Termination Proof of S-Expression Rewriting Systems with Recursive Path Relations

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Abstract. S-expression rewriting systems were proposed by the author (RTA 2004) for termination analysis of Lisp-like untyped higher-order functional programs. This paper presents a short and direct proof for the fact that every finite S-expression rewriting system is terminating if it is compatible with a recursive path relation with status. By considering well-founded binary relations instead of well-founded orders, we give a much simpler proof than the one depending on Kruskal's tree theorem.

1 Introduction

An important syntactical method to prove termination of a first-order term rewriting system is the one using recursive path orders [1, 4, 5, 8, 16] relying on Kruskal's tree theorem [9]. Higher-order rewriting systems are rewriting systems to accommodate higher-order functions and several syntactical methods for proving termination of them are presented [7, 10, 11, 15, 17].

In the previous paper [17] the author proposed Lisp-like untyped higher-order rewriting systems without λ-abstraction, called S-expression rewriting systems, and proved that every finite S-expression rewriting system is terminating if it is compatible with a lexicographic path order, by using the notion of the simplification order based on Kruskal's tree theorem. However, our proof in [17] is neither direct nor simple because of *unbounded-variadic* and *higher-order* feature of S-expressions, and it is not easy to extend it to more expressive orders like recursive path orders with lexicographic and multiset status.

The purpose of this paper is to present a short and direct proof for the main result in [17] and to extend it to recursive path relations with lexicographic and multiset status on S-expressions. Like modern proofs in Buchholz [2], Dawson and Goré [3], Goubault-Larrecq [6], Jouannaud and Rubio [7], our termination proof does not use Kruskal's tree theorem, and it uses well-founded binary relations instead of well-founded orders as in Goubault-Larrecq [6], Mellies [12]. Thus the recursive path relation is not required to be well-ordered nor transitive for proving well-foundedness. Our proof technique is based on Goubault-Larrecq's work [6] in which he presents a general termination proof of abstract recursive path relations, independent on term-structure. However, since Goubault-Larrecq

[6] defines the recursive path relation as the greatest fixed point of the monotonic operator by the co-induction principle, his proof of the stability does not work for the least fixed point. We modify his proof for treating the recursive path relation defined by the induction principle in common use and for adapting it to *unbounded-variadic* and *higher-order* feature of S-expressions.

The remainder of this paper is organized as follows. After a preliminary section, in Section 3 we give a general termination proof of abstract recursive path relations. Section 4 studies lexicographic path relation on S-expressions and discusses termination of S-expression rewriting systems compatible with lexicographic path relations. Section 5 extends our result to recursive path relations with lexicographic and multiset status on S-expressions.

2 Preliminaries

We mainly follow the notation of [1, 16, 17]. Let $\mathcal{C}$ be a (finite or infinite) set of *constants*, denoted by $f, g, h, \cdots$, and $\mathcal{V}$ a countably infinite set of *variables*, denoted by $x, y, z, \cdots, \alpha, \beta, \gamma, \cdots$, where $\mathcal{C} \cap \mathcal{V} = \phi$.

The set $\mathcal{S}(\mathcal{C}, \mathcal{V})$ of terms, called *symbolic expressions* (*S-expressions* for short) [17], is recursively defined by: (i) $\mathcal{C} \cup \mathcal{V} \subseteq \mathcal{S}(\mathcal{C}, \mathcal{V})$, and (ii) $(s_1\ s_2\ \cdots\ s_n) \in \mathcal{S}(\mathcal{C}, \mathcal{V})$ for $s_1, \cdots, s_n \in \mathcal{S}(\mathcal{C}, \mathcal{V})$ $(n \geq 0)$. Terms are denoted by $s, t, r, \cdots$. The set of variables in a term t is denoted by $\mathcal{V}(t)$.

A *substitution* θ is a mapping from $\mathcal{V}$ into $\mathcal{S}(\mathcal{C}, \mathcal{V})$. Substitutions are extended into homomorphisms from $\mathcal{S}(\mathcal{C}, \mathcal{V})$ into $\mathcal{S}(\mathcal{C}, \mathcal{V})$. Following common usage, we write $t\theta$ instead of $\theta(t)$.

Consider an extra constant $\Box$ called a hole. Then $C \in \mathcal{S}(\mathcal{C} \cup \{\Box\}, \mathcal{V})$ is called a context. We use the notation $C[\ ]$ for the context containing precisely one hole, and $C[t]$ denotes the result of placing a term t in the hole of $C[\ ]$. We use the notation $(\cdots t \cdots)$ for $C[t]$ if $C[\ ] = (s_1 \cdots s_{k-1}\ \Box\ s_{k+1} \cdots s_n)$. A term s is called a subterm of t if $t = C[s]$ and an immediate subterm if $t = (\cdots s \cdots)$.

A rewrite rule is a pair $\langle l, r \rangle$ of terms such that $l \notin \mathcal{V}$ and $\mathcal{V}(r) \subseteq \mathcal{V}(l)$. We write $l \to r$ for $\langle l, r \rangle$. A *S-expression rewriting system* (SRS for short) $\mathcal{R}$ is a set of rewrite rules [17]. The rewrite rules of a SRS $\mathcal{R}$ define a reduction relation $\to_{\mathcal{R}}$ on $\mathcal{S}(\mathcal{C}, \mathcal{V})$ as follows: $t \to_{\mathcal{R}} s$ if and only if there exist a rewrite rule $l \to r \in \mathcal{R}$, a context $C[\ ]$ and a substitution θ such that $t = C[l\theta]$ and $s = C[r\theta]$. A SRS $\mathcal{R}$ is *terminating* if there exists no infinite reduction sequence $t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \cdots$.

