Reduction Strategies for Left-Linear Term Rewriting Systems*

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Abstract. Huet and Lévy (1979) showed that needed reduction is a normalizing strategy for orthogonal (i.e., left-linear and non-overlapping) term rewriting systems. In order to obtain a decidable needed reduction strategy, they proposed the notion of strongly sequential approximation. Extending their seminal work, several better decidable approximations of left-linear term rewriting systems, for example, NV approximation, shallow approximation, growing approximation, etc., have been investigated in the literature. In all of these works, orthogonality is required to guarantee approximated decidable needed reductions are actually normalizing strategies. This paper extends these decidable normalizing strategies to left-linear overlapping term rewriting systems. The key idea is the balanced weak Church-Rosser property. We prove that approximated external reduction is a computable normalizing strategy for the class of left-linear term rewriting systems in which every critical pair can be joined with root balanced reductions. This class includes all weakly orthogonal left-normal systems, for example, combinatory logic CL with the overlapping rules $pred \cdot (succ \cdot x) \to x$ and $succ \cdot (pred \cdot x) \to x$, for which leftmost-outermost reduction is a computable normalizing strategy.

1 Introduction

Normalizing reduction strategies of reduction systems, such as leftmost-outermost evaluation of lambda calculus [2, 11], combinatory logic [7, 11], ordinal recursive program schemata [25] and left-normal term rewriting systems [8, 17, 22] guarantee a *safe* evaluation which reduces a given expression to its normal form whenever it exists. Hence, normalizing reduction strategies play an important role in the implementation of functional programming languages based on reduction systems.

Strong sequentiality formalized by Huet and Lévy [8] is a well-known practical criterion guaranteeing an efficiently computable normalizing reduction strategy for orthogonal (i.e., left-linear and non-overlapping) term rewriting systems. They showed that for every strongly sequential orthogonal term rewriting system R, strongly needed reduction is a computable normalizing strategy, that is,

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by rewriting a redex called a *strongly needed redex* at each step, every reduction starting with a term having a normal form eventually terminates at the normal from. Here, the strongly needed redex is defined as a *needed redex* concerning an approximation of R which is obtained by analyzing the left-hand sides only of the rewrite rules of R. Moreover, Huet and Lévy [8] proved the decidability of strong sequentiality. A simpler proof by Klop and Middeldorp can be found in [12] and a proof based on second order monadic logic and tree automata by Comon in [3].

Inspired by the seminal work by Huet and Lévy [8], several better decidable approximations of left-linear term rewriting systems, for example, NV approximation [21], shallow approximation [3], growing approximation [9, 15], etc., have been investigated in the literature. Moreover, Durand and Middeldorp [6] presented a simple uniform framework for normalizing reduction strategies based on decidable approximations. In all of these works [6, 9, 10, 15], however, the non-overlapping restriction is still required to guarantee that approximated decidable needed reductions are actually normalizing strategies; hence, they cannot be applied to term rewriting systems with overlapping rules such as

$$\begin{cases} \mathit{pred}(\mathit{succ}(x)) \to x \\ \mathit{succ}(\mathit{pred}(x)) \to x. \end{cases}$$

Though it is known [6, 9, 10, 15] that only the left-linearity restriction is necessary for considering decidability issues, the question whether there exists an approximated decidable normalizing strategy for left-linear overlapping term rewriting systems has received quite a bit of attention.

The main purpose of this paper develops decidable normalizing reduction strategies for left-linear overlapping term rewriting systems. The notion of sequentiality defined by Huet and Lévy [8] is naturally adapted to that of externality. An external term rewriting system R guarantees that every reducible term contains an outer needed redex, called an external redex, which remains at an outer position until it is rewritten. Under this new framework, we show that external reduction is normalizing for the class of external *root balanced joinable* term rewriting systems. A root balanced joinable term rewriting system is defined as a term rewriting system in which every critical pair can be joined with *root balanced reductions*. We also show that for weakly orthogonal left-normal systems, the leftmost-outermost reduction strategy is normalizing. For example, the leftmost-outermost reduction strategy is normalizing for combinatory logic CL $\cup \{\mathit{pred}\cdot(\mathit{succ}\cdot x) \to x,\ \mathit{succ}\cdot(\mathit{pred}\cdot x) \to x\}$. Here, combinatory logic CL [2, 7, 11] is the orthogonal term rewriting system having the following rewrite rules:

$$\text{CL} \quad \begin{cases} ((S\cdot x)\cdot y)\cdot z \to (x\cdot z)\cdot(y\cdot z) \\ (K\cdot x)\cdot y \to x. \end{cases}$$

Moreover, our result can be applied to term rewriting systems not having the Church-Rosser property too. For example, the leftmost-outermost reduction strategy is again normalizing for CL $\cup$

$$\begin{cases} (K \cdot A) \cdot y \to (K \cdot B) \cdot y \\ (K \cdot B) \cdot y \to (K \cdot A) \cdot y \\ A \to A \\ B \to B, \end{cases}$$

though the system is not Church-Rosser since $(K \cdot A) \cdot y$ can be reduced into two constants A and B which cannot be joined.

The approach presented here is more accessible than that based on sequentiality of orthogonal term rewriting systems by Huet and Lévy [8]. The key idea is the balanced weak Church-Rosser property, which was first considered by Toyama [24] for analyzing normalizing reduction strategies of strongly sequential left-linear overlapping term rewriting systems. We first explain this idea in an abstract framework. Section 2 introduces preliminary concepts of abstract reduction systems. In Section 3, we introduce the *balanced weak Church-Rosser property* of abstract reduction systems and explain how this property is related to a normalizing reduction strategy. Our results are carefully partitioned between abstract properties depending solely on the reduction relation and properties depending on term structure. In Section 4 we present preliminary concepts for term rewriting systems and in the next section we introduce the notion of *externality* of (possibly) overlapping term rewriting systems. In Section 6, by using the *balanced weak Church-Rosser property* of *external reduction*, we prove that external reduction of *root balanced joinable* term rewriting systems is normalizing. Section 7 extends external reduction to *quasi-external reduction*. In Section 8, we present *computable normalizing strategies* based on *decidable approximations*. Finally, Section 9 discusses a syntactic characterization of external overlapping term rewriting systems.

2 Reduction Systems

Assuming that the reader is familiar with the basic concepts and notations concerning reduction systems in [1, 18, 22], we briefly present notations and definitions.

A *reduction system* (or an *abstract reduction system*) is a structure $A = \langle D, \to \rangle$ consisting of some set D and some binary relation $\to$ on D (i.e., $\to \ \subseteq D \times D$), called a *reduction relation*. A *reduction* (starting with x_0) in A is a finite or infinite sequence $x_0 \to x_1 \to x_2 \to \cdots$. The identity of elements x, y of D is denoted by $x \equiv y$. $\to^{\equiv}$ is the reflexive closure of $\to$, $\leftrightarrow$ is the symmetric closure of $\to$, $\to^{+}$ is the transitive closure of $\to$, $\to^{*}$ is the transitive reflexive closure of $\to$, and $=$ is the equivalence relation generated by $\to$ (i.e., the transitive reflexive symmetric closure of $\to$). $x \to^m y$ denotes a reduction of m ($m \geq 0$) steps from x to y. $x \leftrightarrow^m y$ denotes a chain $x \leftrightarrow^* y$ of length m, i.e., there exists a sequence $x = x_0 \leftrightarrow x_1 \leftrightarrow \cdots \leftrightarrow x_m = y$ of m steps.

If $x \in D$ is minimal with respect to $\to$, i.e., $\neg \exists y \in D, [x \to y]$, then we say that x is a *normal form*; let NF be the set of all normal forms. If $x \to^* y$ and $y \in NF$ then we say x has a normal form y and y is a normal form of x. We say x is reducible if $x \notin NF$.

A reduction system $A = \langle D, \rightarrow \rangle$ ($\rightarrow$ for short) is *strongly normalizing* (or *terminating*) if every reduction in A terminates, i.e., there is no infinite sequence $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots$. A is *Church-Rosser* (or *confluent*) if $\forall x, y, z \in D, [x \rightarrow^* y \wedge x \rightarrow^* z \Rightarrow \exists w \in D, y \rightarrow^* w \wedge z \rightarrow^* w]$. A is *weakly Church-Rosser* (or *locally confluent*) if $\forall x, y, z \in D, [x \rightarrow y \wedge x \rightarrow z \Rightarrow \exists w \in D, y \rightarrow^* w \wedge z \rightarrow^* w]$. A is *complete* if A is Church-Rosser (confluent) and strongly normalizing. A has the *normal form property* if $\forall x \in D, \forall y \in NF, [x = y \Rightarrow x \rightarrow^* y]$. A has the *unique normal form property* if $\forall x, y \in NF, [x = y \Rightarrow x \equiv y]$. Note that the normal form property implies the unique normal form property.

The notions of *confluent*, *strongly normalizing*, *complete* on systems are related to the notions on elements. An element $x \in D$ is *confluent* if $\forall y, z \in D, [x \rightarrow^* y \wedge x \rightarrow^* z \Rightarrow \exists w \in D, y \rightarrow^* w \wedge z \rightarrow^* w]$. x is *strongly normalizing* if every reduction starting with x terminates. x is *complete* if x is confluent and strongly normalizing.

Definition 1 (Reduction Strategy). *Let $A = \langle D, \rightarrow \rangle$ and let $\rightarrow_s$ be a sub-relation of $\rightarrow^+$ (i.e., if $x \rightarrow_s y$ then $x \rightarrow^+ y$) such that a normal form concerning $\rightarrow_s$ is also a normal form concerning $\rightarrow$ (i.e., the two binary relations $\rightarrow_s$ and $\rightarrow$ have the same domain). Then, we say that $\rightarrow_s$ is a reduction strategy for A (or for $\rightarrow$). If $\rightarrow_s$ is a sub-relation of $\rightarrow$ then we call it a one step reduction strategy; otherwise $\rightarrow_s$ is called a many step reduction strategy.*

Definition 2 (Normalizing Strategy). *A reduction strategy $\rightarrow_s$ is normalizing iff for each x having a normal form concerning $\rightarrow$, there exists no infinite sequence $x \equiv x_0 \rightarrow_s x_1 \rightarrow_s x_2 \rightarrow_s \cdots$ (i.e., every $\rightarrow_s$ reduction starting with x must eventually terminate at a normal form of x).*

3 Balanced Weak Church-Rosser Property

This section introduces the balanced weak Church-Rosser property. Though in later sections this concept will play an important role for analyzing normalizing strategies of term rewriting systems, our results concerning the balanced weak Church-Rosser property can be presented in an abstract framework depending solely on the reduction relation.

