Semantics and Strong Sequentiality of Priority Term Rewriting Systems

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Abstract. This paper gives an operational semantics of priority term rewriting systems (PRS) by using conditional systems, whose reduction is decidable and stable under substitution. We also define the class of strong sequential PRSs and show that this class is decidable. Moreover, we show that the index rewriting of strong sequential PRSs gives a normalizing strategy.

1 Introduction

Baeten, Bergstra and Klop[BBKW89] proposed the formalism of priority term rewriting systems (PRSs) to capture the theoretical basis of order dependency in rules. A PRS differs from an ordinary TRS in that whenever a rule matches a given redex, it can only be chosen if none of the rules with higher priority can match the (internally rewritten) term, where a reduction step is internal if it proceeds entirely within the arguments of the leftmost symbol of the term. For example, the factorial function can be defined as follows:

$$\mathcal{R} = \left\{ \Big\downarrow \begin{array}{l} fac(0) \to 1 \\ fac(x) \to x * fac(x-1). \end{array} \right.$$

Here, the down arrow sign indicates that the top rule has higher priority than the bottom rule. If the argument of fac is 0 then the first rule is applicable. However, if it is somehow known that the argument can never be reduced to 0, then the priority of the rules makes it possible to choose the second rule without actually reducing the argument. Hence, the second rule can be applied on $fac(1+1)$ but not on $fac(1-1)$.

On the other hand, most functional languages admit priorities in rules and evaluation methods such as the functional strategy[PvE93, TSvEP93], an implementation method for lazy evaluation[Hen80, Jon87], possess the ability to handle priorities implicitly. However, it is hard to use PRSs as a theoretical computation model for analyzing the functional strategy. The main reason is that the PRSs in [BBKW89, Moh89] have limitations, for example “bounded”, to ensure well-defined operational semantics.

This paper develops a new semantics for PRSs. This new semantics is natural as a model of actual computation systems such as the functional strategy. The key idea is translating PRSs to conditional term rewriting systems (CTRSs) with direct approximants in conditions. Our operational semantics of PRSs are then defined as that of the corresponding CTRSs. This technique allows us to discuss behaviors of PRSs both theoretically and in practice. By using our framework of PRSs, we discuss a normalizing strategy of PRSs and provide a class of PRSs with a decidable normalizing strategy.

2 Term Rewriting Systems

We mainly follow the notation of [Klo92] and assume readers are familiar with abstract reduction systems and conditional term rewriting systems (TRS)[Hue80, Klo92, DJ90].

A reduction system is a structure $A = \langle D, \rightarrow \rangle$ for some set D and binary relation $\rightarrow$ on D called a reduction relation. A reduction is a finite sequence $x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_n$ $(n \geq 0)$ or an infinite sequence $x_0 \rightarrow x_1 \rightarrow \cdots$ of reduction steps. The identity of elements $x, y \in D$ is denoted by $x \equiv y$. $\xrightarrow{+}$ is the transitive closure of $\rightarrow$, $\xrightarrow{*}$ is the reflexive and transitive closure of $\rightarrow$, and $=$ is the equivalence relation generated by $\rightarrow$. $\xrightarrow{k}$ denotes a reduction of k steps. If there are no element y such that $x \rightarrow y$, then we say x is a normal form (w.r.t. $\rightarrow$); let $NF_{\rightarrow}$ be the set of normal forms. If y is a normal form such that $x \xrightarrow{*} y$ then we say y is a normal form of x (w.r.t. $\rightarrow$). If there is a normal form of x, we say x has a normal form.

We say a reduction system $A = \langle D, \rightarrow \rangle$ is terminating, if and only if there are no infinite reduction. A is confluent, if and only if $x \xrightarrow{*} y \ \wedge \ x \xrightarrow{*} z$ imply $\exists w \in D[y \xrightarrow{*} w \ \wedge \ z \xrightarrow{*} w]$ for any $x, y, z \in D$.

Let $A = \langle D, \rightarrow \rangle$ and $\rightarrow_s$ be a sub-relation of $\xrightarrow{+}$. If $NF_{\rightarrow_s} = NF_{\rightarrow}$, we say that $\rightarrow_s$ is a reduction strategy for $\rightarrow$. A reduction strategy $\rightarrow_s$ is normalizing, if and only if there exists no infinite sequence $x \equiv x_0 \rightarrow_s x_1 \rightarrow_s \cdots$ for any x having a normal form w.r.t. $\rightarrow$.

Let $\Sigma = V \cup F$ be a signature, where F and V are a set of function symbols with arity and a set of variables, respectively. T_Σ(or simply T) denotes the set of terms well constructed by symbols in Σ. $|t|$ denotes the size of a term t. Letting $\Box$ be an extra constant, a term $C \in T_{\Sigma \cup \{\Box\}}$ is called a context denoted by $C[\ ,\ldots,\ ]$. For a $C[\ ,\ldots,\ ]$ which contains n $\Box$'s and for $t_1, \ldots, t_n \in T_\Sigma$, $C[t_1, \ldots, t_n]$ denotes the obtained term by replacing $\Box$'s with $t_1, \ldots, t_n$ from left to right order. A context that possesses exactly one $\Box$ is denoted by $C[\ ]$.

Let σ be a substitution. We denote the application of σ to term t by $t\sigma$. $\subseteq$ indicates a subterm relation. If a term t has some symbol e, we write $e \in t$. The list $t_1, \ldots, t_n$ of terms for some $n \geq 0$ is often abbreviated by $\mathbf{t}$.

A conditional term rewriting system (CTRS) R is a finite set of conditional rewrite rules, each of which is $l \rightarrow r\ if\ P(\mathbf{x})$, where l and r are terms and $P(\mathbf{x})$ is a (decidable) condition such that $x_i \in l$ for each i. A conditional rewrite rule must satisfy two requirements: (a) l is not a variable, (b) variables in r must appear in l. A term $l\sigma$ is called a redex if $P(x_1\sigma, \ldots, x_n\sigma)$ holds.

We say that the term t reduces to s by rule $l \rightarrow r\ if\ P(\mathbf{x})$, if and only if there exist a substitution σ and context $C[\ ]$ such that $t \equiv C[l\sigma]$, $s \equiv C[r\sigma]$, and

$l\sigma$ is a redex. We write $t \to_R s$ when t reduces to s. Sometimes we write $t \stackrel{\Delta}{\longrightarrow}_R s$ to describe redex $\Delta \equiv l\sigma$ explicitly.

A CTRS is called left-linear, if every variable of l occurs only once for every rule $l \to r\ if\ P(\mathbf{x})$. We say that rules $l \to r\ if\ P(\mathbf{x})$ and $l' \to r'\ if\ P'(\mathbf{x})$ are overlapping, if there exist a term $s \subseteq l'$ ($s \notin V$) and substitutions σ and σ' such that $l\sigma \equiv s\sigma'$(except the trivial case, i.e., the rules are the same and $l \equiv s$). There is root overlap if $s \equiv l'$. A CTRS has overlap if it has overlapping rules. A CTRS has root overlap if the only root overlapping rules are overlapping.

