Confluence and Critical-Pair-Closing Systems*

Michio Oyamaguchi[1] and Nao Hirokawa[2]

[1] Nagoya University, Japan
[2] JAIST, Japan

Abstract

In this note we introduce *critical-pair-closing systems* which are aimed at analyzing confluence of term rewriting systems. Based on the notion two new confluence criteria are presented. One is a generalization of weak orthogonality based on relative termination, and another is a partial solution to the RTA Open Problem #13.

1 Introduction

For confluence of a left-linear term rewriting system (TRS) joinability of every critical pair is necessary but not sufficient [4]. In this note we focus attention on rewrite steps that are used for closing critical pairs. For brevity we adopt Dershowitz's critical pair notation [2].

Definition 1.1. A TRS $\mathcal{C}$ is *critical-pair-closing* for a TRS $\mathcal{R}$ if $\mathcal{C} \subseteq \widehat{\mathcal{R}}$ and ${}_{\mathcal{R}}\!\leftarrow\!\rtimes\!\rightarrow_{\mathcal{R}} \subseteq \downarrow_{\mathcal{C}}$. Here $\widehat{\mathcal{R}} = \{\ell\sigma \to r\sigma \mid \ell \to r \in \mathcal{R}$ and $x\sigma$ is a ground normal form for all $x \in \mathcal{D}\mathsf{om}(\sigma)\}$.

We present two confluence criteria. One is the criterion that a left-linear TRS $\mathcal{R}$ is confluent if there exists a confluent critical-pair-closing system $\mathcal{C}$ for $\mathcal{R}$ such that $\rightarrow_{\mathcal{C}_\mathsf{d}/\mathcal{R}}$ is terminating. Here $\mathcal{C}_\mathsf{d}$ is the set of duplicating rewrite rules of $\mathcal{C}$ and $\rightarrow_{\mathcal{C}_\mathsf{d}/\mathcal{R}} = \rightarrow^*_{\mathcal{R}} \cdot \rightarrow_{\mathcal{C}_\mathsf{d}} \cdot \rightarrow^*_{\mathcal{R}}$. In other words, if a left-linear TRS $\mathcal{R}$ that admits a confluent critical-pair-closing system $\mathcal{C}$ is not confluent, there exists an infinite rewrite sequence of $\mathcal{R}$ that contains infinitely many $\mathcal{C}_\mathsf{d}$-steps. Another criterion is that a left-linear TRS $\mathcal{R}$ is confluent if there is a terminating critical-pair-closing system $\mathcal{C}$ for $\mathcal{R}$ with ${}_{\mathcal{R}}\!\xleftarrow{>\epsilon}\!\rtimes\!\longrightarrow_{\mathcal{R}} \subseteq \mathrel{+\!\!\!\!\!\rightarrow}_{\mathcal{C}} \cdot {}^{\;*}_{\mathcal{C}}\!\leftarrow$. The symbol ${}_{\mathcal{R}}\!\xleftarrow{>\epsilon}\!\rtimes\!\longrightarrow_{\mathcal{R}}$ stands for the set of all *inside* critical pairs of $\mathcal{R}$ induced from non-root-overlaps. As a corollary, a left-linear TRS $\mathcal{R}$ is confluent if ${}_{\mathcal{R}}\!\leftarrow\!\rtimes\!\rightarrow_{\mathcal{R}} \subseteq \mathrel{+\!\!\!\!\!\rightarrow}_{\mathcal{C}} \cup {}_{\mathcal{C}}\!\mathrel{\leftarrow\!\!\!\!\!+}$ holds for some terminating subsystem $\mathcal{C}$ of $\mathcal{R}$. This is regarded as a partial solution to one of variations of the RTA Open Problem 13: *Is a left-linear TRS $\mathcal{R}$ confluent if ${}_{\mathcal{R}}\!\leftarrow\!\rtimes\!\rightarrow_{\mathcal{R}} \subseteq \mathrel{+\!\!\!\!\!\rightarrow}_{\mathcal{R}} \cup {}_{\mathcal{R}}\!\mathrel{\leftarrow\!\!\!\!\!+}$ holds?* Both criteria subsume weak orthogonality, considering the case $\mathcal{C} = \varnothing$.