Let $A = \langle D, \rightarrow \rangle$ be an abstract reduction system.

Definition 3. *$A = \langle D, \rightarrow \rangle$ (or $\rightarrow$) is balanced weakly Church-Rosser (BWCR) iff $\forall x, y, z \in D, [x \rightarrow y \wedge x \rightarrow z \Rightarrow \exists w \in D, \exists k \geq 0, y \rightarrow^k w \wedge z \rightarrow^k w]$ (Figure 1).*

Lemma 1 (BWCR Lemma). *Let $A = \langle D, \rightarrow \rangle$ be BWCR. Let $x = y$ and $y \in NF$. Then,*

(1) *x is complete,*
(2) *all the reductions from x to y have the same length (i.e., the same number of reduction steps).*

$$\begin{tikzcd}
x \arrow[r] \arrow[d] & y \arrow[d, dashed, "k"'] \\
z \arrow[r, dashed, "k"] & w
\end{tikzcd} \quad (k \geq 0)$$

Fig. 1.

Proof. We first prove the following claim: if $x \to^n y$ and $y \in NF$ then x satisfies the properties (1) and (2).

Proof of the claim. We show the claim by induction on n. The case $n = 0$ is trivial. Let $x \to x' \to^{n-1} y \in NF$. Take any one step reduction $x \to z$ starting with x. By the balanced weak Church-Rosser property, there exists some w and k such that $z \to^k w$ and $x' \to^k w$. By the induction hypothesis, the properties (1) and (2) hold at x'; hence $x' \to^k w \to^* y$ must have $n - 1$ steps in length. Thus, $w \to^{n-1-k} y$; see Figure 2. Since $z \to^k w$, we obtain $z \to^{n-1} y$. By the induction hypothesis, z satisfies the properties (1) and (2). Therefore, the claim follows.

$$\begin{tikzcd}
x \arrow[r] \arrow[d] \arrow[dr, phantom, "BWCR"] & x' \arrow[r, "n-1"] \arrow[d, dashed, "k"'] \arrow[r, phantom, shift right=4, "I.H."] & y \in NF \\
z \arrow[r, dashed, "k"] & w \arrow[ur, dashed, "n-1-k"'] &
\end{tikzcd}$$

Fig. 2.

We next show that if $x \leftrightarrow^n y$ and $y \in NF$ then $x \to^* y$. The proof is by induction on n. The case $n = 0$ is trivial. Let $x \leftrightarrow x' \leftrightarrow^{n-1} y$. By the induction hypothesis, we have $x' \to^* y$. The case $x \to x'$ is trivial. Let $x \leftarrow x'$. By applying the claim to $x' \to^* y \in NF$, it is obtained that x' is complete. Thus, $x \to^* y$.

Therefore, from the claim it follows that if $x = y$ and $y \in NF$ then x satisfies the properties (1) and (2). $\square$

Lemma 1 (BWCR Lemma) is a generalization to Theorem 2 and Corollary 2.1 of Newman [16], which requires the following property instead of BWCR:

$\forall x, y, z \in D, [x \to y \wedge x \to z \wedge y \not\equiv z \Rightarrow \exists w \in D, y{\to}w \wedge z{\to}w]$. An extension of BWCR is discussed in Van Oostrom [20].

Corollary 1. *If an abstract reduction system A is BWCR then A has the normal form property.*

Proof. From the BWCR Lemma, it is trivial. $\square$

Next we will explain how the balanced weak Church-Rosser property is related to a normalizing reduction strategy. Let $d(x)$ denote the length of a reduction from x to a normal form if it exists. Note that if $\to$ is balanced weakly Church-Rosser and x has a normal form then $d(x)$ is well-defined according to the BWCR Lemma. We write $x{\leftarrow}^m y$ if $y{\to}^m x$.

Lemma 2. *Let $\to$ be balanced weakly Church-Rosser. Let $x \longrightarrow^{m_1} \cdot \longleftarrow^{n_1} \cdot \longrightarrow^{m_2} \cdot \longleftarrow^{n_2} \cdots \longrightarrow^{m_p} \cdot \longleftarrow^{n_p} y$ for some p, $m_1, \cdots, m_p$, $n_1, \cdots, n_p \geq 0$ and let x have a normal form. Then y has a normal form and $d(x) - d(y) = \sum m_i - \sum n_i$.*

Proof. By the BWCR Lemma it is clear that y has a normal form. We prove $d(x) - d(y) = \sum m_i - \sum n_i$ by induction on p. The case $p = 0$ is trivial. Let $x \longrightarrow^{m_1} \cdot \longleftarrow^{n_1} \cdot \longrightarrow^{m_2} \cdot \longleftarrow^{n_2} \cdots \longrightarrow^{m_{p-1}} \cdot \longleftarrow^{n_{p-1}} y' \longrightarrow^{m_p} z \longleftarrow^{n_p} y$. By the BWCR Lemma, $d(y')$ and $d(z)$ are well-defined and $d(y') - m_p = d(z) = d(y) - n_p$. Thus, we have $d(y') - d(y) = m_p - n_p$. From the induction hypothesis, $d(x) - d(y') = \sum_{i=1}^{p-1} m_i - \sum_{i=1}^{p-1} n_i$. Therefore, $d(x) - d(y) = \sum m_i - \sum n_i$. $\square$

We write $x \longleftrightarrow\!\!\!\!\!\rightarrow y$ if there exists a connection $x \longrightarrow^{m_1} \cdot \longleftarrow^{n_1} \cdot \longrightarrow^{m_2} \cdot \longleftarrow^{n_2} \cdots \longrightarrow^{m_p} \cdot \longleftarrow^{n_p} y$ such that $\sum m_i > \sum n_i$. We sometimes write $x \twoheadleftarrow\!\!\!\longrightarrow y$ instead of $y \longleftrightarrow\!\!\!\!\!\rightarrow x$.

Lemma 3. *Let $\to$ be balanced weakly Church-Rosser. Let $x \longleftrightarrow\!\!\!\!\!\rightarrow y$ and let x have a normal form. Then y has a normal form and $d(x) > d(y)$.*

Proof. It is trivial from Lemma 2. $\square$

The following lemma and corollary explain how the BWCR Lemma implies the normalizing property of a reduction strategy $\to_s$ for $\to$ (i.e., $\to_s \subseteq \to$ and the two reduction relations $\to_s$ and $\to$ have the same set of normal forms.)

Lemma 4. *Let $\to_s$ be a reduction strategy for $\to$ such that:*

(1) *$\to_s$ is balanced weakly Church-Rosser,*
(2) *if $x \to y$ then;*
   (i) *$x {=}_s y$ or,*
   (ii) *$x \longleftrightarrow\!\!\!\!\!\rightarrow_s \cdot \leftrightarrow \cdot \twoheadleftarrow\!\!\!\longrightarrow_s y$.*

If $x = y$ and $y \in NF$ then we have $x {\to}_s^* y$.

Proof. We first show the claim: if $x{\leftrightarrow}\cdot{\to}_s^m y$ and $y \in NF$, then we have $x{=}_s y$. The proof is by induction on m. For the base step we let $m = 0$. Then $x{\leftrightarrow}y \in NF$. Suppose that it satisfies the condition (ii), i.e., $x \longleftarrow\!\!\!\twoheadrightarrow_s x'{\leftrightarrow}y' \twoheadleftarrow\!\!\!\longrightarrow_s y$ holds for some x' and y'. Then by Lemma 3 and $y \in NF$ we have $d(y') < d(y) = 0$; it contradicts $d(y') \geq 0$. Thus $x{\leftrightarrow}y$ must satisfy the condition (i). Induction Step: Let $x \leftrightarrow z{\to}_s^m y \in NF$ $(m > 0)$. Then $x \leftrightarrow z$ must satisfy (i) or (ii) as each condition is symmetric. If $x{=}_s z$ then $x{=}_s y$ is trivial. Assume that $x \longleftarrow\!\!\!\twoheadrightarrow_s x' \leftrightarrow z' \twoheadleftarrow\!\!\!\longrightarrow_s z$. By applying Lemma 3 to $z \longleftarrow\!\!\!\twoheadrightarrow_s z'$, we have $z'{\to}_s^{m'} y$ with $m' < m$; see Figure 3. Applying the induction hypothesis of the claim to x', we have $x'{=}_s y$; thus, $x{=}_s y$ because of $x \longleftarrow\!\!\!\twoheadrightarrow_s x'{=}_s y$.

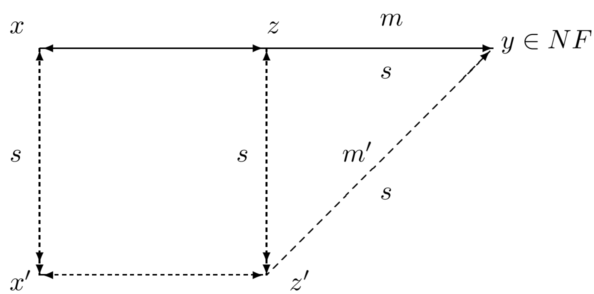

Fig. 3.

We next prove that if $x{\leftrightarrow}^n y$ and $y \in NF$, then $x{\to}_s^* y$. The proof is by induction on n. The case $n = 0$ is trivial. Let $x \leftrightarrow x'{\leftrightarrow}^{n-1} y \in NF$. From the induction hypothesis, we have $x'{\to}_s^* y$. Thus, from the claim, $x{=}_s y$. From the BWCR Lemma, it follows that $x{\to}_s^* y$. □

Corollary 2. *Let $\to_s$ be a reduction strategy for $\to$ such that:*

(1) *$\to_s$ is balanced weakly Church-Rosser,*
(2) *if $x \to y$ then;*
   (i) *$x{=}_s y$ or,*
   (ii) *$x \longleftarrow\!\!\!\twoheadrightarrow_s \cdot \leftrightarrow \cdot \twoheadleftarrow\!\!\!\longrightarrow_s y$.*

Then $\to$ has the normal form property and $\to_s$ is a normalizing strategy.