Example 1 (SRS). The higher-order function *map* is presented by the following SRS $\mathcal{R}$, where *map*, *cons*, *nil*, *plus*, *s*, 0 are constants and α, x, y variables.

$$\mathcal{R} \begin{cases} (map\ \alpha\ nil) \to nil \\ (map\ \alpha\ (cons\ x\ y)) \to (cons\ (\alpha\ x)\ (map\ \alpha\ y)) \\ ((plus\ (s\ x))\ y) \to (s\ ((plus\ x)\ y)) \\ ((plus\ 0)\ y) \to y \end{cases}$$

Then we have the reduction sequence:
$(map\ (plus\ (s\ 0))\ (cons\ (s\ 0)\ (cons\ (s\ (s\ 0))\ nil))) \xrightarrow{+}_{\mathcal{R}}$

$(cons\ (s\ (s\ 0))\ (map\ (plus\ (s\ 0))\ (cons\ (s\ (s\ 0))\ nil))) \xrightarrow{+}_{\mathcal{R}}$
$(cons\ (s\ (s\ 0))\ (cons\ (s\ (s\ (s\ 0)))\ nil))$. $\square$

Let $\succ$ be a binary relation on $\mathcal{S}(\mathcal{C},\mathcal{V})$. For short, we write $\prec$ for its inverse, $\succeq$ for its reflexive closure, $\succ^+$ for its transitive closure, $\succ^*$ for its transitive-reflexive closure, $\succ^n$ for its n-fold composition.

A term $t \in \mathcal{S}(\mathcal{C},\mathcal{V})$ is strongly normalizing with respect to $\succ$ if there is no infinite decreasing sequence $t = t_0 \succ t_1 \succ t_2 \succ \cdots$. $SN(\succ)$ is the set of all terms that are strongly normalizing with respect to $\succ$. $\succ$ is well-founded if $SN(\succ) = \mathcal{S}(\mathcal{C},\mathcal{V})$. A binary relation $\succ$ is well-founded on a term set $T \subseteq \mathcal{S}(\mathcal{C},\mathcal{V})$ if there is no infinite decreasing sequence $t_0 \succ t_1 \succ t_2 \succ \cdots$ on T.

A binary relation $\succ$ is monotonic if $s \succ t$ implies $(\cdots s \cdots) \succ (\cdots t \cdots)$ (i.e., closed under context), and it is stable if $s \succ t$ implies $s\theta \succ t\theta$ (i.e., closed under substitution).

A binary relation $\succ$ is called a rewrite relation if it is monotonic and stable. A SRS $\mathcal{R}$ is compatible with a binary relation $\succ$ if $l \succ r$ for every rule $l \to r \in \mathcal{R}$. It is easy to show that a SRS $\mathcal{R}$ is terminating if and only if it is compatible with a well-founded rewrite relation.

3 Termination Theorem

Goubault-Larrecq [6] gives a general constructive termination proof of abstract recursive path relations. We adapt his general proof to recursive path relations on S-expressions. Our termination proof is non-constructive, different from [6], but simpler.

Let $\succ$, $\succ^{\Lambda}$, $\rhd$ be three binary relations on $\mathcal{S}(\mathcal{C},\mathcal{V})$ and $\rhd$ well-founded. Assume that $\succ$ has the following property [6].

Property 1. $s \succ t$ if and only if
(P1) $\exists u.[s \rhd u \wedge u \succeq t]$, or
(P2) $\forall u.[t \rhd u \Rightarrow s \succ u] \ \wedge \ s \succ^{\Lambda} t$.

Let $\overline{SN(\succ)} = \{\ s \mid \forall u \lhd s.\ u \in SN(\succ)\}$ and a term set $S \subseteq \mathcal{S}(\mathcal{C},\mathcal{V})$ be closed with respect to $\rhd$ (i.e., if $s \in S$ and $s \rhd t$ then $t \in S$). Inspired by Ferreira [5] we say that $\succ^{\Lambda}$ is a *term lifting* on S (with respect to $\succ$ and $\rhd$) if $\succ^{\Lambda}$ is well-founded on $\overline{SN(\succ)} \cap S$. (Note that in [5] a term lifting is defined as a specific lifting, i.e., $\succ^{\Lambda}$ is well-founded on first order terms if $\succ$ is well-founded on their arguments. Our definition is essentially equivalent to that in [5] if $\rhd$ means the immediate subterm relation.) Then we have the following termination theorem.

Theorem 1 (Well-Foundedness). Let $\succ^{\Lambda}$ be a term lifting on S. Then $\succ$ is well-founded on S.

Proof. We prove the claim by contradiction. The proof is similar to a *minimal bad sequence argument* [14]. Suppose $\succ$ is not well-founded on S and consider a minimal infinite sequence $s_0 \succ s_1 \succ s_2 \succ \cdots$ on S with respect to $\rhd$ in the following sense:

(i) if $s_0 \rhd t$ then t is strongly normalizing with respect to $\succ$;

(ii) if $s_{i+1} \rhd t$ and $s_i \succ t$ then t is strongly normalizing with respect to $\succ$.

We show that every $s_k \succ s_{k+1}$ satisfies (P2). Suppose $s_0 \succ s_1$ satisfies (P1). Then $u \succeq s_1$ for some $u \lhd s_0$; contradicting minimality of the infinite sequence on S. Thus $s_0 \succ s_1$ must satisfy (P2). Suppose $s_0 \succ s_1 \succ s_2 \succ \cdots \succ s_k$ $(k > 0)$ satisfy (P2) and $s_k \succ s_{k+1}$ satisfies (P1). Then $u \succeq s_{k+1}$ for some $u \lhd s_k$. Since $s_{k-1} \succ s_k$ satisfies (P2), we have $s_{k-1} \succ u \succeq s_{k+1}$; again contradicting minimality of s_k.

We next show every s_k in $\overline{SN(\succ)} \cap S$. Suppose $s_m \notin \overline{SN(\succ)} \cap S$. Then we have some $u \lhd s_m$ such that u is not in $SN(\succ)$. Since $s_{m-1} \succ s_m$ satisfies (P2), $s_{m-1} \succ u$. Thus we have an infinite sequence $s_0 \succ s_1 \succ s_2 \succ \cdots \succ s_{m-1} \succ u \cdots$; contradicting minimality of s_m.