2 Confluence Criteria

In this section we prove the two criteria stated in the introduction. Both rely on the following confluence criterion for abstract rewriting systems (ARSs). Let $>_1, >_2$ be strict orders and $\gtrsim_1$ a preorder such that $\gtrsim_1 \cdot >_1 \cdot \gtrsim_1 \subseteq >_1$. We define the *lexicographic product* $((>_1, \gtrsim_1), >_2)_{\text{lex}}$ as follows: $(a, b)\ ((>_1, \gtrsim_1), >_2)\ (c, d)$ if either $a >_1 c$, or $a \gtrsim_1 c$ and $b >_2 d$. The lexicographic product is well-founded, whenever $>_1$ and $>_2$ are well-founded. Let $\mathcal{A} = (A, \{\rightarrow_\alpha\}_{\alpha \in I})$ be a labeled ARS equipped with a well-founded order $>$ on the label set I. The union of $\rightarrow_\beta$ for all $\beta < \alpha$ is denoted by $\rightarrow_{\vee\alpha}$.

Lemma 2.1. *Let $\mathcal{B}$ be a confluent ARS with $\rightarrow_{\mathcal{B}} \subseteq \rightarrow^*_{\mathcal{A}}$. The labeled ARS $\mathcal{A}$ is confluent if*

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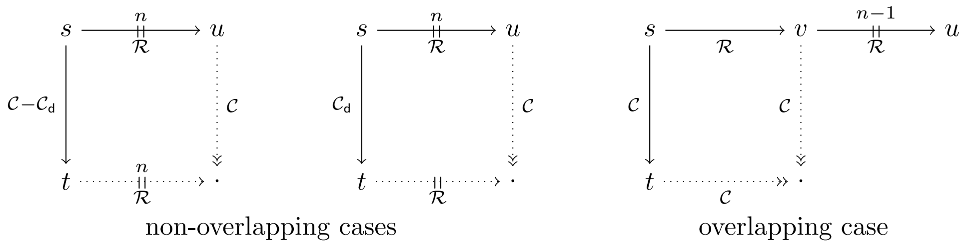


Figure 1: The proof of Lemma 2.3.

- ${}_{\alpha}\!\leftarrow \cdot \to_{\beta} \subseteq (\to_{\mathcal{B}}^{*} \cdot \to_{\mathcal{A}} \cdot {}_{\mathcal{A}}\!\leftarrow \cdot {}_{\mathcal{B}}^{*}\!\leftarrow) \cup ({}_{\vee\alpha}\!\leftarrow \cdot \leftrightarrow_{\mathcal{B}}^{*} \cdot \to_{\vee\beta})$ *for all labels* $\alpha, \beta \in I$, *and*
- ${}_{\mathcal{B}}\!\leftarrow \cdot \to_{\alpha} \subseteq (\to_{\alpha} \cdot {}_{\mathcal{B}}^{*}\!\leftarrow) \cup (\leftrightarrow_{\mathcal{B}}^{*} \cdot \to_{\vee\alpha} \cdot {}_{\mathcal{B}}^{*}\!\leftarrow)$ *for all labels* $\alpha \in I$.