A rewrite rule is a rule $l \to r\ if\ P(\mathbf{x})$ whose condition $P(\mathbf{x})$ alway holds(i.e. $P(\mathbf{x}) \equiv true$), and we abbreviate rewrite rules as $l \to r$. A term rewriting system(TRS) is a CTRS that has only rewrite rules.

3 Priority Term Rewriting Systems and their Operational Semantics

The definition of P-rewriting was given by Baeten et.al.[BBKW89] as a reduction of PRSs. Since P-rewriting is not always decidable, they introduced the notion of "bounded". As bounded is too strong a restriction, Mohan[Moh89] introduced the notion of I-rewrite, which is a decidable definition of reduction. However, since I-rewrite includes the notion of inner most reduction strategy, it cannot be used as a normalizing strategy.

In this section, we give an operational semantics of PRSs that is decidable and is independent from the notion of strategy.

Definition 1 (PRS). A priority term rewriting system(PRS) $\mathcal{R}$ is a pair $\langle R_0, \sqsupset\rangle$ of an underlying TRS R_0 and an acyclic relation $\sqsupset$ on R_0.

Example 1. The "parallel or" is described in PRS as follows:

$$POR = \left\{ \begin{array}{|l} or(tt,x) \to tt \\ or(x,tt) \to tt \\ \downarrow or(x,y) \to ff. \end{array} \right.$$

Before we define a semantics of PRS, we need the notion of an approximation of PRS by ignoring priorities and right-hand sides.

Letting Ω be an extra constant, we represent prefixes of terms by Ω-terms in $T_{\Sigma\cup\{\Omega\}}$ (also denoted by T_Ω simply). t_Ω denotes the Ω-term obtained from a term t by replacing each variable with Ω. The prefix ordering $\succeq$ on T_Ω is defined as follows [HL79, Klo92, KM91]: (i) $t \succeq \Omega$, (ii) $t \succeq t$, (iii) $f(t_1,\ldots,t_n) \succeq f(s_1,\ldots,s_n)$ if $t_i \succeq s_i$ for $i = 1,\ldots,n$.

t and s are compatible, written by $t \uparrow s$, if $u \succeq t$ and $u \succeq s$ for some u; otherwise they are incompatible, denoted by $t\#s$. $s \sqcup t$ denotes a minimal Ω-term u such that $u \succeq s$ and $u \succeq t$ if it exists. We say an Ω-term t and a set S of Ω-term are incompatible, denoted by $t\#S$, if t and s are incompatible for any $s \in S$.

Definition 2 (ω-systems). Let R_0 be a TRS. The ω-system of R_0 is a CTRS $R_\omega = \{f(\mathbf{x}) \to \Omega\ if\ f(\mathbf{x}) \uparrow f(\mathbf{l})_\Omega \mid f(\mathbf{l}) \to r \in R_0\}$. The reduction relation $\to_{R_\omega}$ on T_Ω is called ω-reduction and often abbreviated as $\to_\omega$.

Example 2. ω-system of POR in example 1 is

$$POR_\omega = \begin{cases} or(x,y) \to \Omega\ if\ or(x,y) \uparrow or(tt,\Omega) \\ or(x,y) \to \Omega\ if\ or(x,y) \uparrow or(\Omega,tt) \\ or(x,y) \to \Omega\ if\ or(x,y) \uparrow or(\Omega,\Omega) \end{cases}$$

$f(or(a,tt)) \to_\omega f(\Omega)$ by the second or the last rule.

Lemma 3 [Klo92]. *ω-reduction is confluent and terminating.*

The direct approximant $\omega(t)$ of an Ω-term t is the normal form of t w.r.t. ω-reduction. $\overline{\omega}(f(t_1,\ldots,t_n))$ denotes $f(\omega(t_1),\ldots,\omega(t_n))$.

Definition 4. The set $Hlhs(e)$ of left-hand-side skeletons of rules each of which has higher priority than a rule e is defined by

$$Hlhs(e) = \{l_\Omega \mid l \to r \in R_0,\ l \to r \sqsupset e\}$$

Now we can define an operational semantics of PRSs.

Definition 5 (a semantics of PRS). Let $\mathcal{R} = \langle R_0, \sqsupset\rangle$.

(a) $Sem_\omega(\mathcal{R}) = \{l \to r\ if\ \overline{\omega}(l_\Omega)\#Hlhs(l \to r) \mid l \to r \in R_0\}$.
(b) The reduction relation $\to_\mathcal{R}$ is defined as $\to_{Sem_\omega(\mathcal{R})}$.

Example 3. For POR defined in example 1,

$$Sem_\omega(POR) = \begin{cases} or(tt,y) \to tt\ if\ \overline{\omega}(or(tt,y_\Omega))\#\{\} \\ or(x,tt) \to tt\ if\ \overline{\omega}(or(x_\Omega,tt))\#\{or(tt,\Omega)\} \\ or(x,y) \to ff\ if\ \overline{\omega}(or(x_\Omega,y_\Omega))\#\{or(tt,\Omega), or(\Omega,tt)\} \end{cases}$$

(i) $or(ff,tt) \to_{POR} tt$ by using the second rule. But, the last rule cannot be applied to $or(ff,tt)$. (ii) $or(x,tt)$ is a normal form. (iii) $or(ff,ff) \to_{POR} ff$ by using the last rule. On the other hand, $or(x,y)$ is a normal form. (iv) $or(or(ff,ff),tt) \to_{POR} or(ff,tt) \to_{POR} tt$ by using the last rule and the second rule. But $or(or(ff,ff),tt)$ cannot directly be reduced to tt.

Remark. (i) According to definition of t_Ω, x_Ω and y_Ω in $Sem_\omega(POR)$ seem to be Ω. However this is not the case as the variables x and y are bounded in the rules. Hence, the replacements of x_Ω and y_Ω should take place after they are substituted.

(ii) $Sem_\omega(\mathcal{R})$ does not have limitations like "bounded" in [BBKW89] or "ground" in [Moh89] for $\mathcal{R}$. Thus, $Sem_\omega(\mathcal{R})$ can give well-defined semantics to larger class of PRSs.

(iii) Rewrite relations of our semantics are included in those of the original semantics in [BBKW89]. However, both semantics have the same ground normal forms.

Proposition 6. *Let $\mathcal{R} = \langle R_0, \sqsupset \rangle$ be a PRS. $\rightarrow_\mathcal{R}$ is stable under substitution, i.e., $s \rightarrow_\mathcal{R} t$ implies $s\sigma \rightarrow_\mathcal{R} t\sigma$ for any substitution σ.*

Proof. Let $s \equiv C[l\theta] \rightarrow_\mathcal{R} C[r\theta] \equiv t$ for some context $C[\ ]$, substitution θ and $l \rightarrow r\ if\ \overline{\omega}(l_\Omega)\#S \in Sem_\omega(\mathcal{R})$. Then, we have $\overline{\omega}((l\theta)_\Omega)\#S$. Since $l_\Omega \preceq (l\theta)_\Omega \preceq (l\theta\sigma)_\Omega$, we can show $\overline{\omega}((l\theta)_\Omega) \preceq \overline{\omega}((l\theta\sigma)_\Omega)$ by the definition of $\overline{\omega}$. It follows $\overline{\omega}((l\theta\sigma)_\Omega)\#S$, which concludes $(C\sigma)[l\theta\sigma] \rightarrow_\mathcal{R} (C\sigma)[r\theta\sigma]$. □

Example 4. Consider the following PRS:

$$\mathcal{R} = \left\{ \begin{array}{l} \Big\downarrow \begin{array}{l} f(a) \rightarrow b \\ f(x) \rightarrow c \end{array} \\ \quad d \rightarrow h(d) \end{array} \right.$$

The conditional TRS giving semantics of above $\mathcal{R}$ is

$$Sem_\omega(\mathcal{R}) = \begin{cases} f(a) \rightarrow b\ if\ \overline{\omega}(f(a))\#\{\} \\ f(x) \rightarrow c\ if\ \overline{\omega}(f(x_\Omega))\#\{f(a)\} \\ d \rightarrow h(d)\ if\ \overline{\omega}(d)\#\{\} \end{cases}$$

We have $f(g(y)) \rightarrow_\mathcal{R} c$. Thus, $f(g(d)) \rightarrow_\mathcal{R} c$ by proposition 6.

