Reduction Strategies for Term Rewriting Systems

by

Takashi NAGAYA

submitted to
Japan Advanced Institute of Science and Technology
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

Supervisor: Professor Dr. Yoshihito Toyama

School of Information Science
Japan Advanced Institute of Science and Technology

March 1999

Abstract

Term rewriting systems have been widely studied as a model for computation. In a term rewriting system, they may exist an infinite reduction sequence starting with a term having normal forms. In order to get a normal form for a given term, we require a normalizing strategy guaranteeing to find a normal form of terms whenever their normal forms exist. Huet and Lévy (1979) showed that a call-by-need strategy is normalizing for every orthogonal (i.e., left-linear and non-overlapping) term rewriting systems. Unfortunately, in general a call-by-need strategy is undecidable. They formalized strong sequentiality guaranteeing a decidable normalizing call-by-need strategy for orthogonal term rewriting system. The work of Huet and Lévy has been extended to several kinds of systems.

In this thesis we first extend the class of left-linear term rewriting systems having a decidable call-by-need strategy. We present the class of NVNF-sequential systems. This class properly includes the class of NV-sequential systems which was introduced by Oyamaguchi (1993). We prove that every orthogonal NVNF-sequential system has a decidable normalizing call-by-need strategy. Then we give growing approximations of term rewriting systems without the assumption of the right-linearity whereas Jacuemard (1993) assumed the right-linearity . We show that our approximations extend the class of orthogonal term rewriting systems having a decidable normalizing call-by-need strategy.

Secondly, we investigate the normalizability of a call-by-need strategy for left-linear overlapping term rewriting systems. We first introduced the notion of stable balanced joinability. It is shown that a call-by-need strategy is normalizing for every stable balanced joinable strongly sequential system. This is a generalization of Toyama's result (1992). We next introduce the notion of NV-stable balanced joinability and prove that every NV-stable balanced joinable NV-sequential system has a decidable normalizing call-by-need strategy.

Finally, we apply the results on call-by-need strategy to the E-strategy adopted by the OBJ algebraic specification languages. The E-strategy chooses a redex according to local strategies which are given to each function symbol. We consider how to give local strategies to make the E-strategy normalizing. For this purpose, we introduced the notion index-transitivity and carefulness. We show that for every index-transitive orthogonal term rewriting system, if careful local strategies are given to each function symbol then the E-strategy is normalizing.

Acknowledgments

I am very grateful to my supervisor Professor Yoshihito Toyama of Japan Advanced Institute of Science and Technology for his guidance. I am also grateful to Associate Professor Masahiko Sakai of Nagoya University for his suggestions and kind encouragements. Part of the work on this thesis has been done under his supervision.

I would like to thank Professor Hiroakira Ono and Associate Professor Hajime Ishihara of Japan Advanced Institute of Science and Technology for their suggestions. I would also like to thank Professor Michio Oyamaguchi of Mie University for his useful comments.

I would like to thank Professor Kokichi Futatugi and Associate Kazuhiro Ogata of Japan Advanced Institute of Science and Technology and Michihiro Matsumoto of PFU Limited for their helpful discussions.

I also wish to thank Eiichi Horita of NTT Software Laboratories and Ken Mano of NTT Communication Science Laboratories for their guidance in training at NTT Communication Science Laboratories.

Contents

Chapter 1

Introduction

A *term rewriting system* consists of a set of directed equations, called *rewrite rules.* If a term t contains an instance of the left-hand side of a rewrite rules $l \to r$, so-called *redex* then t can be rewritten to the term obtained from t by replacing this instance with the corresponding instance of the right-hand side r. A term which cannot be further rewritten is the result of the computation, and called a *normal form.* Term rewriting systems play an important role in various fields of computer science such as abstract data type specifications, implementations of functional programming languages, programming verification and automated deduction. The fundamental properties of term rewriting systems are *strongly normalizing property* (or *termination*) and *Church-Rosser property* (or *confluence*). A term rewriting system is said to be strongly normalizing if there exists no infinite reduction sequence. In a strongly normalizing term rewriting system, every computation eventually ends in a normal form. We call a term rewriting system Church-Rosser if any two terms that are reduced from some term can reach same term by the reduction. If a term rewriting systems is Church-Rosser then every term can have at most one normal form. In a term rewriting system being Church-Rosser, there may exist infinite reduction sequences starting with a term having the normal form. In order to compute the normal form of a given term, we require some strategies telling us which redex to contract. A reduction strategy is said to be *normalizing* if we can always find the normal form of a term having a normal form by it. It is well-known that the leftmost-outermost reduction is a normalizing strategy in the λ-calculus and the combinatory logic. However, it has been shown that the leftmost-outermost strategy is not normalizing for arbitrary term rewriting systems.