Proof. It is trivial from the BWCR Lemma and Lemma 4. □

In Lemma 4 and Corollary 2 we cannot relax the condition (ii) $x \longleftarrow\!\!\!\twoheadrightarrow_s \cdot \leftrightarrow \cdot \twoheadleftarrow\!\!\!\longrightarrow_s y$ to $x \longleftarrow\!\!\!\twoheadrightarrow_s \cdot \leftrightarrow^+ \cdot \twoheadleftarrow\!\!\!\longrightarrow_s y$. Consider the abstract reduction system A with the reduction relation $\to$ and the reduction strategy $\to_s$ for $\to$ presented in Figure 4. Then A does not have the normal form property. Note that $c{\to}b$ satisfies $c \longleftarrow\!\!\!\twoheadrightarrow_s \cdot \leftrightarrow^+ \cdot \twoheadleftarrow\!\!\!\longrightarrow_s b$ as $c{\to}_s c \to b{\to}a{\leftarrow}_s b$, and $c{\to}d$ satisfies $c \longleftarrow\!\!\!\twoheadrightarrow_s \cdot \leftrightarrow^+ \cdot \twoheadleftarrow\!\!\!\longrightarrow_s d$ as $c{\to}_s c \to d{\to}e{\leftarrow}_s d$.

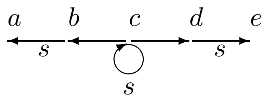

Fig. 4.

In Corollary 2 if we need only to show that $\rightarrow_s$ is a normalizing strategy for $\rightarrow$, we may replace the symmetric condition (ii) $x \longleftarrow\!\!\twoheadrightarrow_s \cdot \leftrightarrow \cdot \twoheadleftarrow\!\!\longrightarrow_s y$ with an asymmetric weaker condition as follows.

Corollary 3. *Let $\rightarrow_s$ be a reduction strategy for $\rightarrow$ such that:*

(1) *$\rightarrow_s$ is balanced weakly Church-Rosser,*
(2) *if $x \rightarrow y$ then;*
 (i) *$x {=_s} y$ or,*
 (ii) *$x {=_s} \cdot \rightarrow \cdot \twoheadleftarrow\!\!\longrightarrow_s y$.*

Then $\rightarrow_s$ is a normalizing strategy.

Proof. Similarly to the proof of Lemma 4, we can show the claim: if $x \rightarrow^* y \in NF$ then $x \rightarrow_s^* y$. Thus from the BWCR Lemma the corollary holds. □

In Corollary 3 the normal form property of $\rightarrow$ need not hold. Consider the abstract reduction system A with the reduction relation $\rightarrow$ and the reduction strategy $\rightarrow_s$ for $\rightarrow$ presented in Figure 5. Then A does not have the normal form property though $\rightarrow_s$ is a normalizing strategy for $\rightarrow$. Note that $b \rightarrow c$ satisfies $b {=_s} \cdot \rightarrow \cdot \twoheadleftarrow\!\!\longrightarrow_s c$ as $b \rightarrow c \rightarrow_s c$, and $d \rightarrow c$ satisfies $d {=_s} \cdot \rightarrow \cdot \twoheadleftarrow\!\!\longrightarrow_s c$ as $d \rightarrow c \rightarrow_s c$.

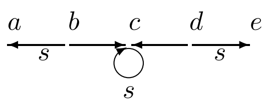

Fig. 5.

4 Term Rewriting Systems

We assume familiarity with the basis of term rewriting systems [1, 18, 22]. Let $\mathcal{F}$ be a set of function symbols denoted by $f, g, h, \cdots$, and let $\mathcal{V}$ be a countably infinite set of variable symbols denoted by $x, y, z, \cdots$ where $\mathcal{F} \cap \mathcal{V} = \varnothing$. By $T(\mathcal{F}, \mathcal{V})$, we denote the set of all terms constructed from $\mathcal{F}$ and $\mathcal{V}$. Terms not containing variables are called *ground* terms. The set of all ground terms built from $\mathcal{F}$ is denoted by $T(\mathcal{F})$. A term t is *linear* if every variable in t occurs only once.

Consider an extra constant □ called a *hole* and the set $T(\mathcal{F} \cup \{\Box\}, \mathcal{V})$. Then $C \in T(\mathcal{F} \cup \{\Box\}, \mathcal{V})$ is called a *context* over $\mathcal{F}$. We use the notation $C[\ ,\ldots,\ ]$

for the context containing n holes ($n \geq 0$), and if $t_1, \ldots, t_n \in T(\mathcal{F}, \mathcal{V})$, then $C[t_1, \ldots, t_n]$ denotes the result of placing $t_1, \ldots, t_n$ in the holes of $C[\ , \ldots, \ ]$ from left to right. In particular, $C[\ ]$ denotes a context containing precisely one hole. A term s is called a *subterm* of t if $t \equiv C[s]$, denoted by $s \trianglelefteq t$. A subterm s of t is *proper*, denoted by $s \triangleleft t$, if $s \not\equiv t$. If a term t has an occurrence of some (function or variable) symbol e, we write $e \in t$.

A *substitution* θ is a mapping from $\mathcal{V}$ to $T(\mathcal{F}, \mathcal{V})$. Substitutions are extended into homomorphisms from $T(\mathcal{F}, \mathcal{V})$ into $T(\mathcal{F}, \mathcal{V})$. We write $t\theta$ instead of $\theta(t)$. We write $s \preceq t$ if $s\theta \equiv t$ for some substitution θ.

A *rewrite rule* over $\mathcal{F}$ is a pair $\langle l, r \rangle$ of terms in $T(\mathcal{F}, \mathcal{V})$ such that $l \notin \mathcal{V}$ and any variable in r also occurs in l. We write $l \to r$ for $\langle l, r \rangle$. A *redex* is a term $l\theta$, where $l \to r$.

A *term rewriting system* (TRS for short) R over $\mathcal{F}$ is a set of rewrite rules over $\mathcal{F}$. (We often simply write R when $\mathcal{F}$ can be inferred from the context.) A TRS R over $\mathcal{F}$ is *finite* if both R and $\mathcal{F}$ are finite. The rewrite rules of R over $\mathcal{F}$ define a *reduction relation* $\to_R$ on $T(\mathcal{F}, \mathcal{V})$ as follows: $t \to_R s$ iff $t \equiv C[l\theta]$ and $s \equiv C[r\theta]$ for some $l \to r \in R$, $C[\ ]$ and θ. When we want to specify the *redex occurrence* $\Delta \equiv l\theta$ of t in this reduction, we write $t \xrightarrow{\Delta}_R s$. All the notions defined in the previous sections for abstract reduction systems carry over to TRSs by associating a reduction system $\langle T(\mathcal{F}, \mathcal{V}), \to_R \rangle$ with R. We will simply write $\to$ instead of $\to_R$ when no confusion arises.

Let $l \to r$ and $l' \to r'$ be two rules in R. We assume that they are renamed to have no common variables. Suppose that $s \notin \mathcal{V}$ is a subterm occurrence in l, i.e., $l \equiv C[s]$, such that s and l' are unifiable with a *most general unifier* θ. Then we say that $l \to r$ and $l' \to r'$ are *overlapping*, and that the pair $\langle C[r']\theta, r\theta \rangle$ of terms is *critical* in R [22]. We may choose $l \to r$ and $l' \to r'$ to be the same rule, but in this case we shall not consider the case $s \equiv l$.

If R has a critical pair, then we say that R is *overlapping*; otherwise, *non-overlapping*. We say that R is *left-linear* if for any $l \to r \in R$, l is linear. R is *orthogonal* if R is left-linear and non-overlapping. R is *weakly orthogonal* if R is left-linear and every critical pair $\langle s, t \rangle$ of R is trivial (i.e., $s \equiv t$).

From here on we assume that R is a *finite left-linear* TRS over $\mathcal{F}$ which may have *overlapping* rules. Furthermore, we view R as a TRS over $\mathcal{F} \cup \{\Box\}$ when we consider a reduction relation on $T(\mathcal{F} \cup \{\Box\}, \mathcal{V})$.

5 Externality

The fundamental concept of *neededness* for orthogonal TRSs was introduced by Huet and Lévy [8]. In an orthogonal TRS, every reducible term contains a *needed* redex and needed reduction (i.e., *call-by-need evaluation*) is a normalizing strategy [8]. This section presents a similar framework of *externality* for left-linear overlapping TRSs. An *external* TRS R guarantees that every reducible term contains an outer needed redex, called an *external redex*, which remains at an outer position until R rewrites it. In the next section we shall show that external reduction works as a normalizing strategy for a class of left-linear overlapping TRSs, like needed reduction for orthogonal TRSs.

Consider a left-linear TRS R over $\mathcal{F}$.

Definition 4 (Outer redex). *A context $C[\ ]$ is outer if $C[\ ]$ has no redex occurrence Δ' such that $\Box \in \Delta'$. A redex occurrence Δ of $C[\Delta]$ is called outer if $C[\ ]$ is outer. The set of all outer contexts with respect to R is denoted by $OUT(R)$. An outer redex Δ of a term t is outermost if there exists no redex Δ' of t such that $\Delta \lhd \Delta'$.*

Definition 5 (External context). *An outer context $C[\ ]$ is external with respect to R if any s obtained by $C[\ ]\rightarrow_R^* s$ is an outer context. The set of all external contexts with respect to R is denoted by $EXT(R)$.*

If the hole in $C[\ ]$ is deleted or duplicated through a reduction then $C[\ ]$ is not external, since some non-outer context must arise previous to deletion or duplication of the hole.

Definition 6 (External redex). *Let Δ be a redex occurrence in $C[\Delta]$ such that $C[\ ]$ is external. Then the redex occurrence Δ is called external. If Δ is an external redex of $C[\Delta]$ then we write $C[\Delta_E]$; otherwise $C[\Delta_{NE}]$.*

The notion of externality for orthogonal TRSs originates with Huet and Lévy [8]. Externality for non-orthogonal TRSs is presented in Van Oostrom and De Vrier [19], which defines externality as a *reduction step* from a term whose residuals are not nested by other redexes. The definition in Van Oostrom and De Vrier is slightly more abstract than ours, but the two notions are externally same (see 9.2.3 in [19]). The following example is given in [19].

Example 1. Let $R = \{f(x,b)\rightarrow x, a\rightarrow b\}$. Then the context $f(a,\Box)$ is external but $f(\Box,a)$ is not, since $f(\Box,a)\rightarrow f(\Box,b) \notin OUT(R)$. Thus, in the term $f(a,a)$ the rightmost redex occurrence a is external but the leftmost occurrence a is not, i.e., $f(a_{NE},a_E)$.

From the definition of external redex it is obvious that in orthogonal TRSs any two external redex occurrences in a term must be disjoint. On the other hand, if a left-linear TRS is overlapping then two external redexes may be overlapping as follows.