Remark. P-rewriting[BBKW89] is not stable under substitution. Mohan[Moh89] modified P-rewriting by restricting the application of rules without highest priority to only ground term, so that P-rewriting has stability. However, this restriction is not natural. For example, consider the PRS in example 4. It is natural that $f(g(y))$ can be reduced to c, since g of $g(y)$ is not changed even if y is substituted for any term. $f(g(y))$ cannot be reduced to c by modified P-rewriting, but it can be reduced in our definition.

Definition 7 (orthogonality).

(a) We say that $\mathcal{R}$ has complete priority, if $r \sqsupset r'$ or $r' \sqsupset r$ for any root overlapping rules r and r' in R_0.
(b) $\mathcal{R}$ is orthogonal, if it has complete priority, R_0 is left-linear and R_0 possibly has root overlap.

A orthogonal PRS $\mathcal{R}$ may have root overlapping rules in R_0, but two different rules cannot apply to the same redex because of priority between the rules. From this 'non-overlapping property' and the stability under substitution indicated in the above proposition, PRS $\mathcal{R}$ works like orthogonal (i.e., left-linear and non-overlapping) TRSs. Thus, the confluent property of orthogonal PRSs can be easily proven in the same way as for orthogonal TRSs, by tracing the classical proof in [Hue80].

Theorem 8. *A orthogonal PRS is confluent.*

4 Strong Sequentiality of PRSs

The fundamental concept of strong sequentiality for orthogonal TRSs was introduced by Huet and Lévy[HL79]. In this section, we first explain the basic notions related to strong sequentiality, according to [HL79], for orthogonal PRSs. We next describe a useful decision procedure that determines the index of a given Ω-term with respect to $R_?$.

Definition 9 (arbitrary-systems). Let $\mathcal{R}$ be a PRS. The arbitrary-system of $\mathcal{R}$ is a CTRS $R_? = \{l \to q\ if\ P \mid q \in T, l \to r\ if\ P \in Sem_\omega(\mathcal{R})\}$. The reduction relation $\to_{R_?}$ is called an arbitrary reduction, which is abbreviated as $\to_?$.

Definition 10 (strongly sequential PRS). Displayed occurrence Ω in term $C[\Omega]$ is an index (denoted by $C[\Omega_I]$), if the following condition holds for any t and context $C'[\ ]$ s.t. $C'[\ ] \succeq C[\ ]$: If $C'[t]$ has an Ω-free normal form w.r.t. ?-reduction then $t \not\equiv \Omega$. $C[\Omega_{NI}]$ denotes that the occurrence Ω in $C[\Omega]$ is not an index. A orthogonal PRS $\mathcal{R}$ is strongly sequential if every Ω-term t with Ω that is in normal forms w.r.t. $\to_\mathcal{R}$ has an index.

For analyzing the decidability of strong sequentiality of orthogonal TRSs, Klop and Middeldorp[KM91] proposed a useful procedure that determines whether a given Ω occurrence in a term t is an index, by using Ω-reduction and the test symbol $\bullet$. We now develop a similar procedure that determines index with respect to $R_?$. The most difficult problem in constructing Ω-reduction for PRSs is how to handle the test symbol $\bullet$. Although $\bullet$ is introduced for preventing reductions, $\bullet$ has an opposite effect in conditions of rules. Hence, wrong application of rewriting rule occurs if we handle $\bullet$ as a mere constant symbol as in [KM91]. To overcome this difficulty, we exchange $\bullet$ and Ω in a term t when we compute normal forms w.r.t. ω-reduction for deciding applicability of rules.

Definition 11 (Ω-systems). Let $\mathcal{R}$ be a PRS. Let $\bullet$ be a new constant. $\overline{t}$ denotes the Ω-term obtained from an Ω-term t by exchanging $\bullet$ and Ω. The Ω-system of $\mathcal{R}$ is a CTRS

$$\begin{aligned} R_\Omega = \{f(\mathbf{x}) \to \Omega\ if\ f(\mathbf{x}) \uparrow f(\mathbf{l})_\Omega\ \wedge \overline{\omega}((\overline{f(\mathbf{x}) \sqcup f(\mathbf{l})_\Omega})_\Omega) \# Hlhs(f(\mathbf{l}) \to r) \\ \mid f(\mathbf{l}) \to r \in R_0\}. \end{aligned}$$

The reduction relation $\to_{R_\Omega}$ on T_Ω is called Ω-reduction and often abbreviated as $\to_\Omega$.

Example 5. For POR defined in example 1,

$$POR_\Omega = \begin{cases} or(x,y) \to \Omega\ if\ or(x,y) \uparrow or(tt,\Omega) \\ \qquad\qquad \wedge\ \overline{\omega}((\overline{or(x,y) \sqcup or(tt,\Omega)})_\Omega)\#\{\} \\ or(x,y) \to \Omega\ if\ or(x,y) \uparrow or(\Omega,tt) \\ \qquad\qquad \wedge\ \overline{\omega}((\overline{or(x,y) \sqcup or(\Omega,tt)})_\Omega)\#\{or(tt,\Omega)\} \\ or(x,y) \to \Omega\ if\ or(x,y) \uparrow or(\Omega,\Omega) \\ \qquad\qquad \wedge\ \overline{\omega}((\overline{or(x,y) \sqcup or(\Omega,\Omega)})_\Omega)\#\{or(tt,\Omega), or(\Omega,tt)\} \end{cases}$$

Proposition 12. *Let s, t and u be Ω-terms.*

(a) If u is $\bullet$-free, $s \preceq t$ and $\omega((\overline{t})_\Omega)\#u$, then $\omega((\overline{s})_\Omega)\#u$.
(b) If s is $\bullet$-free and $s \preceq t$, then $s \preceq \overline{t}$.

We can show the following lemma similarly to the proof of lemma 3.

Lemma 13. *Ω-reduction is confluent and terminating.*

Definition 14. The approximant $\Omega(t)$ of an Ω-term t is the normal form of t w.r.t. Ω-reduction.