Since every $s_k \succ s_{k+1}$ satisfies (P2), we have an infinite sequence $s_0 \succ^{\Lambda} s_1 \succ^{\Lambda} s_2 \succ^{\Lambda} \cdots$. It contradicts the assumption that $\succ^{\Lambda}$ is well-founded on $\overline{SN(\succ)} \cap S$. □

Lemma 1 (Monotonicity). Let $\rhd$ be the immediate subterm relation defined as $(\cdots u \cdots) \rhd u$. Assume that if $s \succ t$ then $(\cdots s \cdots) \succ^{\Lambda} (\cdots t \cdots)$. Then the relation $\succ$ is monotonic, i.e., $s \succ t$ implies $(\cdots s \cdots) \succ (\cdots t \cdots)$.

Proof. Let $s \succ t$. From the assumption we have $(\cdots s \cdots) \succ^{\Lambda} (\cdots t \cdots)$. Thus, in order to obtain $(\cdots s \cdots) \succ (\cdots t \cdots)$ by (P2), we show that for every u, $(\cdots t \cdots) \rhd u$ implies $(\cdots s \cdots) \succ u$. Let $(\cdots t \cdots) \rhd u$. If $u \neq t$ then $(\cdots s \cdots) \rhd u$. Since $u \succeq u$, we have $(\cdots s \cdots) \succ u$ by (P1). If $u = t$ then $s \succ u$. Thus we have $(\cdots s \cdots) \rhd s \succ u$. By (P1), it holds that $(\cdots s \cdots) \succ u$. □

Lemma 2 (Subterm). $s \rhd^{+} t$ implies $s \succ t$.

Proof. By induction on $n \geq 1$ we prove the claim that $s \rhd^{n} t$ implies $s \succ t$. Base step: Let $s \rhd t$. Then, by (P1) and $t \succeq t$, we have $s \succ t$. Induction step: Let $s \rhd u \rhd^{n} t$. From induction hypothesis it follows that $u \succ t$. By (P1), we have $s \succ t$. (Note that the claim holds without the assumption that $\rhd$ is the immediate subterm relation.) □

We say that a term pair $\langle s, t \rangle \in \mathcal{S}(\mathcal{C}, \mathcal{V})$ is stable with respect to $\succ$ if $s \succ t$ implies $s\theta \succ t\theta$ for every substitution θ. A term pair set $P \subseteq \mathcal{S}(\mathcal{C}, \mathcal{V})^2$ is stable with respect to $\succ$ if every $\langle s, t \rangle \in P$ is stable with respect to $\succ$. $ST(\succ)$ denotes the set of all term pairs that are stable with respect to $\succ$. Thus, $\succ$ is stable (i.e., closed under substitution) if and only if $ST(\succ) = \mathcal{S}(\mathcal{C}, \mathcal{V})^2$. We define $\langle s, t \rangle \rhd_2 \langle s', t' \rangle$ as $s \trianglerighteq s'$, $t \trianglerighteq t'$ and $\langle s, t \rangle \neq \langle s', t' \rangle$; $\lhd_2$ as its inverse. It is clear that $\rhd_2$ is well-founded. Let $\overline{ST(\succ)} = \{ \langle s, t \rangle \mid \forall \langle u, v \rangle \lhd_2 \langle s, t \rangle.\ \langle u, v \rangle \in ST(\succ) \}$.

Lemma 3 (Stability). Let $\rhd$ be the immediate subterm relation defined as $(\cdots u \cdots) \rhd u$. Let $P \subseteq \mathcal{S}(\mathcal{C}, \mathcal{V})^2$ be closed with respect to $\rhd_2$. Assume that $\overline{ST(\succ)} \cap P$ is stable with respect to $\succ^{\Lambda}$ (i.e., every $\langle s, t \rangle \in \overline{ST(\succ)} \cap P$ is stable with respect to $\succ^{\Lambda}$) and that $s \succ x$ implies $s \rhd^{+} x$ for every variable x. Then P is stable with respect to $\succ$.

Proof. We prove the claim by contradiction. Suppose P is not stable with respect to $\succ$ and consider a minimal $\langle s,t\rangle \in P$ with respect to $\rhd_2$ such that $\langle s,t\rangle \notin ST(\succ)$. Then there exist some θ such that $s \succ t$ and $s\theta \not\succ t\theta$. If $t = x \in \mathcal{V}$ then by the assumption we have $s \rhd^+ x$. Since $\rhd$ is the immediate subterm relation, $s\theta \rhd^+ x\theta$. From Lemma 2 it follows that $s\theta \succ x\theta$; contradiction. Thus we suppose $t \notin \mathcal{V}$. If $s \succ t$ satisfies (P1) then $s \rhd u$ and $u \succeq t$ for some u. Since $\rhd$ is the immediate subterm relation, we have $s\theta \rhd u\theta$. From minimality of $\langle s,t\rangle$ and $\langle s,t\rangle \rhd_2 \langle u,t\rangle$, $\langle u,t\rangle$ is stable with respect to $\succ$. Thus $u\theta \succeq t\theta$. By (P1) we have $s\theta \succ t\theta$; contradiction. If $s \succ t$ satisfies (P2) then $t \rhd u$ implies $s \succ u$ for every u and $s \succ^{\Lambda} t$. Let $t\theta \rhd v$. Since t is not a variable, there exists some t' such that $t \rhd t'$ and $v = t'\theta$. From (P2), $t \rhd t'$ implies $s \succ t'$. From minimality of $\langle s,t\rangle$ and $\langle s,t\rangle \rhd_2 \langle s,t'\rangle$, $\langle s,t'\rangle$ is stable with respect to $\succ$; we have $s\theta \succ t'\theta = v$. Thus it follows that $t\theta \rhd v$ implies $s\theta \succ v$ for every v. From minimality of $\langle s,t\rangle$, $\langle s,t\rangle$ is in $\overline{ST(\succ)} \cap P$. Thus, from the assumption, $s\theta \succ^{\Lambda} t\theta$. By (P2), it follows that $s\theta \succ t\theta$; contradiction. □

Remark. Goubault-Larrecq [6] proves the stability of $\succ$ under the weaker condition that $\succ^{\Lambda}$ is stable whenever $\succ$ is stable, but $\succ$ is defined as the greatest fixed point of the monotonic operator induced from (P1) and (P2) (See Remark 4 in [6]). Thus his proof does not work for $\succ$ recursively defined as the least fixed point in common use, differently from ours.