Example 2. Let $R = \{p(s(x))\rightarrow x, s(p(x))\rightarrow x\}$. Then we have the overlapping external redexes $f(s(p(s(x))_E)_E)$ since $f(\Box)$ and $f(s(\Box))$ are external. Thus, external redexes need not be outermost [19].

One might think that overlapping redex occurrences always make overlapping external redexes if one of them is external, but this is not the case from the following example.

Example 3. Let $R = \{b\rightarrow c, f(b)\rightarrow c, g(f(x),c)\rightarrow x\}$. Then we have $g(f(b_{NE})_E, b_E)$. Note that two redex occurrences $f(b)$ and b are overlapping but the redex b occurring in $f(b)$ is not external since the context $g(f(\Box),b)$ is not external.

In a left-linear overlapping TRS, external redexes need not exist; for example, in $R = \{a \to b, f(b,x) \to c, f(x,b) \to c\}$ the reducible term $f(a,a)$ has no external redexes [19].

Definition 7 (External TRS). *A reduction $t \to^{\Delta} s$ is external if Δ is an external redex of t. We write $t \to_E s$ if there exists an external reduction $t \to^{\Delta} s$ for some external redex Δ; otherwise $t \to_{NE} s$. We say that R is external if for each term $t \notin NF$, t has an external redex.*

We shortly mention the relationship between *neededness* and *externality* of left-linear TRSs. For details of neededness and externality not treated here we refer to Van Oostrom and De Vrier [19]. The following definition of neededness is due to [6].

Definition 8 (Needed redex). *A context $C[\ ]$ is needed with respect to R if any s obtained by $C[\ ] \to_R^* s$ is not a normal form in $T(\mathcal{F}, \mathcal{V})$. A redex occurrence Δ in a term $C[\Delta]$ is called needed if $C[\ ]$ is needed. A reduction $t \to^{\Delta} s$ is needed if Δ is a needed redex of t.*

As external contexts have no normal forms in $T(\mathcal{F}, \mathcal{V})$, external redexes are (outermost) needed redexes; however, the revers need not hold. In Example 1, the leftmost redex occurrence a of the term $f(a,a)$ is (outermost) needed but not external [19]. For orthogonal TRSs we have the following properties of externality (neededness) [19].

- Any reducible term contains an external redex (a needed redex).
- External (needed) reduction is a normalizing strategy.
- Externality (neededness) of a redex is undecidable.

For a left-linear external TRS R, external reduction is a reduction strategy as every reducible term has an external redex. However, external reduction need not be a normalizing strategy if R is non-orthogonal. (See 9.2.4 in [19] too).

Example 4. Consider $R = \{a \to b, f(x) \to f(x), f(b) \to b\}$. Clearly, R is external. In the term $f(a)$ the outermost redex occurrence $f(a)$ is external but the innermost redex occurrence a not. Then, external reduction starting with $f(a)$ produces an infinite sequence $f(a) \to_E f(a) \to_E f(a) \to_E \cdots$. For normalizing $f(a) \to f(b) \to b$, we need a non-external reduction step $f(a) \to_{NE} f(b)$.

Externality of arbitrary left-linear TRSs is not decidable and external reduction is not computable in general. Hence, in order to obtain computable external reduction, we need to strengthen the notion of externality by decidable approximations. We address this problem in Section 8.

6 Normalization of External Reduction

We will now explain how to prove the normalizing property of external reduction for *overlapping* TRSs by using the BWCR Lemma. We first define *root balanced joinable* TRSs.

Root reduction $t{\to}_r s$ is defined as a reduction $t{\to}s$ contracted at the *root* position of t (i.e., $t{\to}^{\Delta}s$ and $\Delta \equiv t$).

Lemma 5. *Let $C[\Delta_E]$ for some Δ and let $t{\to}_r s$. Then $C[t]{\to}_E C[s]$.*

Proof. It is trivial from the definition of the root reduction. □

Definition 9. *A critical pair $\langle s, t\rangle$ is root balanced joinable if $s{\to}_r^k t'$ and $t{\to}_r^k t'$ for some t' and $k \geq 0$. A TRS R is root balanced joinable if every critical pair is root balanced joinable.*

In general it is undecidable whether a critical pair is root balanced joinable. The following example illustrates this problem.

Example 5. Consider a TRS R containing a constant b in normal forms and a ground term s such that reachability of root reduction $s{\to}^*{}_r b$ is undecidable. (Such a TRS R and a ground term s exist due to universal computation capability of TRSs; for example, see an encoding of Turing machine to a TRS in [22]). Let R' be $R \cup \{a{\to}s, a{\to}b, b{\to}b\}$ where a is a fresh constant. Then, the critical pair $\langle s, b\rangle$ of R' is root balanced joinable iff $s{\to}^*{}_r b$; this is undecidable.

Note that every weakly orthogonal TRS is trivially root balanced joinable since every critical pair is root balanced joinable with $k = 0$. We show that the root balanced joinability is sufficient to guarantee the balanced weak Church-Rosser property of left-linear TRSs.

Definition 10. *Let Δ and Δ' be two redex occurrences in a term t, and let $\Delta \equiv C[x_1\theta, \cdots, x_m\theta]$ where $C[x_1, \cdots, x_m]$ is the left-hand side of a rewrite rule and no variables occur in $C[\ ,\cdots,\ ]$. Then Δ and Δ' (or Δ' and Δ) are overlapping if $\Delta' \trianglelefteq \Delta$ and $\Delta' \ntrianglelefteq x_i\theta$ for any subterm occurrence $x_i\theta$.*

Lemma 6. *Let R be left-linear root balanced joinable. Let $t{\to}_E^{\Delta} t'$ and $t{\to}^{\Delta'} t''$, where $\Delta' \trianglelefteq \Delta$ and Δ and Δ' are overlapping. Then, we have $t'{\to}_E^k s$ and $t''{\to}_E^k s$ for some s and $k \geq 0$.*

Proof. Let $t \equiv C[\Delta] \equiv C[C'[\Delta']]$, $t' \equiv C[p]$, $t'' \equiv C[q]$. From the root balanced joinability of the critical pair concerning Δ and Δ', we have $p{\to}_r^k s'$ and $q{\to}_r^k s'$ for some s' and $k \geq 0$, similarly to the Critical Pair Lemma [1, 18, 22]. Thus, from $C[\Delta_E]$ and Lemma 5, it follows that $C[p]{\to}_E^k C[s']$ and $C[q]{\to}_E^k C[s']$. □

Lemma 7. *Let $C[\Delta_E, s]$. Then $C[\Delta_E, t]$ for any $s{\to}^* t$.*

Proof. Since $C[\Box, s]$ is external and $C[\Box, s]{\to}^* C[\Box, t]$, $C[\Box, t]$ is external. Thus we have $C[\Delta_E, t]$. □

Lemma 8. *Let R be left-linear root balanced joinable. Then external reduction ${\to}_E$ has the balanced weak Church-Rosser property.*

$$\begin{tikzcd}[column sep=huge, row sep=huge]
t \arrow[r, "\Delta'", "E"'] \arrow[d, "\Delta"', "E"] & t'' \arrow[d, dashed, "k"', "E"] \\
t' \arrow[r, dashed, "k", "E"'] & s
\end{tikzcd} \quad (k \geq 0)$$

Fig. 6.

Proof. Let $t \rightarrow_E^{\Delta} t'$, $t \rightarrow_E^{\Delta'} t''$. We shall show that $t' \rightarrow_E^{k} s$ and $t'' \rightarrow_E^{k} s$ for some s and $k \geq 0$ (Figure 6). If Δ and Δ' are disjoint, then from Lemma 7 the theorem clearly holds with $k = 1$. Assume that Δ and Δ' are not disjoint, say $\Delta' \trianglelefteq \Delta$. Then, Δ and Δ' must be overlapping as they both are external. Apply Lemma 6. $\square$

We next consider the relation between external reduction $\rightarrow_E$ and arbitrary reduction. Since external reduction $\rightarrow_E$ is balanced weakly Church-Rosser, the normalization of external reduction is obtained if we can apply Corollary 2 by taking $\rightarrow_E$ as $\rightarrow_s$. However, this is impossible as (ii) in the corollary is not satisfied because of duplication of redexes through reduction by a non-right-linear rewrite rule. To overcome this problem, we use *parallel reduction* of disjoint redexes.

Parallel reduction $t \mathrel{+\!\!\!\!+\!\!\!\!\rightarrow} s$ is defined by $t \equiv C[\Delta_1, \cdots, \Delta_n] \rightarrow^{\Delta_1} \cdots \rightarrow^{\Delta_n} s$ for some disjoint redexes $\Delta_1, \cdots, \Delta_n$ ($n \geq 0$). A parallel reduction $t \mathrel{+\!\!\!\!+\!\!\!\!\rightarrow} s$ is *proper* if $n > 0$, and we write $t \mathrel{+\!\!\!\!+\!\!\!\!\rightarrow}' s$. Since $\rightarrow$ and $\mathrel{+\!\!\!\!+\!\!\!\!\rightarrow}'$ have the same set of normal forms and $\rightarrow_E \subseteq \mathrel{+\!\!\!\!+\!\!\!\!\rightarrow}'$, it is obvious that $\rightarrow_E$ is a reduction strategy for $\rightarrow$ iff it is a reduction strategy for $\mathrel{+\!\!\!\!+\!\!\!\!\rightarrow}'$. In the following lemmas we use $\mathrel{+\!\!\!\!+\!\!\!\!\rightarrow}$ instead of $\mathrel{+\!\!\!\!+\!\!\!\!\rightarrow}'$ because of technical convenience.

Lemma 9. *Let R be left-linear root balanced joinable and external, and let $t \mathrel{+\!\!\!\!+\!\!\!\!\rightarrow} s$. Then $t =_E s$ or $t \leftarrow\!\!\twoheadrightarrow_E \cdot \mathrel{+\!\!\!\!+\!\!\!\!\rightarrow} \cdot \twoheadleftarrow\!\!\rightarrow_E s$.*

Proof. Let $t \mathrel{+\!\!\!\!+\!\!\!\!\rightarrow}^{\Delta_1 \cdots \Delta_n} s$ ($n \geq 0$). The proof is by induction on n. The case $n = 0$ is trivial as $t' \equiv t \equiv s \in NF$. Induction Step:

Case 1: Some Δ_i, say without loss of generality Δ_1, is external. We have $t \rightarrow_E^{\Delta_1} t' \mathrel{+\!\!\!\!+\!\!\!\!\rightarrow}^{\Delta_2 \cdots \Delta_n} s$. By applying the induction hypothesis to $t' \mathrel{+\!\!\!\!+\!\!\!\!\rightarrow}^{\Delta_2 \cdots \Delta_n} s$, we obtain the lemma.