Lemma 15. *Let s and t be Ω-terms.*

(a) $s \preceq t$ and $t \to_\Omega \Omega$ imply $s \equiv \Omega$ or $s \to_\Omega \Omega$.
(b) $s \preceq t$ and $t \xrightarrow{*}_\Omega u$ imply $s \xrightarrow{*}_\Omega v$ for some $v \preceq u$.
(c) $s \preceq t$ implies $\Omega(s) \preceq \Omega(t)$.

Lemma 16. *Let q be a $\bullet$-free Ω-term. If $p \xrightarrow{*}_\Omega q$, then $\overline{p'} \xrightarrow{*}_? \overline{q}$ for some Ω-free term $p' \succeq p$.*

Proof. Prove by induction on k in $p \xrightarrow{k}_\Omega q$. In case of $k = 0$, the lemma holds by taking $p' \equiv p$. Next, we consider $p \to_\Omega q_1 \xrightarrow{*}_\Omega q$. By induction hypothesis, there exists an Ω-free term $q_2 \succeq q_1$ such that $\overline{q_2} \xrightarrow{*}_? \overline{q}$. Since $p \equiv C[f(\mathbf{v})] \to_\Omega C[\Omega] \equiv q_1$ for some $C[\ ]$, f and $\mathbf{v}$, there exist a context $C'[\ ] \succeq C[\ ]$ and an Ω-free term q_3 such that $q_2 \equiv C'[q_3]$.

On the other hand, since $f(\mathbf{v}) \to_\Omega \Omega$, we have $f(\mathbf{v}) \uparrow f(\mathbf{t})_\Omega$ and $\overline{\omega}((\overline{f(\mathbf{v}) \sqcup f(\mathbf{t})_\Omega})_\Omega)\#P$ for some rule in R_Ω:

$$f(\mathbf{x}) \to \Omega \ if \ f(\mathbf{x}) \uparrow f(\mathbf{t})_\Omega \ \wedge \overline{\omega}((\overline{f(\mathbf{x}) \sqcup f(\mathbf{t})_\Omega})_\Omega)\#P.$$

Letting $f(\mathbf{r}) \equiv f(\mathbf{v}) \sqcup f(\mathbf{t})_\Omega$, it follows from $f(\mathbf{r}) \succeq f(\mathbf{t})_\Omega$ and proposition 12(b) that $\overline{f(\mathbf{r})} \succeq f(\mathbf{t})_\Omega$. Hence, we have $\overline{f(\mathbf{r})} \to_? \overline{q_3}$ by applying a rule $f(\mathbf{t}) \to \overline{q_3}\ if\ \overline{\omega}(f(\mathbf{t})_\Omega)\#P$ in $R_?$, because of $\overline{\omega}((\overline{f(\mathbf{r})})_\Omega)\#P$. Therefore, we can take $p' \equiv C'[f(\mathbf{r})]$, which leads $\overline{p'} \equiv \overline{C'}[\overline{f(\mathbf{r})}] \to_? \overline{C'}[\overline{q_3}] \equiv \overline{q_2} \xrightarrow{*}_? \overline{q}$. □

Lemma 17.

(a) $\overline{\Delta} \to_\Omega \Omega$.
(b) If Δ is $\bullet$-free, $\Delta \to_\Omega \Omega$
(c) $p \to_? q$ implies $\exists q' \preceq \overline{q}$, $\overline{p} \to_\Omega q'$.
(d) $p \xrightarrow{*}_? q$ implies $\exists q' \preceq \overline{q}$, $\overline{p} \xrightarrow{*}_\Omega q'$.

Proof. (a) We can assume $\Delta \equiv f(\mathbf{r}) \succeq f(\mathbf{t})_\Omega$ and $\overline{\omega}(f(\mathbf{r})_\Omega)\#P$ for some rule $f(\mathbf{t}) \to q_1\ if\ \overline{\omega}(f(\mathbf{t})_\Omega)\#P$ in $Sem_\omega(\mathcal{R})$. Since $f(\mathbf{r}) \succeq f(\mathbf{t})_\Omega$ and $f(\mathbf{t})_\Omega$ is $\bullet$-free, we have $\overline{f(\mathbf{r})} \succeq f(\mathbf{t})_\Omega$ by proposition 12(b). Hence, $\overline{f(\mathbf{r})} \equiv \overline{f(\mathbf{r})} \sqcup f(\mathbf{t})_\Omega$. Since $\overline{\omega}(f(\mathbf{r})_\Omega)\#P$, we have $\overline{\omega}(f(\mathbf{r})_\Omega) \equiv \overline{\omega}((\overline{\overline{f(\mathbf{r})}})_\Omega) \equiv \overline{\omega}((\overline{\overline{f(\mathbf{r})} \sqcup f(\mathbf{t})_\Omega})_\Omega)\#P$. Therefore, $\overline{f(\mathbf{r})} \to_\Omega \Omega$ by a rule in R_Ω:

$$f(\mathbf{x}) \to \Omega \ if \ f(\mathbf{x}) \uparrow f(\mathbf{t})_\Omega \ \wedge \overline{\omega}((\overline{f(\mathbf{x}) \sqcup f(t)_\Omega})_\Omega)\#P.$$

(b) We can show it similarly to (a). Since $f(\mathbf{r})$ is $\bullet$-free and $f(\mathbf{r}) \equiv f(\mathbf{r}) \sqcup f(\mathbf{t})_\Omega$, we have $f(\mathbf{r}) \preceq \overline{f(\mathbf{r}) \sqcup f(\mathbf{t})_\Omega} \equiv (\overline{f(\mathbf{r}) \sqcup f(\mathbf{t})_\Omega})_\Omega$ by proposition 12(b). Hence, we have $\overline{\omega}(f(\mathbf{r}))\#P$ imply $\overline{\omega}((\overline{f(\mathbf{r}) \sqcup f(\mathbf{t})_\Omega})_\Omega)\#P$ by the definitions of $\overline{\omega}$ and $\#$.

(c) We can assume $p \equiv C[\Delta] \to_? C[q_1] \equiv q$ for some $C[\ ]$ and q_1. From (a), we have $\overline{p} \equiv \overline{C}[\overline{\Delta}] \to_\Omega \overline{C}[\Omega] \preceq \overline{C}[\overline{q_1}] \equiv \overline{q}$ that concludes the lemma by taking $q' \equiv \overline{C}[\Omega]$.

(d) We prove by induction on k in $p \xrightarrow{k}_? q$. In case of $k = 0$, the lemma holds by taking $q' \equiv \overline{p}$. Next, we consider $p \to_? q_1 \xrightarrow{*}_? q$. We have $\exists q_2 \preceq \overline{q}$, $\overline{q_1} \xrightarrow{*}_\Omega q_2$. We also have $\exists q_1', q_1' \preceq \overline{q_1}$, $\overline{p} \to_\Omega q_1'$ by (c). Hence, there exists q' such that $q' \preceq q_2$, $q_1' \xrightarrow{*}_\Omega q'$ by lemma 15(b). Therefore, we can get $q' \preceq q_2 \preceq \overline{q}$. $\square$

Lemma 18. *Let $C[\ ]$ be $\bullet$-free context. The followings are equivalent.*

(a) $C[\Omega_I]$.
(b) $\Omega(C[\bullet]) \not\equiv \Omega(C[\Omega])$.
(c) $\bullet \in \Omega(C[\bullet])$.