4 Lexicographic Path Relation

In this section we discuss lexicographic path relation on S-expressions and prove termination of finite S-expression rewriting systems compatible with lexicographic path relations. Thanks to the abstract results presented in the previous section, our proof is much simpler than that in [17]. For a binary relation $\succ$ on $\mathcal{S}(\mathcal{C},\mathcal{V})$, $\succ^{lex}$ denotes the lexicographic extension of $\succ$. The lexicographic path relation on $\mathcal{S}(\mathcal{C},\mathcal{V})$ is defined as follows [17].

Definition 1 (Lexicographic Path Relation). Let $>$ be a well-founded relation on $\mathcal{C}$. The lexicographic path relation $\succ_{lp}$ on $\mathcal{S}(\mathcal{C},\mathcal{V})$ is recursively defined as follows: $s \succ_{lp} t$ if and only if
(L0) $s,t \in \mathcal{C}$ and $s > t$, or
(L1) $s = (s_1 \cdots s_m)$ $(m \geq 1)$ and $\exists s_i.\ s_i \succeq_{lp} t$, or
(L2) $t = (t_1 \cdots t_n)$ $(n \geq 0)$ and $\forall t_i.\ s \succ_{lp} t_i$, and either
(L2-1) $s \in \mathcal{C}$, or
(L2-2) $s = (s_1 \cdots s_m)$ $(m \geq 1)$ and $\langle s_1, \cdots, s_m\rangle \succ_{lp}^{lex} \langle t_1, \cdots, t_n\rangle$.

Note that the lexicographic path relation $\succ_{lp}$ is not well-founded due to *unbounded-variadic* feature of S-expressions; for example, we can have the infinite descending sequence $(1) \succ_{lp} (0\ 1) \succ_{lp} (0\ 0\ 1) \succ_{lp} (0\ 0\ 0\ 1) \succ_{lp} \cdots$ if $0 < 1$. The problem arises from the fact that lexicographic sequences of unbounded size are not well-founded. To overcome this difficulty we introduce the notion of the maximum degree of terms.

The maximum degree $d(t)$ of a term $t \in \mathcal{S}(\mathcal{C}, \mathcal{V})$ is defined by: (i) $d(t) = 0$ for $t \in \mathcal{C} \cup \mathcal{V}$, and (ii) $d(t) = max\{n, d(t_1), \cdots, d(t_n)\}$ for $t = (t_1 \cdots t_n)$ $(n \geq 0)$. The set of terms having the maximum degree not more than p is defined by $\mathcal{S}_p(\mathcal{C}, \mathcal{V}) = \{t \mid d(t) \leq p\}$.

Theorem 2. The lexicographic path relation $\succ_{lp}$ is well-founded on $\mathcal{S}_p(\mathcal{C}, \mathcal{V})$ for every $p \geq 0$, and it is monotonic and stable.

Proof. Let $\rhd$ be the immediate subterm relation. From the definition of $\succ_{lp}$ it is obvious that $s \succ_{lp} x$ implies $s \rhd^+ x$ for every variable x. We define $s \succ_{lp}^{\Lambda} t$ as (L0) $s, t, \in \mathcal{C}$ and $s > t$, or (L2-1) $s \in \mathcal{C}$ and $t = (t_1 \cdots t_n)$ $(n \geq 0)$, or (L2-2) $s = (s_1 \cdots s_m)$ $(m \geq 1)$, $t = (t_1 \cdots t_n)$ $(n \geq 0)$ and $\langle s_1, \cdots, s_m \rangle \succ_{lp}^{lex} \langle t_1, \cdots, t_n \rangle$. Then $\succ_{lp}$ obviously satisfies Property 1. Monotonicity and stability of $\succ_{lp}$ directly follow from Lemma 1 and Lemma 3 by taking $P = \mathcal{S}(\mathcal{C}, \mathcal{V})^2$ respectively. Thus, it remains to prove well-foundedness of $\succ_{lp}$ on $\mathcal{S}_p(\mathcal{C}, \mathcal{V})$. From Theorem 1, it is enough to show that $\succ_{lp}^{\Lambda}$ is actually a term lifting on $\mathcal{S}_p(\mathcal{C}, \mathcal{V})$ with respect to $\succ_{lp}$ and $\rhd$, i.e., it is well-founded on $\overline{SN(\succ_{lp})} \cap \mathcal{S}_p(\mathcal{C}, \mathcal{V})$. Suppose $\succ_{lp}^{\Lambda}$ is not well-founded on $\overline{SN(\succ_{lp})} \cap \mathcal{S}_p(\mathcal{C}, \mathcal{V})$ and consider an infinite sequence $s_0 \succ_{lp}^{\Lambda} s_1 \succ_{lp}^{\Lambda} s_2 \succ_{lp}^{\Lambda} \cdots$ on $\overline{SN(\succ_{lp})} \cap \mathcal{S}_p(\mathcal{C}, \mathcal{V})$. Since $>$ is well-founded on $\mathcal{C}$, there exists some k such that for every $i \geq k$, $s_i \succ_{lp}^{\Lambda} s_{i+1}$ satisfies (L2-2). Thus, we have an infinite sequence $\langle s_{k,1}, \cdots, s_{k,n} \rangle \succ_{lp}^{lex} \langle s_{k+1,1}, \cdots, s_{k+1,n'} \rangle \succ_{lp}^{lex} \langle s_{k+2,1}, \cdots, s_{k+2,n''} \rangle \succ_{lp}^{lex} \cdots$ where all $s_{i,j}$ are in $SN(\succ_{lp})$ and $n, n', n'', \cdots \leq p$. However, since $\succ_{lp}$ on the set of all $s_{i,j}$ is well-founded, its lexicographic extension over the sequences of length bounded by p is well-founded. Thus, we get a contradiction. $\square$

A SRS $\mathcal{R}$ is *bounded* [17] if for any infinite reduction sequence $t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} t_3 \to_{\mathcal{R}} \cdots$ there exists some natural number p such that $d(t_i) \leq p$.