Case 2: No Δ_i is external. From externality there must exist an external redex, say Δ, in t. Let $t \rightarrow_E^{\Delta} t''$ and consider the following two cases.

Case 2-1: Δ and Δ_i ($i = 1 \cdots n$) are non-overlapping. By using left-linearity of R, we can easily show that $t'' \mathrel{+\!\!\!\!+\!\!\!\!\rightarrow} s'$ and $s \rightarrow_E s'$ for some s' (Figure 7).

Case 2-2: Δ and some Δ_i, say without loss of generality Δ_1, are overlapping. Let $t \rightarrow_{NE}^{\Delta_1} t' \mathrel{+\!\!\!\!+\!\!\!\!\rightarrow}^{\Delta_2 \cdots \Delta_n} s$. Note that $\Delta_1 \trianglelefteq \Delta$. From Lemma 6, it follows that

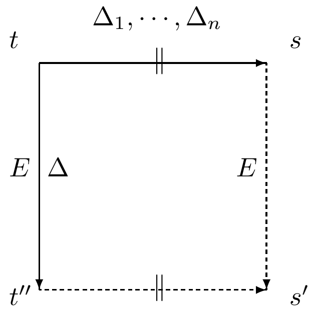

Fig. 7.

$t'' \to_E^k s'$ and $t' \to_E^k s'$ for some s' and $k \geq 0$ (Figure 8). Thus, we can obtain $t \longleftrightarrow\!\!\!\!\twoheadrightarrow_E t'$. Apply the induction hypothesis to $t' \mathrel{-\!\!\!\|\!\!\!\rightarrow}^{\Delta_2 \cdots \Delta_n} s$. $\square$

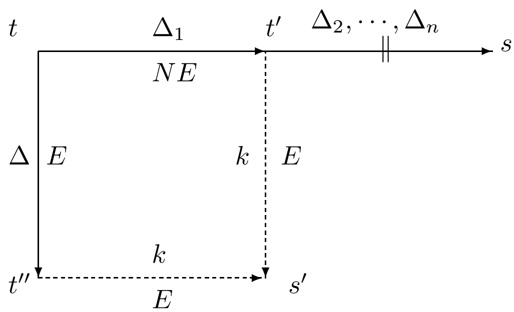

Fig. 8.

Theorem 1. *Let a TRS R be left-linear root balanced joinable and external. Then, R has the normal form property, and external reduction $\to_E$ is a normalizing strategy.*

Proof. Note that by externality, we have $NF = NF_E$ (NF_E denotes the set of the normal forms concerning $\to_E$). Thus, $\to_E$ is a reduction strategy for $\to$, and also for $\mathrel{-\!\!\!\|\!\!\!\rightarrow}'$. From Lemma 8 $\to_E$ is BWCR. From Lemma 9 it follows that if $t \mathrel{-\!\!\!\|\!\!\!\rightarrow}' s$ then $t =_E s$ or $t \longleftrightarrow\!\!\!\!\twoheadrightarrow_E \cdot \mathrel{-\!\!\!\|\!\!\!\rightarrow}' \cdot \twoheadleftarrow\!\!\!\!\longleftrightarrow_E s$. Taking $\to_E$ as $\to_s$ and $\mathrel{-\!\!\!\|\!\!\!\rightarrow}'$ as $\to$ respectively, we can apply Corollary 2. Thus, $\mathrel{-\!\!\!\|\!\!\!\rightarrow}'$ has the normal form property and $\to_E$ is a normalizing strategy for $\mathrel{-\!\!\!\|\!\!\!\rightarrow}'$. From $\to^+ = \mathrel{-\!\!\!\|\!\!\!\rightarrow}'^+$, the theorem follows. $\square$

We remark that in Definition 9 *root reduction* imposed for balanced joinability of critical pairs can be relaxed to *stably external reduction*. The notion of *stable externality* was considered first as *stable index* by Nagaya, Sakai and Toyama [14].

An external context $C[\ ]$ in $EXT(R)$ is *stable* if $C'[C[\ ]\theta]$ is in $EXT(R)$ for any $C'[\ ]$ in $EXT(R)$ and substitution θ. A redex occurrence Δ in $C[\Delta]$ is called *stably external* if $C[\ ]$ is stably external. A reduction $t \to^{\Delta} s$ is *stably external* if Δ is a stably external redex of t. (Note that root reduction is clearly stably external as the hole $\Box$ is a stably external context; thus, root redaction can be viewed as a special case of stably external reduction.) By replacing *root reduction* in Definition 9 with *stably external reduction*, we define *stable balanced joinability.* Then, all the proofs relied on *root balanced joinability* work also for *stable balanced joinability.* For example, Theorem 1 is improved by replacing "*root balanced joinable*" with "*stable balanced joinable*". Unfortunately, stably external redexes are not decidable; thus, appropriate decidable approximations of them are necessary for computable normalizing strategy. For decidable approximations of stable index reduction, we refer to [14].

7 Normalization of Quasi-External Reduction

This section improves the result of Theorem 1 by extending external reduction to quasi-external reduction; that is, there exist no infinite reduction sequences starting with a term having a normal form in which infinitely many external redexes are contracted. *Quasi-external reduction* (or *hyper-external reduction* [19]) is defined as $\to^*_{NE} \cdot \to_E \cdot \to^*_{NE}$ [22]. We first prove the next lemma.

Lemma 10. *Let R be left-linear root balanced joinable and external. Let $t \to^n_E s \in NF$ for some $n \geq 0$ and $t \to^* t'$. Then, we have $t' \to^m_E s$ for some $m \leq n$ (Figure 9).*

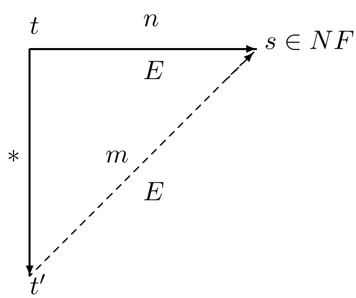

Fig. 9.

Proof. The proof is by induction on n. The case $n = 0$ is trivial as $t \equiv s \in NF$. Induction Step: We first prove the following claim: if $t \to^n_E s \in NF$ and $t \to t'$ then $t' \to^m_E s$ for some $m \leq n$ (Figure 10).

Proof of the claim. Let $t \to^{\Delta}_E t'' \to^{n-1}_E s \in NF$ and $t \to^{\Delta'} t'$ (Figure 11).

Case 1: Δ and Δ' are non-overlapping. By left-linearity of R, we can easily show that $t' \to_E s'$ and $t'' \to^* s'$ for some s'. From the induction hypothesis, it follows that $s' \to^{m'}_E s$ for some $m' \leq n-1$. Thus, we obtain $t' \to^{m'+1}_E s$.

Case 2: Δ and Δ' are overlapping. By using Lemma 6, we have $t' \to_E^i s'$ and $t'' \to_E^i s'$ for some s' and $i \geq 0$. Applying the BWCR Lemma to t'', we have $s' \to_E^{n-1-i} s$. Thus, it holds that $t' \to_E^{n-1} s$. Therefore, the claim follows.

We next prove that if $t \to_E^n s \in NF$ and $t \to^k t'$ then $t' \to_E^m s$ for some $m \leq n$. The proof is by induction on k. The case $k = 0$ is trivial. Induction Step: Let $t \to \hat{t} \to^{k-1} t'$. From the claim we have $\hat{t} \to_E^{n'} s$ for some $n' \leq n$. Thus, from the induction hypothesis with respect to k in case $n' = n$ or with respect to n in case $n' < n$, it follows that $t' \to_E^m s$ for some $m \leq n$. □

Theorem 2. *Let a TRS R be left-linear root balanced joinable and external. Then quasi-external reduction $\to_{NE}^* \cdot \to_E \cdot \to_{NE}^*$ is a normalizing strategy.*

Proof. Let t have a normal form s. Then by Theorem 1 we have $t \to_E^n s$ for some n. By using induction on n we prove that every quasi-external reduction starting with t is normalizing. The case $n = 0$ is trivial as $t \equiv s$. Let $t \to_E^n s$ $(n > 0)$. Take any one-step quasi-external reduction starting with t, say $t \to_{NE}^* t' \to_E t'' \to_{NE}^* \hat{t}$. From Lemma 10 we have $t' \to_E^{n'} s$ for some $n' \leq n$. Thus, by applying the BWCR Lemma to t' we obtain $t'' \to_E^{n''} s$ for some $n'' < n$ as $n'' + 1 = n'$. Again from Lemma 10 it holds that $\hat{t} \to_E^m s$ for some $m \leq n''$. From $m < n$ and the induction hypothesis it follows that every quasi-external reduction starting with $\hat{t}$ is normalizing. Therefore the theorem holds. □

8 Decidable Approximations of Externality

In this section we address the problem to find *decidable approximations* of external reduction. Durand and Middeldorp [6] presented a simple framework of decidable approximations to show normalizing strategies of orthogonal TRSs. We adapt this framework to *left-linear overlapping* (i.e., *non-orthogonal*) TRSs, based on the notions of *balanced weak Church-Rosser property* and *externality*. The framework of decidable approximations presented in [6] heavily relies on *tree automata* techniques. We first recall the basic notions concerning tree automata [4].

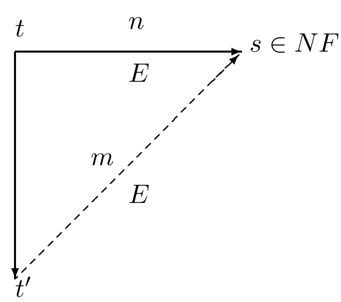

Fig. 10.

$$
\begin{tikzcd}
t \arrow[r, "\Delta", "E"'] \arrow[dd, "\Delta'"'] & t'' \arrow[r, "n-1", "E"'] & s \in NF \\
 & & \\
t' \arrow[uurr, dashed, "m", "E"'] & &
\end{tikzcd}
$$

Fig. 11.