Proof. **(a)⇒(b)** Assuming $\Omega(C[\bullet]) \equiv \Omega(C[\Omega])$, we have $C[\bullet] \xrightarrow{*}_\Omega \Omega(C[\Omega])$. Letting $t \equiv \Omega(C[\Omega])$, t is $\bullet$-free Ω-term in normal forms w.r.t. Ω-reduction. By lemma 16, there is an Ω-free context $C'[\ ]$ such that $C'[\ ] \succeq C[\ ]$ and $\overline{C'[\bullet]} \equiv \overline{C'}[\Omega] \xrightarrow{*}_? \overline{t}$. If $\overline{t}$ is a normal form w.r.t. ?-reduction, we can conclude that $C[\Omega_I]$ does not hold, since $\overline{C'}[\ ] \succeq C[\ ]$ by proposition 12 and $\overline{t}$ is Ω-free. Next we must show $\overline{t}$ is a normal form w.r.t. ?-reduction. If we assume $\overline{t} \to_? u$ for some u, there exists $u' \preceq \overline{u}$ such that $\overline{\overline{t}} \equiv t \to_\Omega u'$ by lemma 17(c), which contradicts the fact that t is the normal form w.r.t. Ω-reduction.

(b)⇒(c) If we assume $\bullet \notin \Omega(C[\bullet])$, we have $\Omega(C[\bullet]) \preceq C[\Omega]$. Since $\Omega(\Omega(t)) \equiv \Omega(t)$ for any t, we have $\Omega(C[\bullet]) \preceq \Omega(C[\Omega])$. On the other hand, it follows from $C[\Omega] \preceq C[\bullet]$ by lemma 15(c) that $\Omega(C[\Omega]) \preceq \Omega(C[\bullet])$. Therefore, we have $\Omega(C[\bullet]) \equiv \Omega(C[\Omega])$.

(c)⇒(a) If we assume that $C[\Omega_I]$ does not hold, we have $C'[\Omega] \xrightarrow{*}_? n$ for some $C'[\ ] \succeq C[\ ]$ and Ω-free term n in normal forms w.r.t ?-reduction. Then we have $\overline{C'[\Omega]} \xrightarrow{*}_\Omega n'$ for some $n' \preceq \overline{n}$ by Lemma 17(d). Moreover, $\overline{C'}[\ ] \succeq C[\ ]$ by proposition 12(b). Hence, it follows by lemma 15(b) that $C[\bullet] \xrightarrow{*}_\Omega n'' \xrightarrow{*}_\Omega \Omega(n'')$ for some $n'' \preceq n'$. Since $\Omega(n'')$ is $\bullet$-free from the fact that $\overline{n}$ is $\bullet$-free and $\overline{n} \succeq n' \succeq n'' \succeq \Omega(n'')$, we get that $\Omega(\bullet)$ is also $\bullet$-free, contradicting to $\bullet \in \Omega(C[\bullet])$. $\square$

Example 6. Consider POR defined in example 1.

- $or(\Omega_I, \Omega_{NI})$ because $or(\bullet, \Omega)$ is normal form w.r.t. Ω-reduction and $or(\Omega, \bullet) \xrightarrow{*}_\Omega \Omega$.
- $or(ff, \Omega_I)$ because $or(ff, \bullet)$ is normal form w.r.t. Ω-reduction.

From lemma 18, we can write $C[\Delta_I]$ for some redex Δ instead of $C[\Omega_I]$. $t \xrightarrow{\Delta}_{\mathcal{R}} s$ is an index reduction if Δ is an index of t. We write $t \to_I s$ if there exists an index reduction $t \xrightarrow{\Delta}_{\mathcal{R}} s$.

5 The normalizability of index reduction

We will now show the normalizing property of index reduction for orthogonal PRSs by using the balanced weak Church-Rosser property proposed in [Toy92]. We first explain the basic notions and properties related to the balanced weakly Church-Rosser property according to [Toy92].

$\rightarrow$ is balanced weakly Church-Rosser(BWCR), if for any term t,s and s', $t \rightarrow s \ \wedge \ t \rightarrow s'$ imply $\exists w, \exists k \geq 0, s \xrightarrow{k} w \ \wedge \ s' \xrightarrow{k} w$.

We write $s \longleftrightarrow\!\!\!\rightarrow t$ if there exists a connection $s \xrightarrow{m_1} \cdot \xleftarrow{n_1} \cdot \xrightarrow{m_2} \cdot \xleftarrow{n_2} \cdots \xrightarrow{m_p} \cdot \xleftarrow{n_p} t$ such that $\Sigma m_i > \Sigma n_i$. We also write $t \twoheadleftarrow\!\!\!\longleftrightarrow s$ instead of $s \longleftrightarrow\!\!\!\rightarrow t$

Lemma 19 [Toy92]. *Let $\rightarrow_s$ be a reduction strategy for $\rightarrow$ such that*

(a) $\rightarrow_s$ is balanced weakly Church-Rosser,
(b) If $s \rightarrow t$ then $s =_s t$ or $s \longleftrightarrow\!\!\!\rightarrow_s \cdot \leftrightarrow \cdot \twoheadleftarrow\!\!\!\longleftrightarrow_s t$.

Then $\rightarrow_s$ is a normalizing strategy.

Now let's show that $\rightarrow_I$ on orthogonal PRS is a normalizing strategy.

Lemma 20. *Let $C[\Delta_I, \Delta']$. Then $C[\Delta_I, t]$ for any t.*

Lemma 21. *Let $\mathcal{R}$ be orthogonal. Let $\Delta \equiv C[\Delta']$ and $\Delta \not\equiv \Delta'$. Then, $C[t]$ is a redex for any t.*

Proof. Since Δ is a redex, we have $l_\Omega \preceq \Delta$ and $\overline{\omega}(\Delta_\Omega)\#P$ for some rule $l \rightarrow r\ if\ \overline{\omega}(l_\Omega)\#P$ in $Sem_\omega(\mathcal{R})$. Since $\omega(\Delta') \equiv \Omega$ and Δ and Δ' are not overlapping, we can show $l_\Omega \preceq C[t]$ and $\overline{\omega}(C[\Delta']_\Omega) \preceq \overline{\omega}(C[t]_\Omega)$ from left-linearity. Hence $C[t]$ is a redex. □

The parallel reduction $t \;+\!\!\!\!\!\!\rightarrowtail\; s$ is defined with $t \equiv C[\Delta_1, \ldots, \Delta_n] \xrightarrow{\Delta_1} \cdots \xrightarrow{\Delta_n} s \equiv C[s_1, \ldots, s_n]$ $(n \geq 0)$.