Proof. Let $\rightarrowtail \; = \; \to_{\mathcal{B}}^{*} \cdot \to_{\mathcal{A}}$. Since $\to_{\mathcal{A}} \subseteq \rightarrowtail \subseteq \to_{\mathcal{A}}^{*}$, it is enough to prove the diamond property of $\rightarrowtail$. The inclusion ${}_{\alpha}\!\leftarrow \cdot \to_{\mathcal{B}}^{m} \cdot {}_{\mathcal{B}}^{n}\!\leftarrow \cdot \to_{\beta} \subseteq \rightarrowtail \cdot \leftarrowtail$ is shown by induction on $(\{\alpha, \beta\}, m+n)$ with respect to $((>^{\mathrm{mul}}, =), >)_{\mathrm{lex}}$. Hence, $\leftarrowtail \cdot \rightarrowtail \subseteq {}_{\mathcal{A}}\!\leftarrow \cdot \to_{\mathcal{B}}^{*} \cdot {}_{\mathcal{B}}^{*}\!\leftarrow \cdot \to_{\mathcal{A}} \subseteq \rightarrowtail \cdot \leftarrowtail$. ☐

2.1 A criterion based on relative termination

We prove correctness of the first criterion. For a set U of parallel redex occurrences, the parallel step of $\mathcal{R}$ that contracts U is referred to as $\overset{U}{\mathrel{+\!\!\!\to}}_{\mathcal{R}}$. We write $s \overset{n}{\mathrel{+\!\!\!\to}}_{\mathcal{R}} t$ if $s \overset{U}{\mathrel{+\!\!\!\to}}_{\mathcal{R}} t$ and $|U| \leqslant n$ for some U. For technical convenience we assume $\overset{n}{\mathrel{+\!\!\!\to}}_{\mathcal{R}} = \varnothing$ when $n < 0$.

Lemma 2.2. *Suppose that a left-linear TRS* $\mathcal{R}$ *admits a critical-pair-closing system* $\mathcal{C}$. *If* $t \; {}_{\mathcal{R}}\!\overset{m}{\mathrel{\leftarrow\!\!\!+}} s \overset{n}{\mathrel{+\!\!\!\to}}_{\mathcal{R}} u$ *then* $t \mathrel{+\!\!\!\to}_{\mathcal{R}} \cdot {}_{\mathcal{R}}\!\mathrel{\leftarrow\!\!\!+} u$, *or there exist* v *and* w *such that* $t \; {}_{\mathcal{R}}\!\overset{m-1}{\mathrel{\leftarrow\!\!\!+}} v \leftrightarrow_{\mathcal{C}}^{*} w \overset{n-1}{\mathrel{+\!\!\!\to}}_{\mathcal{R}} u$ *and* $v \; {}_{\mathcal{R}}\!\leftarrow s \to_{\mathcal{R}} w$.

Lemma 2.3. *Suppose that* $\mathcal{C}$ *is a confluent critical-pair-closing system for a left-linear* $\mathcal{R}$. *If* $t \; {}_{\mathcal{C}}\!\leftarrow s \overset{n}{\mathrel{+\!\!\!\to}}_{\mathcal{R}} u$ *then one of the following conditions holds.*

(a) $t \overset{n}{\mathrel{+\!\!\!\to}}_{\mathcal{R}} \cdot {}_{\mathcal{C}}^{*}\!\leftarrow u$,

(b) $t \leftrightarrow_{\mathcal{C}}^{*} v \mathrel{+\!\!\!\to}_{\mathcal{R}} \cdot {}_{\mathcal{C}}^{*}\!\leftarrow u$ *and* $s \to_{\mathcal{C}_{\mathsf{d}}/\mathcal{R}} v$ *hold for some* v,

(c) $t \leftrightarrow_{\mathcal{C}}^{*} v \overset{n-1}{\mathrel{+\!\!\!\to}}_{\mathcal{R}} \cdot {}_{\mathcal{C}}^{*}\!\leftarrow u$ *and* $s \to_{\mathcal{R}}^{*} v$ *hold for some* v.

