let R be combinatory logic CL +

$$\begin{cases} pred \cdot (succ \cdot x) \to x \\ succ \cdot (pred \cdot x) \to x \end{cases}$$

It is clear that R is weakly orthogonal left-normal. Thus, leftmost-outermost reduction strategy is normalizing for R.

Example 8.6 *Let R' be combinatory logic CL +*

$$\begin{cases} (K \cdot A) \cdot y \to (K \cdot B) \cdot y \\ (K \cdot B) \cdot y \to (K \cdot A) \cdot y \\ A \to A \\ B \to B, \end{cases}$$

It is clear that R' is not weakly orthogonal, but root balanced joinable and left-normal. Thus, from Theorem 8.3, leftmost-outermost reduction strategy is normalizing for R'. Note that R' is not Church-Rosser since $(K \cdot A) \cdot y$ can be reduced into two terms A and B which cannot be joined.

Acknowledgments

The author would like to thank H. P. Barendregt and J. W. Klop for offering a stimulating environment for this research and for valuable discussions. The author would also like to thank M. C. J. D. van Eekelen, J. R. Kennaway, M. J. Plasmeijer, M. R. Sleep, J. E. W. Smetsers, R. C. de Vrijer for valuable discussions. Specially thanks to A. Middeldorp and F. J. de Vries for carefully reading a previous version of this paper and for useful comments. Thanks are also due to the referees for useful comments.

References

[1] H. P. Barendregt, *The lambda calculus, its syntax and semantics* (North-Holland, 1981).

[2] G. Berry and J. J. Lévy, Minimal and optimal computations of recursive programs, *J. ACM 26* (1979) 148-175.

[3] N. Dershowitz and J. P. Jouannaud, Rewrite Systems, in: J. V. Leeuwen, ed., *Handbook of Theoretical Computer Science B* (North-Holland, 1990) 244-320.

[4] P. J. Downey and R. Sethi, Correct computation rules for recursive languages, *SIAM J. Comput. 5* (1976) 378-401.

[5] G. Huet and J. J. Lévy, Call-by-need computations in non-ambiguous linear term rewriting systems, *Rapport INRIA 359* (1979).

[6] G. Huet, Confluent reductions: Abstract properties and applications to term rewriting systems, *J. ACM 27* (1980) 797-821.

[7] G. Huet and D. C. Oppen, Equations and rewrite rules: A survey, in: R.V. Book, ed., *Formal languages: perspectives and open problems*, (Academic Press, 1980) 349-405.

[8] J. R. Kennaway, Sequential evaluation strategies for parallel-or and related reduction systems, *Annals of Pure and Applied Logic 43* (1989) 31-56.

[9] J. W. Klop, Combinatory reduction systems, Dissertation, Univ. of Utrecht, 1980.

[10] J. W. Klop, Term rewriting systems, *to appear* in: S. Abramsky, D. Gabbay and T. Maibaum, eds., *Handbook of Logic in Computer Science I* (Oxford University Press, 1991).

[11] J. W. Klop and A. Middeldorp, Sequentiality in orthogonal term rewritng systems, *Jounal of Symbolic Computation 12* (1991) 161-195.

[12] M. J. O'Donnell, Computing in systems described by equations, *Lecture Notes in Comput. Sci. Vol. 58* (Springer-Verlag, 1977).

[13] M. Oyamaguchi, Sufficent Sequentiality: A decidable condition for call-by-need computations in term rewriting systems, *Report, Mie University* (1987).

[14] S. Thatte, A refinement of strong sequentiality for term rewriting with constructors, *Information and Computation 72* (1987) 46-65.

[15] J. Vuillemin, Correct and optimal implementations of recursion in a simple programming language, *J. Compt. and Syst. Sci. 9* (1974) 332-354.