Lemma 22. *Let $\mathcal{R}$ be orthogonal and strong sequential. If $t \;+\!\!\!\!\!\!\rightarrowtail\; s$ then $t =_I s$ or $t \longleftrightarrow\!\!\!\rightarrow_I \cdot \;+\!\!\!\!\!\!\rightarrowtail\; \cdot \twoheadleftarrow\!\!\!\longleftrightarrow_I s$.*

Proof. Let $t \overset{\Delta_1 \cdots \Delta_n}{+\!\!\!\!\!\!\rightarrowtail} s$ $(n \geq 0)$. Prove by induction on n. If $n = 0$, then we have $t =_I s$ trivially. Next we consider $n > 0$:

- In case that one of Δ_i, say Δ_1 is an index, we have $t \xrightarrow{\Delta_1}_I t' \overset{\Delta_2 \cdots \Delta_n}{+\!\!\!\!\!\!\rightarrowtail} s$. By induction hypothesis, we can show the lemma.
- Consider the case that none of Δ_i is an index. From strong sequentiality, there exists an index Δ in t. From orthogonality, Δ is not overlapping to any Δ_i. If Δ is disjoint from any Δ_i, we obtain the lemma by lemma 20. In case that Δ_i appear in Δ for some i, we can assume $\Delta \equiv C[\Delta_1, \Delta_2] \overset{\Delta_1, \Delta_2}{+\!\!\!\!\!\!\rightarrowtail} C[t_1, t_2]$ without loss of generality. From lemma 21, $\Delta' \equiv C[t_1, t_2]$ is a redex. We can show $t \xrightarrow{\Delta}_I \cdot \;+\!\!\!\!\!\!\rightarrowtail\; \cdot \xleftarrow{\Delta'}_I s$ by using lemma 20.

$\square$

Lemma 23. *Let $\mathcal{R}$ be a orthogonal PRS. Let $t \xrightarrow{\Delta}_I s$ and $t \xrightarrow{\Delta'}_I s'$. Then, we have $s \rightarrow_I u$ and $s' \rightarrow_I u$ for some term u.*

Proof. From orthogonality, $\Delta \not\equiv \Delta'$. Hence, the redex Δ and Δ' are disjoint from definition of index. Thus, the lemma holds trivially by lemma 20. $\square$

Theorem 24. *Let $\mathcal{R}$ be orthogonal and strongly sequential. Then index reduction $\rightarrow_I$ is a normalizing strategy.*

Proof. Since $\rightarrow_I$ is a reduction strategy of $\rightarrow$, it is also a reduction strategy of $\dashv\!\!\!\vdash\!\!\!\rightarrow$. Taking $\rightarrow_I$ and $\dashv\!\!\!\vdash\!\!\!\rightarrow$ as $\rightarrow_s$ and $\rightarrow$, the premises of lemma 19 are satisfied by lemma 23 and lemma 22. Hence, we have $\rightarrow_I$ is a normalizing strategy for $\dashv\!\!\!\vdash\!\!\!\rightarrow$. Since $\xrightarrow{*}=\overset{*}{\dashv\!\!\!\vdash\!\!\!\rightarrow}$, the theorem follows. $\square$

6 Decidability

In this section, we show strong sequentiality of orthogonal PRS is decidable according to Comon's method[Com95]. This method depends on two results relating to weak second-order monadic logic(WSkS)[Rab77, Tho90]. One is that WSkS is decidable. The other is that a set of finite trees is definable in WSkS if and only if it is recognized by a finite tree automaton. By using these results, Comon showed decidability of several kind of sequentialities.

Proposition 25 [Com95]. *If a predicate P is definable in a tree automaton, the sequentiality of P is decidable.*

According to proposition 25, if we have a finite tree automaton that accepts terms having Ω-free normal form w.r.t. ?-reduction, we can prove the decidability of sequentiality of PRS.

Definition 26. A finite tree automaton is $A = \langle \mathcal{F}, Q, Q_f, \mathcal{T} \rangle$, where $\mathcal{F}$ is a finite set of symbols with arity, Q is a finite set of states, Q_f is a subset of Q called final states, and T is a set of transition rules.

A transition rule is the one of the forms:

- $f(q_1, \ldots, q_n) \rightarrow q$ with $f \in \mathcal{F}$, $q_1, \ldots, q_n, q \in Q$,
- $q \rightarrow q'$ with $q, q' \in Q$.

Reduction relation $\rightarrow_A$ is naturally defined on $T(\mathcal{F} \cup Q) \times T(\mathcal{F} \cup Q)$. A tree automaton accepts $t \in T(\mathcal{F})$, if and only if there is a reduction $t \xrightarrow{+}_A q \in Q_f$.

We can assume that every variable contained in object terms is only x, since rewrite rules are linear. Thus, in the following we fix $\mathcal{F} = F \cup \{x, \Omega\}$.

First, we define a finite set MR of terms that are used as states maintaining information for matching and compatibility check.

$$
\begin{aligned}
MR_0 &= \{t \mid t \subset l_\Omega, l \to r \in R_0\} \\
MR_{n+1} &= MR_n \cup \{t \mid t \preceq (s \sqcup s'),\ s, s' \in MR_n \ s.t.\ s \uparrow s'\},
\end{aligned}
$$

Let $MR_\Omega = \bigcup_{i \geq 0} MR_i$. Let $\bot$ be a fresh constant. Then, MR is the set of terms each of which is obtained from $t \in MR_\Omega$ by replacing arbitrary number of Ω occurrences in t with $\bot$. $MR_\bot$ is the set of Ω-free terms in MR. $t \preceq_\bot s$ displays that s can be obtained from t by replacing occurrences of $\bot$ in t with (possibly different) terms.

Lemma 27. *Let s be maximal in MR w.r.t. $\preceq_\bot$ such that $s \preceq_\bot t$. Then s is unique.*

First we construct a tree automaton A^ω that characterize ω-reduction.

Definition 28. Tree automaton A^ω is $\langle \mathcal{F}, Q^\omega, Q^\omega, \mathcal{T}^\omega \rangle$, where $Q^\omega = \{\langle t \rangle \mid t \in MR\}$, and $\mathcal{T}^\omega$ is the following transition rules:

(a) $\Omega \to \langle \Omega \rangle$.
(b) $x \to \langle \Omega \rangle$.
(c) $f(\langle t_1 \rangle, \ldots, \langle t_n \rangle) \to \langle \Omega \rangle$, if $f(t_1, \ldots, t_n) \uparrow l_\Omega$ for some $l \to r \in R_0$.
(d) $f(\langle t_1 \rangle, \ldots, \langle t_n \rangle) \to \langle t \rangle$, if $f(t_1, \ldots, t_n) \# l_\Omega$ for any $l \to r \in R_0$, and t is maximal in MR w.r.t. $\preceq_\bot$ such that $t \preceq_\bot f(t_1, \ldots, t_n)$.

Note that A^ω is deterministic by lemma 27. To show properties on A^ω, we need several definitions and lemmas.