A *tree automaton* is a tuple $\mathcal{A} = (\mathcal{F}, Q, Q^f, \Pi)$ where $\mathcal{F}$ is a finite set of function symbols, Q is a finite set of *states*, $Q^f \subseteq Q$ is a set of *final* states and Π is a set of ground rewrite rules of the form $f(q_1, \ldots, q_n) \rightarrow q$ or $q \rightarrow q'$ where $f \in \mathcal{F}$, $q_1, \ldots, q_n, q, q' \in Q$. We use $\rightarrow_{\mathcal{A}}$ for the reduction relation $\rightarrow_{\Pi}$ on $T(\mathcal{F} \cup Q)$. A term $t \in T(\mathcal{F})$ is *accepted* by $\mathcal{A}$ if $t \rightarrow^*_{\mathcal{A}} q$ for some $q \in Q^f$. The tree language $L(\mathcal{A})$ recognized by $\mathcal{A}$ is the set of all terms accepted by $\mathcal{A}$. A set L is *regular* if there exists a tree automaton $\mathcal{A}$ such that $L = L(\mathcal{A})$. The following properties of tree automata are well-known [4]:

- The class of regular languages is effectively closed under union, intersection, difference, and complementation.
- The *membership* and the *emptiness* problems for regular languages are decidable.

Consider a TRS R over $\mathcal{F}$. We denote the set of all normal forms of R in $T(\mathcal{F}, \mathcal{V})$ by $NF(R)$ and the set of all redexes of R in $T(\mathcal{F}, \mathcal{V})$ by $RED(R)$. We introduce a fresh constant $\circ \notin \mathcal{F}$ and let $\mathcal{F}^\circ = \mathcal{F} \cup \{\circ\}$. We view R as a TRS over $\mathcal{F}^\circ$ when a reduction relation on $T(\mathcal{F}^\circ)$ is considered. Note that $NF(R)$ and $RED(R)$ have no terms containing $\circ$ since they are defined as subsets of $T(\mathcal{F}, \mathcal{V})$.

Let t° denote the term in $T(\mathcal{F}^\circ)$ obtained from a term $t \in T(\mathcal{F}, \mathcal{V})$ by replacing each variable in t with the constant $\circ$. We write $T^\circ = \{t^\circ \mid t \in T\}$ for a set $T \subseteq T(\mathcal{F}, \mathcal{V})$. We say a term set $T \subseteq T(\mathcal{F}, \mathcal{V})$ is *variable insensitive* if, for all $t \in T(\mathcal{F}, \mathcal{V})$, $t \in T$ iff $t^\circ \in T^\circ$. These notions are naturally extended to contexts over $\mathcal{F}$. Note that if T is a variable insensitive set of terms (or contexts) over $\mathcal{F}$ and T° is regular then T is decidable.

Lemma 11. *Let R be a left-linear TRS. Then, $RED(R)$, $NF(R)$ and $OUT(R)$ are variable insensitive. Moreover, $NF(R)^\circ$, $RED(R)^\circ$ and $OUT(R)^\circ$ are regular.*

Proof. From left-linearity of R it holds that s is a redex iff s° is a redex for any $s \in T(\mathcal{F}, \mathcal{V})$. Thus, $RED(R)$ is variable insensitive. Similarly we can show that $NF(R)$ and $OUT(R)$ are variable insensitive. From [4] it is clear that $NF(R)^\circ$, $RED(R)^\circ$ and $OUT(R)^\circ$ are regular. $\square$

Definition 11 (Externality approximation mapping). *An externality approximation mapping α is a function mapping from a TRS R to a set of contexts such that $\alpha(R) \subseteq EXT(R)$. We say that α is decidable if $\alpha(R)$ is decidable for all R, and α is regular if, for all R, (i) $\alpha(R)^\circ$ is regular and (ii) $\alpha(R)$ is variable insensitive.*

Note that if an externality approximation mapping α is regular then it is decidable.

Definition 12 (α-external TRS). *We say a context $C[\,]$ is α-external with respect to R if $C[\,] \in \alpha(R)$. A redex occurrence Δ in $C[\Delta]$ is called α-external if $C[\,]$ is α-external. A reduction $t \to^{\Delta} s$ is α-external if Δ is an α-external redex of t. We say that a TRS R is α-external if each term $t \in T(\mathcal{F}, \mathcal{V}) - NF(R)$ has an α-external redex (i.e., an α-external reduction of R is a reduction strategy).*

As an *α-external redex occurrence Δ* is an *external redex* and an *α-external* TRS R is *external*, we have the following theorem.

Theorem 3. *Let α be an externality approximation mapping (resp. an externality regular approximation mapping), and let a TRS R be left-linear root balanced joinable and α-external. Then, R has the normal form property, and α-external reduction is a normalizing strategy (resp. a computable normalizing strategy).*

Proof. From Theorem 1 it is trivial. □

The following theorem shows that the class of α-external TRSs is decidable if α is regular.

Theorem 4. *Let α be an externality regular approximation mapping. Then it is decidable whether a left-linear TRS R is α-external.*

Proof. Let R be a left-linear TRS over $\mathcal{F}$. Let $L = \{\, C[\Delta] \mid \Delta \in RED(R)$ and $C[\,] \in \alpha(R)\}$. Since $RED(R)$ and $\alpha(R)$ are variable insensible, L is variable insensible and we can write $L^\circ = \{\, C'[\Delta'] \mid \Delta' \in RED(R)^\circ$ and $C'[\,] \in \alpha(R)^\circ\}$. Since $RED(R)^\circ$ and $\alpha(R)^\circ$ are regular, there exist two tree automata $\mathcal{A}_{red} = (\mathcal{F}^\circ, Q_{red}, {Q^f}_{red}, \Pi_{red})$ and $\mathcal{A}_\alpha = (\mathcal{F}^\circ \cup \{\square\}, Q_\alpha, {Q^f}_\alpha, \Pi_\alpha)$, where $Q_{red} \cap Q_\alpha = \varnothing$, which recognize $RED(R)^\circ$ and $\alpha(R)^\circ$ respectively. Without loss of generality we may suppose ${Q^f}_{red} = \{q_{red}\}$ and ${Q^f}_\alpha = \{q_\alpha\}$ [4]. Let $\mathcal{A} = (\mathcal{F}^\circ, Q_{red} \cup Q_\alpha, \{q_\alpha\}, \Pi_L)$ where $\Pi_L = \Pi_{red} \cup (\Pi_\alpha - \{\square \to p \mid \square \to p \in \Pi_\alpha\}) \cup \{q_{red} \to p \mid \square \to p \in \Pi_\alpha\}$. Then it can be shown that $L^\circ = L(\mathcal{A})$; thus L° is regular. From the definition of α-externality, the TRS R is α-external iff $(T(\mathcal{F}, \mathcal{V}) - NF(R)) - L = \varnothing$. Since $T(\mathcal{F}, \mathcal{V})$, $NF(R)$ and L are variable insensitive, we have that $(T(\mathcal{F}, \mathcal{V}) - NF(R)) - L$ is variable insensitive. Thus, it holds that $(T(\mathcal{F}, \mathcal{V}) - NF(R)) - L = \varnothing$ iff $(T(\mathcal{F}^\circ) - NF(R)^\circ) - L^\circ = \varnothing$. Since $(T(\mathcal{F}^\circ) - NF(R)^\circ) - L^\circ$ is regular, the emptiness of $(T(\mathcal{F}^\circ) - NF(R)^\circ) - L^\circ$ is decidable. □

We next consider *decidable approximations* of TRSs to find an *externality regular approximation mapping* α. An *extended* TRS (eTRS for short) R over $\mathcal{F}$ is a finite set of *extended* rewrite rules $l{\rightarrow}r$ in which the right-hand side r may have *extra variables* not occurring in the left-hand side l. Similarly to TRSs, we can view R as an eTRS over $\mathcal{F}^\circ \cup \{\Box\}$ when a reduction relation on $T(\mathcal{F}^\circ \cup \{\Box\})$ is considered, imposing the restriction that when an extended rewrite rule $l \rightarrow r$ is applied, the extra variables in r are instantiated by arbitrary terms in $T(\mathcal{F}^\circ)$; thus, a reduction does not generate new holes.

Following Durand and Middeldorp [6], we say that an eTRS R over $\mathcal{F}$ is *regularity preserving* if $(R^{-1})^*(L) = \{\ t \mid \exists s \in L\ t{\rightarrow}^*_R s\ \}$ is regular for every regular tree language L over $\mathcal{F}$ (resp. $\mathcal{F}^\circ \cup \{\Box\}$), where $\rightarrow_R$ is the reduction relation on $T(\mathcal{F})$ (resp. $T(\mathcal{F}^\circ \cup \{\Box\})$) defined by R.

An eTRS R_a over $\mathcal{F}$ is an *approximation* of a TRS R over $\mathcal{F}$ if $\rightarrow^*_R\ \subseteq\ \rightarrow^*_{R_a}$. Note that the definition of approximation is slightly different from that in [6] which imposes the extra condition $NF(R) = NF(R_a)$. The following definition is due to [6].

Definition 13 (Regularity preserving approximation mapping). *A regularity preserving approximation mapping τ is a function mapping from a left-linear TRS R to a left-linear eTRS $\tau(R)$ such that (i) $\tau(R)$ is an approximation of R and (ii) $\tau(R)$ is regularity preserving.*

Definition 14 (τ-external context). *Let τ be a regularity preserving approximation mapping. An outer context $C[\ ]$ is τ-external with respect to R if any s obtained by $C[\ ]{\rightarrow}^*_{\tau(R)} s$ is in $OUT(R)$. The set of all τ-external contexts with respect to R is denoted by $\alpha_\tau(R)$.*

Lemma 12. *Let τ be a regularity preserving approximation mapping and R a left-linear TRS over $\mathcal{F}$. Let $s \in T(\mathcal{F} \cup \{\Box\}, \mathcal{V})$ and $s^\circ{\rightarrow}^*_{\tau(R)} u$. Then there exists some $t \in T(\mathcal{F} \cup \{\Box\}, \mathcal{V})$ such that $s{\rightarrow}^*_{\tau(R)} t$ and $t^\circ \equiv u$.*

Proof. Let $s^\circ{\rightarrow}^k_{\tau(R)} u$. We prove the claim by induction on k. The case $k = 0$ is trivial. Induction Step: Let $s^\circ{\rightarrow}_{\tau(R)} p{\rightarrow}^{k-1}_{\tau(R)} u$. From left-linearity of $\tau(R)$, there exists some $q \in T(\mathcal{F} \cup \{\Box\}, \mathcal{V})$ such that $s{\rightarrow}_{\tau(R)} q$ and $q^\circ \equiv p$. Thus, form $q^\circ{\rightarrow}^{k-1}_{\tau(R)} u$ and the induction hypothesis, we have $t \in T(\mathcal{F} \cup \{\Box\}, \mathcal{V})$ such that $q{\rightarrow}^*_{\tau(R)} t$ and $t^\circ \equiv u$. $\Box$