Lemma 4. Every finite SRS $\mathcal{R}$ is bounded.

Proof. Let $t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} t_3 \to_{\mathcal{R}} \cdots$ be an infinite reduction sequence and let $c = max\{d(r) \mid l \to r \in \mathcal{R}\}$. Consider a reduction $s \to_{\mathcal{R}} t$ where $s = C[l\theta]$, $t = C[r\theta]$ and $l \to r \in \mathcal{R}$. Then we have $d(t) = d(C[r\theta]) =$
$max\{d(C[\ ]), d(r), d(x\theta) \mid x \in \mathcal{V}(r)\} \leq$
$max\{d(C[\ ]), d(l), d(r), d(x\theta) \mid x \in \mathcal{V}(r)\} \leq$
$max\{d(C[l\theta]), d(r)\} = max\{d(s), d(r)\} \leq$
$max\{d(s), c\}$. Thus it holds that $d(t_{i+1}) \leq max\{d(t_i), c\}$ for every $i \geq 0$. By taking $p = max\{d(t_0), c\}$ we conclude the claim. $\square$

Theorem 3. Let $\mathcal{R}$ be a finite SRS compatible with a lexicographic path relation $\succ_{lp}$. Then $\mathcal{R}$ is terminating.

Proof. The claim follows from Theorem 2 and Lemma 4. $\square$

By replacing Conditions (L2), (L2-1) and (L2-2) in Definition 1 of $\succ_{lp}$ with Conditions (L2-a), (L2-b) and (L2-c), we obtain the following practical definition in [17], which is a bit easier to apply. It is easy to show that the two definitions of $\succ_{lp}$ are equivalent.

Definition 2 (Lexicographic Path Relation). Let $>$ be a well-founded relation on $\mathcal{C}$. The lexicographic path relation $\succ_{lp}$ on $\mathcal{S}(\mathcal{C},\mathcal{V})$ is recursively defined as follows: $s \succ_{lp} t$ if and only if

(L0) $s, t, \in \mathcal{C}$ and $s > t$, or

(L1) $s = (s_1 \cdots s_m)$ $(m \geq 1)$ and $\exists s_i.\ s_i \succeq_{lp} t$, or

(L2-a) $s \in \mathcal{C}$, $t = (t_1 \cdots t_n)$ $(n \geq 0)$ and $\forall t_i.\ s \succ_{lp} t_i$, or

(L2-b) $s = (s_1 \cdots s_m)$, $t = (t_1 \cdots t_n)$ $(m > n \geq 0)$,
and $s_1 = t_1, \cdots, s_n = t_n$, or

(L2-c) $s = (s_1 \cdots s_m)$, $t = (t_1 \cdots t_n)$ $(m, n \geq 1)$
and $\exists i.\ s_1 = t_1, \cdots, s_{i-1} = t_{i-1}$, $s_i \succ_{lp} t_i$, $s \succ_{lp} t_{i+1}, \cdots, s \succ_{lp} t_n$.

Now we verify termination of S-expression rewriting systems by Theorem 3. Various higher-order rewriting systems in the literature [7, 10, 11, 15] are naturally expressed in SRSs and termination of them is easily proven as that of SRSs without using the notion of *type*. In the following examples, we denote variables acting as higher-order variables by $\alpha, \beta, \gamma, \cdots$ in distinction from $x, y, z, \cdots$ for readability.

Example 2 (map). Let $\mathcal{C} = \{map, cons, nil\}$ with $map > cons > nil$ and consider the following SRS $\mathcal{R}$.

$$\mathcal{R} \begin{cases} ((map\ \alpha)\ nil) \rightarrow nil \\ ((map\ \alpha)\ (cons\ x\ y)) \rightarrow (cons\ (\alpha\ x)\ ((map\ \alpha)\ y)) \end{cases}$$

For the first rule $((map\ \alpha)\ nil) \succ_{lp} nil$ is trivial by the subterm property. For the second rule we have (i) $(map\ \alpha) \succ_{lp} cons$ by (L1) because $map \succ_{lp} cons$ by (L0), (ii) $((map\ \alpha)\ \ (cons\ x\ y)) \succ_{lp} (\alpha\ x)$ by (L2-c) because $(map\ \alpha) \succ_{lp} \alpha$ and $((map\ \alpha)\ (cons\ x\ y)) \succ_{lp} x$ by the subterm property, and (iii) $((map\ \alpha)\ (cons\ x\ y))$ $\succ_{lp} ((map\ \alpha)\ y)$ by (L2-c) because $(cons\ x\ y) \succ_{lp} y$ by the subterm property. From (i), (ii) and (iii) we have $((map\ \alpha)\ (cons\ x\ y)) \succ_{lp} (cons\ (\alpha\ x)\ ((map\ \alpha)\ y))$ by (L2-c). Thus $\mathcal{R}$ is terminating since it is compatible with $\succ_{lp}$. □

In the above example, *map* is represented as a currying notation $((map\ s)\ t)$ by the extra parentheses as in [10, 17], instead of the usual flat notation $(map\ s\ t)$. This currying is necessary for guaranteeing the compatibility with $\succ_{lp}$ because of $(map\ \alpha\ (cons\ x\ y)) \not\succ_{lp} (cons\ (\alpha\ x)\ (map\ \alpha\ y))$ for the flat notation of the second rewrite rule. For a currying-transformation technique to strengthen the power of the lexicographic path relation, see [17].