Proof. If $n = 0$ then (a) holds trivially. Otherwise, one of the three diagrams in Figure 2.1 holds. In any case one of conditions (a–c) holds. ☐

Theorem 2.4. *A left-linear TRS* $\mathcal{R}$ *is confluent if* $\mathcal{C}_{\mathsf{d}}/\mathcal{R}$ *is terminating for some confluent critical-pair-closing system* $\mathcal{C}$ *for* $\mathcal{R}$.

Proof. We define the labeled ARS $\mathcal{A}$ on terms as follows: $s \to_{\alpha} t$ if $s \overset{n}{\mathrel{+\!\!\!\to}}_{\mathcal{R}} t$ and $\alpha = (s, n)$. Moreover, we define the ARS $\mathcal{B}$ as $\to_{\mathcal{B}} = \to_{\mathcal{C}}$. Lemmata 2.2 and 2.3 ensure the conditions of Lemma 2.1 for the well-founded order $((\to_{\mathcal{C}_{\mathsf{d}}/\mathcal{R}}^{+}, \to_{\mathcal{R}}^{*}), >)_{\mathrm{lex}}$ on labels. Therefore, confluence of $\mathcal{A}$ is obtained. Hence, $\mathcal{R}$ is confluent. ☐

We illustrate use of Theorem 2.4 with an example for which ACP v0.40 [1] and CSI v0.4.1 [7] fail to show confluence.

Example 2.5. We show confluence of the left-linear TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{h}(x,\mathsf{d}),y) \to \mathsf{f}(\mathsf{h}(y,\mathsf{d}),x) \qquad \mathsf{h}(\mathsf{c},x) \to \mathsf{h}(x,x)$$

by successive application of Theorem 2.4.

(i) The left-linear TRS $\mathcal{C}$

$$1\colon\ \mathsf{f}(\mathsf{h}(x,\mathsf{d}),\mathsf{c}) \to \mathsf{f}(\mathsf{h}(\mathsf{c},\mathsf{d}),x) \qquad 2\colon\ \mathsf{h}(\mathsf{c},\mathsf{d}) \to \mathsf{h}(\mathsf{d},\mathsf{d})$$

is critical-pair-closing for $\mathcal{R}$, and $\mathcal{C}_\mathsf{d}/\mathcal{R}$ is terminating as $\mathcal{C}_\mathsf{d} = \varnothing$. Thus, it is sufficient to show that $\mathcal{C}$ is confluent.

(ii) Let $\mathcal{C}' = \{2\}$. The TRS $\mathcal{C}'$ is critical-pair-closing for $\mathcal{C}$. Because $\mathcal{C}'_\mathsf{d} = \varnothing$, termination of $\mathcal{C}'_\mathsf{d}/\mathcal{R}$ is trivial. It remains to show that $\mathcal{C}'$ is confluent.

(iii) Since $\mathcal{C}'$ admits no critical pair, the empty TRS $\varnothing$ is a confluent critical-pair-closing system for $\mathcal{C}'$. Termination of $\varnothing_\mathsf{d}/\mathcal{R}_1$ is trivial, and therefore $\mathcal{C}'$ is confluent.

Hence, $\mathcal{R}$ is confluent. Note that taking $\mathcal{C}$ from *instances* of rules in $\mathcal{R}$ is essential.

The following examples explain why none of confluence of $\mathcal{C}$, termination of $\mathcal{C}_\mathsf{d}/\mathcal{R}$, or the ground normal form condition of $\widehat{\mathcal{R}}$ can be removed from the conditions of Theorem 2.4.

Example 2.6. Consider the *non-confluent* left-linear TRS $\mathcal{R}$ taken from [4]:

$$\mathsf{b} \to \mathsf{a} \qquad \mathsf{b} \to \mathsf{c} \qquad \mathsf{c} \to \mathsf{b} \qquad \mathsf{c} \to \mathsf{d}$$

Let $\mathcal{C} = \mathcal{R}$. While $\mathcal{C}$ is not confluent, $\mathcal{C}$ is critical-pair-closing and termination of $\mathcal{C}_\mathsf{d}/\mathcal{R}$ follows from $\mathcal{C}_\mathsf{d} = \varnothing$.