Lemma 13. *Let τ be a regularity preserving approximation mapping and R a left-linear TRS over $\mathcal{F}$. Then $\alpha_\tau(R)$ is variable insensitive.*

Proof. We show that $C[\,] \in \alpha_\tau(R)$ iff $C[\,]^\circ \in \alpha_\tau(R)^\circ$. *If-part*: Let $C[\,]^\circ \in \alpha_\tau(R)^\circ$ and $C[\ ]{\rightarrow}^*_{\tau(R)} s$. Then we have $C[\ ]^\circ{\rightarrow}^*_{\tau(R)} s^\circ$. Since $C[\ ]^\circ \in \alpha_\tau(R)^\circ$, it holds that $s^\circ \in OUT(R)^\circ$. As $OUT(R)$ is variable insensitive, $s \in OUT(R)$. Thus, $C[\ ] \in \alpha_\tau(R)$. *Only-if-part*: Let $C[\ ] \in \alpha_\tau(R)$ and $C[\ ]^\circ{\rightarrow}^*_{\tau(R)} u$. By Lemma 12 there exists some t such that $C[\,]{\rightarrow}^*_{\tau(R)} t$ and $t^\circ \equiv u$. Since $C[\,] \in \alpha_\tau(R)$, it holds that $t \in OUT(R)$. As $OUT(R)$ is variable insensitive, $u \in OUT(R)^\circ$. Thus, $C[\ ]^\circ \in \alpha_\tau(R)^\circ$. $\Box$

Lemma 14. *Let τ be a regularity preserving approximation mapping. Then α_τ is an externality regular approximation mapping. Hence, $\alpha_\tau(R)$ is a computable approximation of $EXT(R)$.*

Proof. Since $\to_R^* \subseteq \to_{\tau(R)}^*$, it is trivial that $\alpha_\tau(R) \subseteq EXT(R)$. By Lemma 13, $\alpha_\tau(R)$ is variable insensitive. Hence, we shall show that $\alpha_\tau(R)^\circ$ is regular. Let $CONT$ be the set of all contexts over $\mathcal{F}$ containing precisely one hole and let $NOUT(R) = CONT - OUT(R)$. Then we have $NOUT(R)^\circ = CONT^\circ - OUT(R)^\circ$. Since $CONT^\circ$ and $OUT(R)^\circ$ are regular, $NOUT(R)^\circ$ is regular. Let $L_\tau = \{s \in T(\mathcal{F} \cup \{\Box\}, \mathcal{V}) \mid \exists C[\,] \in NOUT(R),\ s \to_{\tau(R)}^* C[\,]\}$. Then, by Lemma 12 it is easily shown that $s \to_{\tau(R)}^* C[\ ]$ for some $C[\ ] \in NOUT(R)$ iff $s^\circ \to_{\tau(R)}^* C'[\ ]$ for some $C'[\,] \in NOUT(R)^\circ$. Thus, we have $L_\tau^\circ = \{s' \in T(\mathcal{F}^\circ \cup \{\Box\}) \mid \exists C'[\,] \in NOUT(R)^\circ,\ s' \to_{\tau(R)}^* C'[\,]\}$. Since $\tau(R)$ is regularity preserving, we have $L_\tau^\circ = (\tau(R)^{-1})^*(NOUT(R)^\circ)$; thus, L_τ° is regular. From $\alpha_\tau(R)^\circ = CONT^\circ - L_\tau^\circ$, it follows that $\alpha_\tau(R)^\circ$ is regular. $\Box$

Definition 15 (τ-external TRS). *We say $C[\ ]$ is τ-external with respect to R if $C[\ ] \in \alpha_\tau(R)$. A redex occurrence Δ in $C[\Delta]$ is called τ-external if $C[\ ]$ is τ-external. A reduction $t \to^\Delta s$ is τ-external if Δ is a τ-external redex of t. We say that a TRS R is τ-external if each term $t \in T(\mathcal{F}, \mathcal{V}) - NF(R)$ has a τ-external redex.*

Theorem 5. *Let τ be a regularity preserving approximation mapping and let a TRS R be left-linear root balanced joinable and τ-external. Then, R has the normal form property, and τ-external reduction is a computable normalizing strategy.*

Proof. From Theorem 3 and Lemma 14 it is clear. $\Box$

The following theorem shows that the class of τ-external TRSs is decidable.

Theorem 6. *Let τ be a regularity preserving approximation mapping. Then it is decidable whether a left-linear TRS R is τ-external.*

Proof. From Theorem 4 and Lemma 14 it is clear. $\Box$

The first idea of regularity preserving approximations was proposed by Huet and Lévy [8] as the *strong* approximation of orthogonal TRSs, which is obtained by replacing the right-hand side of every rewrite rule with a fresh variable not occurring in the left-hand side. Oyamaguchi [21] gave a better approximation, the *NV* approximation, which is obtained by replacing all variables in the right-hand side of every rewrite rule with distinct fresh variables. Jacquemard [9], Nagaya and Toyama [15] introduced the *growing* approximation, which is obtained by replacing all variables in the *left-hand* sides of every rewrite rule that occur at a depth greater than 1 with distinct fresh variables [15]. In these approximations, the regularity preserving property depends only on *left-linearity*, but not on *orthogonality* [9, 15, 6]. Thus, we can use them as regularity preserving approximations for arbitrary left-linear TRSs.

An approximation mapping τ is *strong* (resp. *NV*, *growing*) if $\tau(\mathcal{R})$ is a strong (resp. *NV*, *growing*) approximation of $\mathcal{R}$ for every TRS $\mathcal{R}$. Then, from Theorems 5 and 6, the following corollaries hold.

Corollary 4. *Let a TRS R be left-linear root balanced joinable and strong-external (resp. NV-external, growing-external). Then, R has the normal form property, and strong-external (resp. NV-external, growing-external) reduction is a computable normalizing strategy.*

Corollary 5. *It is decidable whether a left-linear TRS R is strong-external (resp. NV-external, growing-external).*

Example 6. Let R be combinatory logic CL $\cup$

$$\begin{cases} f(g(x, K), K) \to (K \cdot ((K \cdot x) \cdot x)) \cdot x \\ f(g(x, K), K) \to ((S \cdot K) \cdot x) \cdot x \\ f(g(K, y), S) \to g(y, y) \\ g(S, S) \to S. \end{cases}$$

As R has overlapping redexes at the root of $f(g(x, K), K)$, we obtain the critical pair $\langle (K \cdot ((K \cdot x) \cdot x)) \cdot x,\ ((S \cdot K) \cdot x) \cdot x \rangle$. The critical pair meets by root reductions $(K \cdot ((K \cdot x) \cdot x)) \cdot x \to_r (K \cdot x) \cdot x \to_r x$ and $((S \cdot K) \cdot x) \cdot x \to_r (K \cdot x) \cdot (x \cdot x) \to_r x$. Thus R is root balanced joinable. Let the strong approximation of combinatory logic CL be $\tau(\text{CL})$:

$$\begin{cases} ((S \cdot x) \cdot y) \cdot z \to w \\ (K \cdot x) \cdot y \to z. \end{cases}$$

Then the strong approximation $\tau(R)$ of R is $\tau(\text{CL}) \cup$

$$\begin{cases} f(g(x, K), K) \to z \\ f(g(x, K), K) \to z \\ f(g(K, y), S) \to z \\ g(S, S) \to z. \end{cases}$$

Since R is *transitive* [14, 23] (*forward-branching* [5]), it is strong-external. Thus, from Corollary 4, R has the normal form property, and strong-external reduction is a computable normalizing strategy. Consider a term of the form $f(g(\Delta_1, \Delta_2), \Delta_3)$ in R, where Δ_i $(i = 1, 2, 3)$ are redex occurrences. Then neither Δ_1 nor Δ_2 is a strong-external redex, as $f(g(\Box, \Delta_2), \Delta_3) \to_{\tau(R)} f(g(\Box, K), \Delta_3) \to_{\tau(R)} f(g(\Box, K), K) \notin OUT(R)$ and $f(g(\Delta_1, \Box), \Delta_3) \to_{\tau(R)} f(g(K, \Box), \Delta_3) \to_{\tau(R)} f(g(K, \Box), S) \notin OUT(R)$ respectively. The rightmost redex occurrence Δ_3 is strong-external since one can easily check $f(g(\Delta_1, \Delta_2), \Box) \to^*_{\tau(R)} s \in OUT(R)$ for any term s.

Example 7. Let R be combinatory logic CL $\cup$

$$\begin{cases} f(x, S) \to x \cdot S \\ f(S, y) \to S \cdot y \\ f(x, y) \to x \cdot y. \end{cases}$$

Since R is weakly orthogonal, it is trivially root balanced joinable. Let the strong approximation $\tau(R)$ of R be $\tau(\mathrm{CL})\cup$

$$\begin{cases} f(x,S) \to z \\ f(S,y) \to z \\ f(x,y) \to z. \end{cases}$$

Since a term of the form $f(s,t)$ in R certainly gives a redex independent on s and t, one can easily check strong-externality of R, ignoring the first two rules $f(x,S) \to z$ and $f(S,y) \to z$. Thus, from Corollary 4, R has the normal form property, and strong-external reduction is a computable normalizing strategy. Note that if the third rule $f(x,y) \to z$ does not exist, then R is not strong-external as $f(\Delta_1, \Delta_2)$ has no strong-external redexes.

9 Left-Normal Systems

In this section we discuss a syntactic characterization of *external overlapping* TRSs. Such a syntactical characterization was found by O'Donnell [17] for orthogonal TRSs. He proved that if an orthogonal TRS R is *left-normal* then *leftmost-outermost reduction* is normalizing. We show that his result can be naturally extended to root balanced joinable TRSs.

Definition 16. *The set $T_L(\mathcal{F},\mathcal{V})$ of the left-normal terms constructed from $\mathcal{F}$ and $\mathcal{V}$ is inductively defined as follows:*

1. $x \in T_L(\mathcal{F},\mathcal{V})$ **if** $x \in \mathcal{V}$,
2. $f(t_1,\cdots,t_{p-1},s_p,x_{p+1}\cdots,x_n) \in T_L(\mathcal{F},\mathcal{V}) \quad (0 \le p \le n)$
   if $f \in \mathcal{F}$, $t_1,\cdots,t_{p-1} \in T(\mathcal{F})$, $s_p \in T_L(\mathcal{F},\mathcal{V})$, $x_{p+1},\cdots,x_n \in \mathcal{V}$, *and* $f(t_1,\cdots,t_{p-1},s_p,x_{p+1}\cdots,x_n)$ *is linear.*

A TRS R over $\mathcal{F}$ is *left-normal* [8, 17, 22] if for any rule $l \to r$ in R, l is a left-normal term in $T_L(\mathcal{F},\mathcal{V})$. From the definition of left-normal terms, a left-normal TRS R is left-linear, and it may be overlapping.