Example 3 (maplist). Let $\mathcal{C} = \{fmap, cons, nil\}$ with $fmap > cons > nil$ and consider the following SRS $\mathcal{R}$.

$$\mathcal{R} \begin{cases} ((fmap\ nil)\ x) \rightarrow nil \\ ((fmap\ (cons\ \alpha\ \beta))\ \ x) \rightarrow (cons\ (\alpha\ x)\ ((fmap\ \beta)\ x)) \end{cases}$$

Then $\mathcal{R}$ is terminating since it is compatible with $\succ_{lp}$. □

Example 4 (twice). Let $\mathcal{C} = \{twice, dapply\}$ with $twice > dapply$ and consider the following SRS $\mathcal{R}$.

$$\mathcal{R}\begin{cases}((\mathit{dapply}\ \alpha\ \beta)\ x) \to (\alpha\ (\beta\ x))\\ (\mathit{twice}\ \alpha) \to (\mathit{dapply}\ \alpha\ \alpha)\end{cases}$$

Then $\mathcal{R}$ is terminating since it is compatible with $\succ_{lp}$. □

Example 5 (applylist). Let $\mathcal{C} = \{\mathit{lapply}, \mathit{cons}, \mathit{nil}\}$ with $\mathit{lapply} > \mathit{cons} > \mathit{nil}$ and consider the following SRS $\mathcal{R}$.

$$\mathcal{R}\begin{cases}((\mathit{lapply}\ \mathit{nil})\ x) \to x\\ ((\mathit{lapply}\ (\mathit{cons}\ \alpha\ \beta))\ x) \to (\alpha\ ((\mathit{lapply}\ \beta)\ x))\end{cases}$$

Then $\mathcal{R}$ is terminating since it is compatible with $\succ_{lp}$. □

Example 6 (recursor). Let $\mathcal{C} = \{\mathit{rec}, s, 0\}$ with $\mathit{rec} > s > 0$ and consider the following SRS $\mathcal{R}$.

$$\mathcal{R}\begin{cases}((\mathit{rec}\ \alpha\ x)\ 0) \to x\\ ((\mathit{rec}\ \alpha\ x)\ (s\ y)) \to (\alpha\ (s\ y)\ ((\mathit{rec}\ \alpha\ x)\ y))\end{cases}$$

Then $\mathcal{R}$ is terminating since it is compatible with $\succ_{lp}$. □

5 Recursive Path Relation with Status

In this section we discuss the recursive path relation with status on S-expressions, which is a higher-order extension of the well-known one on the first-order terms [1, 5, 8, 13, 16]. For a binary relation $\succ$ on $\mathcal{S}(\mathcal{C}, \mathcal{V})$, $\succ^{mul}$ denotes the multiset extension of $\succ$. We assume a status (function) τ as a mapping from $\mathcal{S}(\mathcal{C}, \mathcal{V})$ to $\{\mathit{lex}, \mathit{mul}\}$. Then, the recursive path relation with status on $\mathcal{S}(\mathcal{C}, \mathcal{V})$ is defined as follows.

Definition 3 (Recursive Path Relation with Status). Let $>$ be a well-founded relation on $\mathcal{C}$. The recursive path relation $\succ_{rp}$ with status on $\mathcal{S}(\mathcal{C}, \mathcal{V})$ is recursively defined as follows: $s \succ_{rp} t$ if and only if
(R0) $s, t, \in \mathcal{C}$ and $s > t$, or
(R1) $s = (s_1 \cdots s_m)$ $(m \geq 1)$ and $\exists s_i.\ s_i \succeq_{rp} t$, or
(R2) $t = (t_1 \cdots t_n)$ $(n \geq 0)$ and $\forall t_i.\ s \succ_{rp} t_i$, and either
  (R2-1) $s \in \mathcal{C}$, or
  (R2-2) $s = (s_1 \cdots s_m)$ $(m \geq 1)$ and $t = ()$, or
  (R2-3) $s = (s_1 \cdots s_m)$ and $t = (t_1 \cdots t_n)$ $(m, n \geq 1)$, and
    (a) $s_1 \succ_{rp} t_1$, or
    (b) $s_1 = t_1$ and $\langle s_2, \cdots, s_m\rangle \succ_{rp}^{\tau(s_1)} \langle t_2, \cdots, t_n\rangle$.

Theorem 4. The recursive path relation $\succ_{rp}$ with status is well-founded on $\mathcal{S}_p(\mathcal{C}, \mathcal{V})$ for every $p \geq 0$, and it is monotonic.

Proof. Let $\rhd$ be the immediate subterm relation and $s \succ_{rp}^{\Lambda} t$ defined as (R0) $s, t, \in \mathcal{C}$ and $s > t$, or (R2-1) $s \in \mathcal{C}$ and $t = (t_1 \cdots t_n)$ $(n \geq 0)$, or (R2-2) $s = (s_1 \cdots s_m)$ $(m \geq 1)$ and $t = ()$, or (R2-3a) $s = (s_1 \cdots s_m)$, $t = (t_1 \cdots t_n)$ $(m, n \geq 1)$ and $s_1 \succ_{rp} t_1$, or (R2-3b) $s = (s_1 \cdots s_m)$, $t = (t_1 \cdots t_n)$ $(m, n \geq 1)$, $s_1 =$