Example 2.7. Consider the *non-confluent* left-linear TRS $\mathcal{R}$ from [3]:

$$1\colon\ \mathsf{f}(\mathsf{a},\mathsf{a}) \to \mathsf{b} \qquad 2\colon\ \mathsf{f}(x,\mathsf{b}) \to \mathsf{f}(x,x) \qquad 3\colon\ \mathsf{f}(\mathsf{b},x) \to \mathsf{f}(x,x) \qquad 4\colon\ \mathsf{a} \to \mathsf{b}$$

Let $\mathcal{C} = \{1,2,3\}$. Then, $\mathcal{C}$ is confluent and critical-pair-closing for $\mathcal{R}$, while $\mathcal{C}_\mathsf{d}/\mathcal{R}$ is not terminating. Let $\mathcal{C}' = \{1,2',3'\}$, where $2'\colon \mathsf{f}(\mathsf{a},\mathsf{b}) \to \mathsf{f}(\mathsf{a},\mathsf{a})$ and $3'\colon \mathsf{f}(\mathsf{b},\mathsf{a}) \to \mathsf{f}(\mathsf{a},\mathsf{a})$. Confluence of $\mathcal{C}'$, termination of $\mathcal{C}'_\mathsf{d}/\mathcal{R}$, and ${}_{\mathcal{R}}{\leftarrow}\!\rtimes\!{\to}_{\mathcal{R}} \subseteq {\downarrow_{\mathcal{C}'}}$ hold, while $\mathcal{C}' \subseteq \widehat{\mathcal{R}}$ does not hold.

2.2 A criterion based on termination

Next, we show the second criterion.

Lemma 2.8. *Every terminating critical-pair-closing system $\mathcal{C}$ for a TRS $\mathcal{R}$ is confluent.*

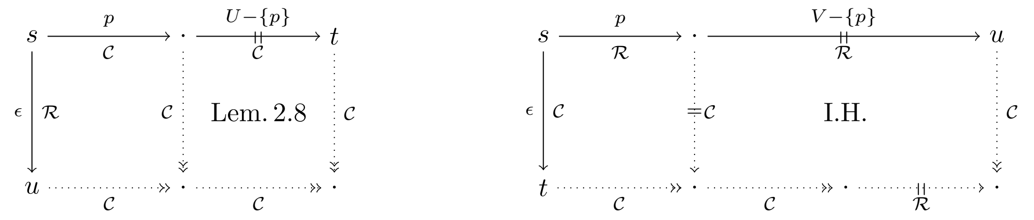
Proof. By the definition of $\widehat{\mathcal{R}}$ every critical pair of $\mathcal{C}$ is an instance of some critical pair of $\mathcal{R}$. Therefore, ${}_{\mathcal{C}}{\leftarrow}\!\rtimes\!{\to}_{\mathcal{C}} \subseteq {\downarrow_{\mathcal{C}}}$. Hence, Knuth and Bendix' criterion entails confluence. □

In order to analyze local peaks we recall Huet's measure which was used for proving correctness of the Parallel Closedness Theorem [4].

Definition 2.9. Let $\gamma : t \;{}_{\mathcal{R}}\!\overset{U}{\nleftarrow\!\!\!\!\nleftarrow}\; s \overset{V}{\twoheadrightarrow\!\!\!\!\!\!+}_{\mathcal{R}} u$. The *weight* $w(\gamma)$ of the local peak γ is given by

$$w(\gamma) = \sum_{p \in W} |s|_p|$$

where, $W = \{p \in U \mid q \leqslant p \text{ for some } q \in V\} \cup \{p \in V \mid q < p \text{ for some } q \in U\}$. Note that if $W = \varnothing$ then $w(\gamma) = 0$.



(a) (b)

Figure 2: The proof of Lemma 2.10.