Lemma 29. *Let s be maximal in $MR_\bot$ w.r.t. $\preceq_\bot$ such that $s \preceq_\bot t$. Then $\forall u \in MR_\Omega[u \preceq s \iff u \preceq t]$.*

Lemma 30. *Let s be maximal in MR w.r.t. $\preceq_\bot$ such that $s \preceq_\bot f(p_1, \cdots, p_n)$, and let every p_i be maximal in MR w.r.t. $\preceq_\bot$ such that $p_i \preceq_\bot t_i$. Then s is maximal in MR w.r.t. $\preceq_\bot$ such that $s \preceq_\bot f(t_1, \cdots, t_n)$.*

Lemma 31. *Let $t \in T(\mathcal{F})$ and $t \xrightarrow{+}_{A^\omega} \langle s \rangle$. Then, s is maximal in MR w.r.t. $\preceq_\bot$ such that $s \preceq_\bot \omega(t_\Omega)$ and $\forall u \in MR_\Omega[u \uparrow s \iff u \uparrow \omega(t_\Omega)]$.*

We next define a tree automaton A^{NF} that accepts Ω-free normal forms w.r.t. ?-reduction. Since each rule in $R_?$ has a condition, the states in A^{NF} must keep extra information associated to ω-reduction. Hence, we design each state as a triple $\langle s, r, q \rangle$ associated to a term t, where s and r keep the structures of u and $\omega(u_\Omega)$, respectively. q shows whether t is an Ω-free normal form or not.

Definition 32. Tree automaton A^{NF} is $\langle \mathcal{F}, Q^{NF}, Q^{NF}_f, \mathcal{T}^{NF} \rangle$, where $Q^{NF} = \{\langle s, r, q \rangle \mid s \in MR_\bot, r \in MR, q \in \{tt, ff\}\}$, $Q^{NF}_f = \{\langle s, r, tt \rangle \in Q^{NF}\}$, and $\mathcal{T}^{NF}$ is the following transition rules;

(a) $\Omega \to \langle \bot, \Omega, ff \rangle$.
(b) $x \to \langle \bot, \Omega, tt \rangle$.
(c) $f(\langle s_1, r_1, q_1 \rangle, \ldots, \langle s_n, r_n, q_n \rangle) \to \langle s, r, q \rangle$, if all of the following conditions hold:
   i. s is maximal in $MR_\bot$ w.r.t. $\preceq_\bot$ such that $s \preceq_\bot f(s_1, \ldots, s_n)$,
   ii. $f(\langle r_1 \rangle, \ldots, \langle r_n \rangle) \to \langle r \rangle \in \mathcal{T}^\omega$;
   iii. $q \equiv tt$, if $q_i \equiv tt$ for any i and $\neg(l_\Omega \preceq f(s_1, \ldots, s_n) \wedge f(r_1, \ldots, r_n) \# P)$ for any rule $l \to r\ if\ \overline{\omega}(l_\Omega) \# P$ in $Sem_\omega(\mathcal{R})$. Otherwise, $q \equiv ff$.

Note that A^{NF} is deterministic by lemma 27.

Lemma 33. *Let $t \in T(\mathcal{F})$ and $t \xrightarrow{+}_{A^{NF}} \langle s, r, q \rangle$. Then,*

(i) s is maximal in $MR_\bot$ w.r.t. $\preceq_\bot$ such that $s \preceq_\bot t$,
(ii) $t \xrightarrow{+}_{A^\omega} \langle r \rangle$,
(iii) t is a Ω-free normal form, iff $q \equiv tt$.

Proof. From the construction of A^{NF}, (ii) is trivial. Prove by induction on $|t|$.

Consider the case $|t| = 1$. If t is Ω then $t \xrightarrow{+}_{A^{NF}} \langle \bot, \Omega, ff \rangle$ by (a). If t is x then $t \xrightarrow{+}_{A^{NF}} \langle \bot, \Omega, tt \rangle$ by (b). If $t \equiv c \not\equiv \Omega$ then $t \xrightarrow{+}_{A^{NF}} \langle s, r, q \rangle$ by (c), which satisfies the lemma.

Consider the case $|t| > 1$. Let $t \equiv f(\mathbf{t})$. We have $f(\mathbf{t}) \xrightarrow{+}_{A^{NF}} f(\langle s_1, r_1, q_1 \rangle, \ldots, \langle s_n, r_n, q_n \rangle) \to_{A^{NF}} \langle s, r, q \rangle$, where s, r and q satisfy the conditions in rule (c). Then, (i) follows from induction hypothesis by lemma 30. Next, consider (iii).

Case 1. t_i is in Ω-free normal forms for any i. From induction hypothesis, we have $q_i \equiv tt$ for any i.

- If t is not in Ω-free normal forms, there exists some rule $l \to r\ if\ \overline{\omega}(l_\Omega) \# P$ such that $l_\Omega \preceq t$ and $\overline{\omega}(t_\Omega) \# P$. From induction hypothesis and lemma 29, we can show $l_\Omega \preceq f(\mathbf{s})$. Moreover, it follows from lemma 31 that $f(\mathbf{r}) \# P$. Hence, we have $q \equiv tt$ from the definition of (c).
- If t is in Ω-free normal forms. We can show that $\neg(l_\Omega \preceq f(\mathbf{s}) \wedge f(\mathbf{r}) \# P)$ for any rule $l \to r\ if\ \overline{\omega}(l_\Omega) \# P$. Hence, $q \equiv ff$.

Case 2. t_i is not in Ω-free normal forms for some i. From induction hypothesis, we have $q_i \equiv ff$. Hence, $q \equiv ff$ from the definition of (c). □

Corollary 34. *Let $f(t_1, \cdots, t_n) \in T(\mathcal{F})$ and $f(t_1, \cdots, t_n) \xrightarrow{+}_{A^{NF}} f(\langle s_1, r_1, q_1 \rangle, \cdots, \langle s_n, r_n, q_n \rangle)$. Then for any rule $l \to r\ if\ \overline{\omega}(l_\Omega) \# P$, $l_\Omega \preceq f(t_1, \cdots, t_n)$ and $\overline{\omega}(f(t_1, \cdots, t_n)_\Omega) \# P$ iff $l_\Omega \preceq f(s_1, \cdots, s_n)$ and $f(r_1, \cdots, r_n) \# P$.*

Proof. Straightforward from lemmas 29, 31 and 33. □

Now, we construct a tree automaton $A^?$ that recognizes terms having Ω-free normal form w.r.t. ?-reduction. We start some preliminary properties of $R_?$.

A position (denoted by $\alpha, \beta, \cdots$) in a term can be viewed as a finite sequence of natural numbers, pointing out a path from the root of this tree [Hue80]. For two positions α and β we write $\alpha \leq \beta$ if there exists some position γ such that $\alpha \cdot \gamma = \beta$. $\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$. We write $t \xrightarrow{\alpha}_R s$ if $t \xrightarrow{\Delta}_R s$ and Δ is the redex at position α in t.

Lemma 35. *Let $t \xrightarrow{\alpha}_? \cdot \xrightarrow{\beta}_? s$ and $\alpha \leq \beta$. Then $t \xrightarrow{\alpha}_? s$.*

We define a tree automaton $A^?$ that recognizes terms having Ω-free normal form w.r.t. ?-reduction as follows.

Definition 36. Tree automaton $A^?$ is $\langle \mathcal{F}, Q^?, Q^?_f, \mathcal{T}^? \rangle$, where $Q^? = Q^{NF}$, $Q^?_f = Q^{NF}_f$. $T^?$ is a set both of transition rules in T^{NF} and the following rules:

(d) $f(\langle s_1, r_1, q_1 \rangle, \ldots, \langle s_n, r_n, q_n \rangle) \rightarrow \langle s, r, q \rangle$, if all of the following conditions hold:
   i. $l_\Omega \preceq f(s_1, \ldots, s_n)$ and $f(r_1, \ldots, r_n) \# P$ for some rule $l \rightarrow r \ if \ \overline{\omega}(l_\Omega) \# P$ in $Sem_\omega(\mathcal{R})$.
   ii. $u \xrightarrow{+}_{A^{NF}} \langle s, r, q \rangle$ for some term u.