t_1 and $\langle s_2,\cdots,s_m\rangle \succ_{rp}^{\tau(s_1)} \langle t_2,\cdots,t_n\rangle$. Then $\succ_{rp}$ obviously satisfies Property 1. Monotonicity of $\succ_{rp}$ directly follows from Lemma 1. Thus, it remains to prove well-foundedness of $\succ_{rp}$ on $\mathcal{S}_p(\mathcal{C},\mathcal{V})$. From Theorem 1, it is enough to show that $\succ_{rp}^{\Lambda}$ is well-founded on $\overline{SN(\succ_{rp})} \cap \mathcal{S}_p(\mathcal{C},\mathcal{V})$. Suppose $\succ_{rp}^{\Lambda}$ is not well-founded on $\overline{SN(\succ_{rp})} \cap \mathcal{S}_p(\mathcal{C},\mathcal{V})$ and consider an infinite sequence $s_0 \succ_{rp}^{\Lambda} s_1 \succ_{rp}^{\Lambda} s_2 \succ_{rp}^{\Lambda} \cdots$ on $\overline{SN(\succ_{rp})} \cap \mathcal{S}_p(\mathcal{C},\mathcal{V})$. Since $>$ is well-founded on $\mathcal{C}$, there exists some k such that for every $i \geq k$, $s_i \succ_{rp}^{\Lambda} s_{i+1}$ satisfies (R2-3). Thus, we have an infinite sequence $(s_{k,1} \ \cdots \ s_{k,n}) \succ_{rp} (s_{k+1,1} \ \cdots \ s_{k+1,n'}) \succ_{rp} (s_{k+2,1} \ \cdots \ s_{k+2,n''}) \succ_{rp} \cdots$, where all $s_{i,j}$ are in $SN(\succ_{rp})$, $n, n', n'', \cdots \leq p$ and $s_{k,1} \succeq_{rp} s_{k+1,1} \succeq_{rp} s_{k+2,1} \succeq_{rp} \cdots$. Thus there exists some $q \geq k$ such that $s_{q,1} = s_{q+1,1} = s_{q+2,1} = \cdots$. Then, we have an infinite sequence $\langle s_{q,1},\cdots,s_{q,m}\rangle \succ_{rp}^{\tau(s_{q,1})} \langle s_{q+1,1},\cdots,s_{q+1,m'}\rangle \succ_{rp}^{\tau(s_{q,1})} \langle s_{q+2,1},\cdots,s_{q+2,m''}\rangle \succ_{rp}^{\tau(s_{q,1})} \cdots$. However, since $\succ_{rp}$ on the set of all $s_{i,j}$ is well-founded, its lexicographic (multiset) extension over the sequences of length bounded by p is well-founded. Thus, we get a contradiction. □

We note that the recursive path relation $\succ_{rp}$ with status on $\mathcal{S}(\mathcal{C},\mathcal{V})$ is not stable due to *higher-order* feature of S-expressions, different from the one on the first-order terms. For example, assume that $\tau(x) = lex$, $\tau(0) = mul$, and let $s = (x\ 1\ 0\ 0)$, $t = (x\ 0\ 1\ 1)$. Then $s \succ_{rp} t$ but $t\theta \succ_{rp} s\theta$ when $\theta = [x := 0]$. This problem arises whenever status τ is not trivial, i.e., $\tau(x) \neq \tau(u)$ for some term u. However, stability of all term pairs is not necessary for proving termination of a SRS $\mathcal{R}$; indeed, it is enough to guarantee that $\mathcal{R}$ is stable with respect to $\succ_{rp}$. Hence, we present criteria for stability of $\mathcal{R}$.

For status τ, we define $(s_1 \cdots s_m) \succ_{rp}^{\tau} (t_1 \cdots t_n)$ $(m, n \geq 1)$ by (R2-3b) $s_1 = t_1$ and $\langle s_2,\cdots,s_m\rangle \succ_{rp}^{\tau(s_1)} \langle t_2,\cdots,t_n\rangle$.

Lemma 5 (Stability of $\succ_{rp}$). Let $\rhd$ be the immediate subterm relation and $P \subseteq \mathcal{S}(\mathcal{C},\mathcal{V})^2$ closed with respect to $\rhd_2$. Let $\overline{ST(\succ_{rp})} \cap P$ be stable with respect to $\succ_{rp}^{\tau}$. Then P is stable with respect to $\succ_{rp}$.

Proof. From the definition of $\succ_{lp}$ it is obvious that $s \succ_{rp} x$ implies $s \rhd^{+} x$ for every variable x. Similarly to the proof of Theorem 4, we define $\succ_{rp}^{\Lambda}$ as (R0), (R2-1), (R2-2), (R2-3a), or (R2-3b). Then the claim directly follows from Lemma 3. □

For a term pair set $Q \subseteq \mathcal{S}(\mathcal{C},\mathcal{V})^2$, we define the closure of Q with respect to $\rhd_2$ as $Q^* = \{\ \langle s,t\rangle \mid \langle u,v\rangle \rhd_2^* \langle s,t\rangle$ for some $\langle u,v\rangle \in Q\ \}$.

Theorem 5. Let $\mathcal{R}$ be a finite SRS compatible with a recursive path relation with status τ. Let $\overline{ST(\succ_{rp})} \cap \mathcal{R}^*$ be stable with respect to $\succ_{rp}^{\tau}$. Then $\mathcal{R}$ is terminating.

Proof. The claim follows from Theorem 4 and Lemma 5. □

We say status τ is stable on a term set $S \subseteq \mathcal{S}(\mathcal{C},\mathcal{V})$ if for every $s \in S$, $\tau(s) = \tau(s\theta)$ for any θ. The head term of $s = (s_1 \cdots s_m)$ $(m \geq 1)$ is denoted by $hd(s) = s_1$. For $Q \subseteq \mathcal{S}(\mathcal{C},\mathcal{V})^2$ we define $H_{eq}(Q) = \{\ hd(s) \mid \langle s,t\rangle \in Q$ and $hd(s) = hd(t)\ \}$.

Corollary 1. Let $\mathcal{R}$ be a finite SRS compatible with a recursive path relation with status τ. Let status τ be stable on $H_{eq}(\mathcal{R}^*)$. Then $\mathcal{R}$ is terminating.

Proof. The claim follows from Theorem 5. □

Example 7 (folding: adapted from [11]). Let $\mathcal{C} = \{sum, prod,\ fold,\ *,\ +,\ s,\ 0,\ cons,\ nil\}$ with $sum, prod > fold > * > + > s > 0 > cons > nil$ and consider the following SRS $\mathcal{R}$.

$$\mathcal{R}\begin{cases} ((fold\ \alpha\ x)\ nil) \to x \\ ((fold\ \alpha\ x)\ (cons\ y\ z)) \to (\alpha\ y\ ((fold\ \alpha\ x)\ z)) \\ sum \to (fold\ +\ 0) \\ prod \to (fold\ *\ (s\ 0)) \\ (+\ 0\ y) \to y \\ (+\ (s\ x)\ y) \to (s\ (+\ y\ x)) \\ (*\ 0\ y) \to 0 \\ (*\ (s\ x)\ y) \to (+\ y\ (*\ y\ x)) \end{cases}$$