Lemma 2.10. *Let $\mathcal{C}$ be a terminating critical-pair-closing system for a left-linear TRS $\mathcal{R}$, and suppose ${}_{\mathcal{R}}\xleftarrow{>\epsilon} \rtimes \longrightarrow_{\mathcal{R}} \subseteq \mathrel{\Vert\!\!\!\rightarrow}_{\mathcal{C}} \cdot {}^{*}_{\mathcal{C}}\!\leftarrow$. The following inclusions hold.*

(a) ${}_{\mathcal{C}}\!\leftarrow \cdot \mathrel{\Vert\!\!\!\rightarrow}_{\mathcal{R}} \subseteq \to^{*}_{\mathcal{C}} \cdot \mathrel{\Vert\!\!\!\rightarrow}_{\mathcal{R}} \cdot {}^{*}_{\mathcal{C}}\!\leftarrow$

(b) ${}_{\mathcal{R}}\!\mathrel{\leftarrow\!\!\!\Vert} \cdot \mathrel{\Vert\!\!\!\rightarrow}_{\mathcal{R}} \subseteq \to^{*}_{\mathcal{C}} \cdot \mathrel{\Vert\!\!\!\rightarrow}_{\mathcal{R}} \cdot {}_{\mathcal{R}}\!\mathrel{\leftarrow\!\!\!\Vert} \cdot {}^{*}_{\mathcal{C}}\!\leftarrow$

Proof. We only prove (a) since (b) can be proved in a similar way. We show the stronger claim that $\gamma : t \; {}_{\mathcal{C}}\!\overset{U}{\mathrel{\leftarrow\!\!\!\Vert}} \; s \overset{V}{\mathrel{\Vert\!\!\!\rightarrow}}_{\mathcal{R}} u$ implies $t \to^{*}_{\mathcal{C}} \cdot \mathrel{\Vert\!\!\!\rightarrow}_{\mathcal{R}} \cdot {}^{*}_{\mathcal{C}}\!\leftarrow u$, by induction on $(w(\gamma), s)$ with respect to $((>, =), \rhd)_{\mathrm{lex}}$. Here $\rhd$ stands for the proper superterm relation. We distinguish four cases, depending on U and V:

- If U and V admit no critical overlap, the Parallel Moves Lemma entails $t \mathrel{\Vert\!\!\!\rightarrow}_{\mathcal{R}} \cdot {}_{\mathcal{C}}\!\mathrel{\leftarrow\!\!\!\Vert}\; u$.
- If neither U nor V contains ϵ, the induction hypotheses for immediate subterms of s apply.
- If some $p \in U$ and $\epsilon \in V$ form a critical overlap, (a) in Figure 2.2 holds.
- If $\epsilon \in U$ and some $p \in V$ with $p > \epsilon$ form a critical overlap, (b) in Figure 2.2 holds. ☐

Theorem 2.11. *A left-linear TRS $\mathcal{R}$ is confluent if ${}_{\mathcal{R}}\xleftarrow{>\epsilon} \rtimes \longrightarrow_{\mathcal{R}} \subseteq \mathrel{\Vert\!\!\!\rightarrow}_{\mathcal{C}} \cdot {}^{*}_{\mathcal{C}}\!\leftarrow$ holds for some terminating critical-pair-closing system $\mathcal{C}$ of $\mathcal{R}$.*

Proof. Define the labeled ARS $\mathcal{A}$ as follows: $s \to_{\alpha} t$ if $s \mathrel{\Vert\!\!\!\rightarrow}_{\mathcal{R}} t$ and $\alpha = s$. The claim follows from Lemmata 2.1, 2.8, and 2.10 by taking $\to_{\mathcal{B}} = \to_{\mathcal{C}}$ and $> = \to^{+}_{\mathcal{C}}$. ☐

Example 2.12. Consider the left-linear TRS $\mathcal{R}$ from [6]:

1:	$x - 0 \to x$	7:	$\mathsf{gcd}(x, 0) \to x$	13:	$\mathsf{if}(\mathsf{false}, x, y) \to x$
2:	$0 - x \to 0$	8:	$\mathsf{gcd}(0, x) \to x$	14:	$\mathsf{if}(\mathsf{true}, x, y) \to x$
3:	$\mathsf{s}(x) - \mathsf{s}(y) \to x - y$	9:	$\mathsf{gcd}(x, y) \to \mathsf{gcd}(y, \mathsf{mod}(x, y))$		
4:	$x < 0 \to \mathsf{false}$	10:	$\mathsf{mod}(x, 0) \to x$		
5:	$0 < \mathsf{s}(y) \to \mathsf{true}$	11:	$\mathsf{mod}(0, y) \to 0$		
6:	$\mathsf{s}(x) < \mathsf{s}(y) \to x < y$	12:	$\mathsf{mod}(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{if}(x < y, \mathsf{s}(x), \mathsf{mod}(x - y, \mathsf{s}(y)))$		

Let $\mathcal{C} = \{7, 8, 10, 11\}$. The system $\mathcal{C}$ is terminating and critical-pair-closing for $\mathcal{R}$. Since $\mathcal{R}$ admits no inside critical pairs, ${}_{\mathcal{R}}\xleftarrow{>\epsilon} \rtimes \longrightarrow_{\mathcal{R}} \subseteq \mathrel{\Vert\!\!\!\rightarrow}_{\mathcal{C}} \cdot {}^{*}_{\mathcal{C}}\!\leftarrow$ holds. Hence, $\mathcal{R}$ is confluent.

The termination assumption of $\mathcal{C}$ cannot be dropped from Theorem 2.11, as seen below.

Example 2.13. Recall the TRS $\mathcal{R}$ of Example 2.6. There are no inside critical pairs. Let $\mathcal{C} = \mathcal{R}$. Then $\mathcal{C}$ is a critical-pair-closing system of $\mathcal{R}$. But $\mathcal{R}$ is not confluent.

Table 1: Experiments on 192 left-linear TRSs

	Th.2.4	Th.2.11	[3, Th.3]
confluence proved	**44**	**37**	**29**
timeout (30 sec)	28	43	0

3 Concluding Remarks

We have introduced the notion of critical-pair-closing subsystems for left-linear TRSs and shown two new confluence criteria for left-linear TRSs. The first criterion (Theorem 2.4) is a generalization of weak orthogonality based on relative termination, and the second criterion (Theorem 2.11) is related to the long-standing open problem stated in the introduction.

Table 1 summaries experimental results of the two criteria on left-linear TRSs in Cops Nos. 1–390[1]. As in Example 2.5, Theorem 2.4 was tested as a stand-alone criterion. For the comparison sake we also tested [3, Theorem 3], which is another generalization of weak orthogonality based on relative termination. A suitable critical-pair-closing system is searched from subsets of an input TRS $\mathcal{R}$ by enumeration. We have not supported use of $\widehat{\mathcal{R}}$ yet. Termination and relative termination were checked by using $\mathsf{T_TT_2}$ v1.16 [5], and joinability of critical pairs for Theorems 2.4 and [3, Theorem 3] were checked by $\to^k \cdot {}^m\!\leftarrow$ for $k, m \leqslant 4$. The tests were single-threaded run on a system equipped with an Intel Core i7-4500U with 1.8 GHz using a timeout of 30 seconds. Although the current implementation is still naive and inefficient, the numbers of proofs in the table clearly show effectiveness of our criteria.

All the three criteria are incomparable. For instance, Cops Nos. 22, 19, and 14 can be handled only by Theorem 2.4, Theorem 2.11, and [3, Theorem 3], respectively. It is worthwhile to investigate whether critical-pair-closing systems and *critical pair systems* [3] can be combined.

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