Remark that $A^?$ is obviously non-deterministic because of the condition ii. in (d), and non-deterministic selection between (c) and (d).

Lemma 37. *Let $t \in T(\mathcal{F})$ and $t \xrightarrow{+}_{A^?} \langle s, r, tt \rangle$. Then there exists some Ω-free normal form t' such that $t \xrightarrow{*}_? t'$.*

Proof. By induction on the number of transitions by rule (d) occurring in $t \xrightarrow{+}_{A^?} \langle s, r, tt \rangle$, we prove the claim.

Base Step: Since no transition by rule (d) occurs in $t \xrightarrow{+}_{A^?} \langle s, r, tt \rangle$, we have $t \xrightarrow{+}_{A^{NF}} \langle s, r, tt \rangle$. Thus, t is Ω-free normal form by lemma 33. Take $t' \equiv t$.

Induction Step: Without loss of generality, we can write $t \equiv C[f(\mathbf{t})] \xrightarrow{+}_{A^?} C[f(\langle s_1, r_1, q_1 \rangle, \ldots, \langle s_n, r_n, q_n \rangle)] \rightarrow_{A^?} C[\langle s', r', q' \rangle] \xrightarrow{*}_{A^?} \langle s, r, tt \rangle$, where the first transition by rule (d) is $C[f(\langle s_1, r_1, q_1 \rangle, \ldots, \langle s_n, r_n, q_n \rangle)] \rightarrow_{A^?} C[\langle s', r', q' \rangle]$. Then, we have $l_\Omega \preceq f(\mathbf{s})$ and $f(\mathbf{r}) \# P$ for some $l \rightarrow r \ if \ \overline{\omega}(l_\Omega) \# P$, and $u \xrightarrow{+}_{A^{NF}} \langle s', r', q' \rangle$ for some u. Hence, the number of transitons by rule (d) in $C[u] \xrightarrow{+}_{A^?} C[\langle s', r', q' \rangle] \xrightarrow{*}_{A^?} \langle s, r, tt \rangle$ is less than that in $t \xrightarrow{+}_{A^?} \langle s, r, tt \rangle$. From induction hypothesis, it follows that $C[u] \xrightarrow{*}_? t'$ for some Ω-free normal form t'. Since $f(\mathbf{t}) \xrightarrow{+}_{A^{NF}} f(\langle s_1, r_1, q_1 \rangle, \ldots, \langle s_n, r_n, q_n \rangle)$, $l_\Omega \preceq f(\mathbf{s})$, and $f(\mathbf{r}) \# P$, we have $l_\Omega \preceq f(\mathbf{t})$ and $\overline{\omega}(f(\mathbf{t})_\Omega) \# P$ by corollary 34. Thus, $C[f(\mathbf{t})] \rightarrow_? C[u]$. Therefore, $t \equiv C[f(\mathbf{t})] \rightarrow_? C[u] \xrightarrow{*}_? t'$. ☐

Lemma 38. *Let $t, t' \in T(\mathcal{F})$ and $t \xrightarrow{*}_? t'$. Then there exist some s, r, q such that $t \xrightarrow{+}_{A^?} \langle s, r, q \rangle$ and $t' \xrightarrow{+}_{A^{NF}} \langle s, r, q \rangle$.*

Proof. By lemma 35, we can suppose that $t \xrightarrow{*}_? t'$ is an innermost reduction. By induction on $|t|$ we prove the claim.

Base Step($|t| = 1$): If $t \equiv t'$, then the claim holds as $t \equiv t' \xrightarrow{+}_{A^{NF}} \langle s, r, q\rangle$. Let $t \equiv c \not\equiv t'$. Then there exists some rule $c \to r\ if\ c\#P$. Hence, by applying transition rule (d) to c we have $c \to_{A^?} \langle s, r, q\rangle$ where $t' \xrightarrow{+}_{A^{NF}} \langle s, r, q\rangle$.

Induction Step:

Case 1. Let $t \equiv f(t_1, \cdots, t_n) \xrightarrow{*}_? t' \equiv f(t'_1, \cdots, t'_n)$ where $t_i \xrightarrow{*}_? t'_i$ for all i. Then, from induction hypothesis we have $t_i \xrightarrow{+}_{A^?} \langle s_i, r_i, q_i\rangle$ and $t'_i \xrightarrow{+}_{A^{NF}} \langle s_i, r_i, q_i\rangle$ for some s_i, r_i, q_i. Let $f(\langle s_1, r_1, q_1\rangle, \ldots, \langle s_n, r_n, q_n\rangle) \to_{A^{NF}} \langle s, r, q\rangle$. Then, we have $t \equiv f(t_1, \cdots, t_n) \xrightarrow{+}_{A^?} f(\langle s_1, r_1, q_1\rangle, \ldots, \langle s_n, r_n, q_n\rangle) \to_{A^{NF}} \langle s, r, q\rangle$ and $t' \equiv f(t'_1, \cdots, t'_n) \xrightarrow{+}_{A^{NF}} \langle s, r, q\rangle$.

Case 2. Let $t \equiv f(t_1, \cdots, t_n) \xrightarrow{*}_? f(u_1, \cdots, u_n) \xrightarrow{\epsilon}_? t'$ (ϵ denotes the root position) where $t_i \xrightarrow{*}_? u_i$ for all i. Then, from induction hypothesis we have $t_i \xrightarrow{+}_{A^?} \langle s_i, r_i, q_i\rangle$ and $u_i \xrightarrow{+}_{A^{NF}} \langle s_i, r_i, q_i\rangle$ for some s_i, r_i, q_i. Since $f(u_1, \cdots, u_n) \xrightarrow{\epsilon}_? t'$, there exists some rule $l \to r\ if\ \overline{\omega}(l_\Omega)\#P$ such that $l_\Omega \preceq f(u_1, \cdots, u_n)$ and $\overline{\omega}(f(u_1, \cdots, u_n)_\Omega)\#P$. Then by corollary 34 we have $l_\Omega \preceq f(s_1, \cdots, s_n)$ and $f(r_1, \cdots, r_n)\#P$. Hence, by applying transition rule (d) it holds that $f(t_1, \cdots, t_n) \xrightarrow{+}_{A^?} f(\langle s_1, r_1, q_1\rangle, \ldots, \langle s_n, r_n, q_n\rangle) \to_{A^?} \langle s, r, q\rangle$ where $t' \xrightarrow{+}_{A^{NF}} \langle s, r, q\rangle$. □

Corollary 39. *Let $t, t' \in T(\mathcal{F})$ and $t \xrightarrow{*}_? t'$. If t' is in Ω-free normal forms, then there exist some s, r such that $t \xrightarrow{+}_{A^?} \langle s, r, tt\rangle$.*

Proof. Straightforward from lemma 38. □

Theorem 40. *The strong sequentiality of PRS is decidable.*

Proof. It follows from lemma 37, corollary 39, and proposition 25. □

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