Definition 17 (Left-outer context). *A context $C[\,]$ is left-outer if every redex Δ' of $C[\,]$ occurs right of $\Box$ (i.e., $C[\,] \equiv C'[\Box, \Delta']$ for some $C'[\,,\,]$) whenever it exists.*

Definition 18 (Left-outer redex). *A redex occurrence Δ of $C[\Delta]$ is called left-outer if $C[\,]$ is left-outer. A reduction $t \to^{\Delta} s$ is left-outer if Δ is a left-outer redex of t.*

Let $\alpha_L(R)$ be the set of all left-outer contexts with respect to a TRS R. Then the decidability of $\alpha_L(R)$ is trivial. We shall show that $\alpha_L(R)$ is a decidable approximation of $EXT(R)$ if R is left-normal.

Lemma 15. *Let a TRS R be left-normal. If $C[\,]$ is left-outer and $C[\,] \to^* s$, then:*

(1) *s is a left-outer context,*
(2) *for any $t \in T_L(\mathcal{F},\mathcal{V})$, if $t \preceq s$ then $t \preceq C[\,]$.*

Proof. By induction on the size of $C[\,]$, we will prove (1) and (2) simultaneously.

Basic step: $C[\,] \equiv \Box$. Then (1) and (2) are trivial.

Induction step: Since $C[\,] \not\equiv \Box$, we can write $C[\,] \equiv f(t_1, \cdots, t_{p-1}, C_p[\,], t_{p+1}, \cdots, t_n)$, where $t_1, \cdots, t_{p-1}$ are normal forms and $C_p[\,]$ is left-outer.

(1) Suppose that s is not left-outer. As $C[\,]$ is a left-outer context, there exists some non-left-outer context $\tilde{C}[\,] \equiv f(t_1, \cdots, t_{p-1}, \tilde{C}_p[\,], \tilde{t}_{p+1}, \cdots, \tilde{t}_n)$ such that $C[\,] \rightarrow^* \tilde{C}[\,] \rightarrow^* s$ where $C_p[\,] \rightarrow^* \tilde{C}_p[\,]$ and $t_{p+1} \rightarrow^* \tilde{t}_{p+1}, \cdots, t_n \rightarrow^* \tilde{t}_n$. From the induction hypothesis with respect to (1) and $C_p[\,] \rightarrow^* \tilde{C}_p[\,]$, $\tilde{C}_p[\,]$ is left-outer. Since $\tilde{C}[\,]$ is not left-outer, there exists a redex Δ such that $\Box \in \Delta \trianglelefteq \tilde{C}[\,]$. As $\tilde{C}_p[\,]$ is left-outer, $\Delta \not\trianglelefteq \tilde{C}_p[\,]$. Thus, we have $\Delta \equiv \tilde{C}[\,]$. Hence, $l \preceq \tilde{C}[\,] \equiv f(t_1, \cdots, t_{p-1}, \tilde{C}_p[\,], \tilde{t}_{p+1}, \cdots, \tilde{t}_n)$ for some $l \rightarrow r \in R$ such that $l \equiv f(t_1, \cdots, t_{q-1}, s_q, x_{q+1}, \cdots, x_n) \in T_L(\mathcal{F}, \mathcal{V})$. Since $C[\,]$ is left-outer, it holds that $l \not\preceq C[\,]$; thus, $q \geq p$. Since $t_i \not\preceq \tilde{C}_p[\,]$ for any ground term t_i, $q \leq p$ holds. So we have $p = q$. From the induction hypothesis with respect to (2) and $s_p \preceq \tilde{C}_p[\,]$, we have $s_p \preceq C_p[\,]$. Thus, $f(t_1, \cdots, t_{p-1}, s_p, x_{p+1}, \cdots, x_n) \preceq f(t_1, \cdots, t_{p-1}, C_p[\,], t_{p+1}, \cdots, t_n) \equiv C[\,]$; it contradicts to the fact that $C[\,]$ is left-outer. Hence, s must be left-outer.

(2) From (1) it follows that every s' must be left-outer for $C[\,] \rightarrow^* s' \rightarrow^* s$; thus, we can write $s \equiv f(t_1, \cdots, t_{p-1}, C'_p[\,], t'_{p+1}, \cdots, t'_n)$ where $C_p[\,] \rightarrow^* C'_p[\,]$ and $t_{p+1} \rightarrow^* t'_{p+1}, \cdots, t_n \rightarrow^* t'_n$. Let $t \preceq s$ for some $t \equiv f(t_1, \cdots, t_{q-1}, s_q, x_{q+1}, \cdots, x_n) \in T_L(\mathcal{F}, \mathcal{V})$. If $q < p$ then it is clear that $t \preceq C[\,]$. If $q = p$ then $s_q \equiv s_p \preceq C'_p[\,]$. From the induction hypothesis with respect to (2) we have $s_p \preceq C_p[\,]$. Thus, $t \preceq C[\,]$. $\Box$

Lemma 16. *Let a TRS R be left-normal. Then $\alpha_L(R) \subseteq EXT(R)$.*

Proof. Note that the left-outer contexts are outer. Thus from Lemma 15 (1) the left-outer contexts are external. $\Box$

Thus, α_L is an externality decidable approximation mapping for the class of left-normal TRSs.

Lemma 17. *Let a TRS R be left-normal. Then R is α_L-external (i.e., every reducible term has a left-outer redex).*

Proof. Trivial. $\Box$

Theorem 7. *Let a TRS R be root balanced joinable and left-normal. Then, R has the normal form property, and left-outer reduction is a computable normalizing strategy.*

Proof. It follows from Theorem 1, Lemmas 16 and 17. $\Box$

Definition 19 (Leftmost-outermost redex). *A redex occurrence Δ of t is called* leftmost-outermost *if Δ is the leftmost of the outermost redexes of t. A reduction $t \rightarrow^{\Delta} s$ is leftmost-outermost if Δ is a leftmost-outermost redex of t.*

As *leftmost-outermost redexes* are clearly left-outer redexes, we have the following corollary.

Corollary 6. *Let a TRS R be root balanced joinable and left-normal. Then, R has the normal form property, and leftmost-outermost reduction is a computable normalizing strategy.*

Note that every weakly orthogonal left-normal TRS is root balanced joinable. Thus the following corollary holds.

Corollary 7. *Let a TRS R be weakly orthogonal and left-normal. Then, R has the normal form property, and leftmost-outermost reduction is a computable normalizing strategy.*

Example 8. Let R be combinatory logic CL $\cup$

$$\begin{cases} \mathrm{pred} \cdot (\mathrm{succ} \cdot x) \to x \\ \mathrm{succ} \cdot (\mathrm{pred} \cdot x) \to x. \end{cases}$$

It is clear that R is weakly orthogonal and left-normal. Thus, from Corollary 7, R has the normal form property, and leftmost-outermost reduction is a computable normalizing strategy.

Example 9. Let R be combinatory logic CL $\cup$

$$\begin{cases} (A \cdot x) \cdot y \to ((x \cdot K) \cdot x) \cdot y \\ (A \cdot S) \to (S \cdot K) \cdot A. \end{cases}$$

Clearly, R is left-normal and it has overlapping redexes in $(A \cdot S) \cdot y$. Thus, we have the critical pair $\langle ((S \cdot K) \cdot A) \cdot y, ((S \cdot K) \cdot S) \cdot y \rangle$. Since the critical pair can join by root reductions of two steps $((S \cdot K) \cdot A) \cdot y \to_r (K \cdot y) \cdot (A \cdot y) \to_r y$ and $((S \cdot K) \cdot S) \cdot y \to_r (K \cdot y) \cdot (S \cdot y) \to_r y$, R is root balanced joinable. Thus, from Corollary 6, R has the normal form property, and leftmost-outermost reduction is a computable normalizing strategy.

Example 10. Let R be combinatory logic CL $\cup$

$$\begin{cases} (K \cdot A) \cdot y \to (K \cdot B) \cdot y \\ (K \cdot B) \cdot y \to (K \cdot A) \cdot y \\ A \to A \\ B \to B. \end{cases}$$

It is clear that R is left-normal and it has the two critical pairs $\langle (K \cdot A) \cdot y, (K \cdot B) \cdot y \rangle$ and $\langle (K \cdot B) \cdot y, (K \cdot A) \cdot y \rangle$. We have root reduction $(K \cdot A) \cdot y \to_r (K \cdot B) \cdot y \to_r B$ and $(K \cdot B) \cdot y \to_r B \to_r B$ for the critical pair $\langle (K \cdot A) \cdot y, (K \cdot B) \cdot y \rangle$, and $(K \cdot B) \cdot y \to_r (K \cdot A) \cdot y \to_r A$ and $(K \cdot A) \cdot y \to_r A \to_r A$ for the critical pair $\langle (K \cdot B) \cdot y, (K \cdot A) \cdot y \rangle$ respectively. Thus, R is root balanced joinable. Therefore, from Corollary 6, R has the normal form property, and leftmost-outermost reduction is a computable normalizing strategy. Note that though R has the unique normal form property due to the normal form property, R is not Church-Rosser as $(K \cdot A) \cdot y$ can be reduced into two constants A and B which cannot be joined.

10 Conclusion

In this paper we have investigated normalizing strategies for left-linear overlapping TRSs. We have introduced the concept of the balanced weak Church-Rosser (BWCR) property and related it to a normalizing strategy based on the BWCR Lemma, which is presented in an abstract framework depending solely on the reduction relation. Applying this abstract framework to TRSs, we have shown that external reduction is a normalizing strategy for the class of left-linear TRSs in which every critical pair can be joined with root balanced reductions and every reducible term has an external redex. Further, we have presented computable normalizing strategies based on decidable approximations of external redexes.

An interesting direction for further research is application to *higher-order* rewriting systems, like Klop's *combinatory reduction system* [11]. We believe the BWCR lemma can provide an accessible means of developing computable normalizing strategies uniformly for various higher-order rewriting systems. Another interesting issue is *root-external* reduction for non-orthogonal TRSs, which is very parallel to *root-needed* reduction for orthogonal TRSs developed by Middeldorp [13]. As root-normalizing strategy is more fundamental and complicated than normalizing strategy, we need to generalize the theoretical framework for dealing with approximated decidable reduction based on root-externality.

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