Note that $\mathcal{R}$ is not compatible with $\succ_{lp}$ because $(+\ (s\ x)\ y) \not\succ_{lp} (s\ (+\ y\ x))$ and $(*\ (s\ x)\ y) \not\succ_{lp} (+\ y\ (*\ y\ x))$. We take status τ as $\tau(t) = mul$ if $t = +, *$ and $\tau(t) = lex$ otherwise. Then it is easily shown that $\mathcal{R}$ is compatible with $\succ_{rp}$. As $H_{eq}(R^*) = \{(fold\ \alpha\ x), fold, *, +, s\}$, τ is stable on $H_{eq}(R^*)$. From Corollary 1, it follows that $\mathcal{R}$ is terminating. □

Example 8 (sorting: adapted from [7]). Let $\mathcal{C} = \{asort, dsort, sort,\ ins,\ max,\ min,\ s,\ 0,\ cons,\ nil\}$ with $asort, dsort > sort > ins > max, min > s > 0 > cons > nil$ and consider the following SRS $\mathcal{R}$.

$$\mathcal{R}\begin{cases} ((sort\ \alpha\ \beta)\ nil) \to nil \\ ((sort\ \alpha\ \beta)\ (cons\ x\ y)) \to ((ins\ \alpha\ \beta)\ ((sort\ \alpha\ \beta)\ y)\ x) \\ ((ins\ \alpha\ \beta)\ nil\ y) \to (cons\ y\ nil) \\ ((ins\ \alpha\ \beta)\ (cons\ x\ z)\ y) \to (cons\ (\alpha\ x\ y)\ ((ins\ \alpha\ \beta)\ z\ (\beta\ x\ y))) \\ (max\ x\ x) \to x \\ (max\ 0\ y) \to y \\ (max\ x\ 0) \to x \\ (max\ (s\ x)\ (s\ y)) \to (s\ (max\ y\ x)) \\ (min\ x\ x) \to x \\ (min\ 0\ y) \to 0 \\ (min\ x\ 0) \to 0 \\ (min\ (s\ x)\ (s\ y)) \to (s\ (min\ y\ x)) \\ (asort\ z) \to ((sort\ min\ max)\ z) \\ (dsort\ z) \to ((sort\ max\ min)\ z) \end{cases}$$

Since $(max\ (s\ x)\ (s\ y)) \not\succ_{lp} (s\ (max\ y\ x))$ and $(min\ (s\ x)\ (s\ y)) \not\succ_{lp} (s\ (min\ y\ x))$, $\mathcal{R}$ is not compatible with $\succ_{lp}$. We take status τ as $\tau(t) = mul$ if $t = max, min$ and $\tau(t) = lex$ otherwise. Then, $\mathcal{R}$ is compatible with $\succ_{rp}$. As $H_{eq}(R^*) = \{(sort\ \alpha\ \beta), sort, (ins\ \alpha\ \beta), ins, cons, max, min, s\}$, τ is stable on $H_{eq}(R^*)$. Thus, from Corollary 1, it follows that $\mathcal{R}$ is terminating. □

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References

1. F. Baader and T. Nipkow, Term Rewriting and All That, Cambridge University Press (1998).
2. W. Buchholz, Proof-theoretic analysis of termination proofs, *Ann. Pure Appl. Logic 75* (1995) 57-65.
3. J. E. Dawson and R. Goré, A general theorem on termination of rewriting. *Proc. of Computer Science Logic 2004, Lecture Notes in Comput. Sci. 3210* (2004) 100-114.
4. N. Dershowitz, Ordering for term-rewriting systems, *Theoretical Comput. Sci. 17* (1982) 279-301.
5. M.C.F. Ferreira, Termination of term rewriting: Well-foundedness, totality and transformation, *PhD thesis*, Dep. of Comput. Sci., Utrecht University (1995).
6. J. Goubault-Larrecq, Well-founded recursive relations. *Proc. of the 15th International Workshop Computer Science Logic, Lecture Notes in Comput. Sci. 2142* (2001) 484-497.
7. J.-P. Jouannaud and A. Rubio, Polymorphic higher-order recursive path orderings, *J. ACM 54, Article 2* (2007) 1-48.
8. S. Kamin and J.-J. Levy. Two generalizations of the recursive path ordering, *Unpublished Manuscript*, University of Illinois (1980).
9. B. Kruskal, Well quasi ordering, the tree theorem and Vazsonyi's conjecture, *Trans. Am. Math. Soc. 95* (1960) 210-225.
10. M. Linfantsev and L. Bachmair, An LPO-based termination ordering for higher-order terms without λ-abstraction, *Proc. of the 11th International Conference on Theorem Proving in Higher Order Logics, Lecture Notes in Comput. Sci. 1479* (1998) 277-293.
11. O. Lysne and J. Piris, A termination ordering for higher order rewrite systems, *Proc. of the 6th International Conference on Rewriting Techniques and Applications, Lecture Notes in Comput. Sci. 914* (1995) 26-40.
12. P.-A. Mellies, On a duality between Kruskal and Dershowitz theorems, *Proc. of the 25th International Colloquium on Automata, Languages and Programming, Lecture Notes in Comput. Sci. 1443* (1998) 518-529.
13. A. Middeldorp and H. Zantema, Simple termination of rewrite systems, *Theoretical Comput. Sci. 175* (1997) 127-158.
14. C. St.J. A. Nash-Williams, On well-quasi-ordering finite trees, *Proc. of the Cambridge Phil. Soc. 59* (1963) 833-835.
15. F. van Raamsdonk, On termination of higher-order rewriting, *Proc. of the 12th International Conference on Rewriting Techniques and Applications, Lecture Notes in Comput. Sci. 2051* (2001) 261–275.
16. Terese, Term Rewriting Systems, Cambridge University Press (2003).
17. Y. Toyama, Termination of S-expression rewriting systems: Lexicographic path ordering for higher-order terms, *Proc. of the 15th International Conference on Rewriting Techniques and Applications, Lecture Notes in Comput. Sci. 3091* (2004) 40-54.