O'Donnell [25] was the first to consider reduction strategies for orthogonal term rewriting systems. He showed that the parallel-outermost reduction strategy is normalizing for orthogonal term rewriting systems. Huet and Lévy investigated one-step reduction strategies for orthogonal term rewriting systems in [13]. Huet and Lévy proved that every term not in normal forms contains a needed redex and repeated rewriting of needed redexes leads to the normal form if it exists. A needed redex is a redex which must be contract in order to reach a normal form. However, it is undecidable whether a redex in a term is needed. Huet and Lévy formulated the notion of strong sequentiality for orthogonal term rewriting systems. They showed that for every strongly sequential orthogonal term rewriting system $\mathcal{R}$, index reduction is a normalizing strategy, that is, by rewriting a redex called an index at each step, every reduction starting with a term having a normal form eventually terminates at the normal form. Here, the index is defined as a needed redex concerning an approximation of $\mathcal{R}$ which is obtained by analyzing the left-hand sides alone of the rewrite rules of term rewriting systems. Oyamaguchi [28] introduced the notion of NV-sequentiality which is a proper extension of strong sequentiality. NV-sequentiality is not only based on the analysis of the left-hand sides of the rewrite rules of term rewriting systems but also on the non-variable parts of the right-hand sides. Extensions of NV-sequentiality were proposed by Nagaya et al. [21], Comon [3] and Jacquemard [14]. The notion of strong sequentiality was extended to left-linear term rewriting systems by Toyama [30]. He showed that index reduction is a normalizing strategy for every root balanced joinable strongly sequential system. Kennaway [16] proved that every almost orthogonal term rewriting system has a decidable one-step normalizing strategy. However, the strategy of Kennaway is complicated. Antony and Middeldorp [1] proposed a simpler and intuitive one-step reduction strategy for every term rewriting systems. They proved that their strategy is normalizing for weakly orthogonal term rewriting systems.

The rest of this chapter gives an overview of this thesis.

Chapter 2 gives the basic definitions of term rewriting systems. We first present abstract reduction systems which are set equipped with a binary relation. Term rewriting systems are special abstract reduction systems. The notions of sequentiality and indices are explained in Section 2.3. In Section 2.4, we introduce tree automata which are generalization of sequential automata.

In Chapter 3, we introduce an extension of NV-sequentiality, which is called NVNF-sequentiality [21]. We first show that the class of NVNF-sequential systems properly includes the class of NV-sequential systems. We next show the decidability of indices with respect to NVNF-sequentiality for left-linear term rewriting systems. Every orthogonal NVNF-sequential system has a decidable normalizing call-by-need strategy. It was shown by Comom [3] that NVNF-sequentiality of left-linear term rewriting systems is decidable.

In Chapter 4, we show that index reduction is normalizing for the class of stable balanced joinable strongly sequential systems [22]. A stable balanced joinable system is a left-linear term rewriting system in which every critical pair is joinable with balanced stable reduction. In stable reduction, transitive index being stable under substitutions is contracted. This class includes all root balanced joinable strongly sequential systems. In stable balanced joinable strongly sequential systems, index reduction has the balanced weakly Church-Rosser property. Thus we can show the normalizability of index reduction by using Toyama's theorem [30] concerning reduction strategies. We next show that every NV-stable balanced joinable NV-sequential system has a normalizing strategy by introducing the notions of transitivity and stability for indices with respect to NV-sequentiality. In Chapter 4, we do not consider more general sequential systems (NVNF-, shallow [3] or growing [14] sequential systems). The reason is that index reduction is not balanced weakly Church-Rosser even if the system is orthogonal.

In Chapter 5, we extend Jacquemard's result in [14] to left-linear growing term rewriting systems [23]. Jacquemard showed that the set of normalizable ground terms is recognized by a tree automaton if the term rewriting system is linear and growing. We first show that the set of reachable terms to some recognizable set by the reduction is recognized by a tree automaton if a term rewriting system is left-linear and growing. We can remove the right-linear condition by constructing a deterministic automaton. This result gives us better approximations of term rewriting systems which are left-linear growing systems obtained by renaming variables in the right-side hands of rewrite rules. These approximations yield the class of left-linear term rewriting systems for which there exists a decidable call-by-need strategy. Moreover, this recognizability result implies the decidability of reachability for left-linear growing systems. It is also shown that reachability and joinability for some subclass of right-linear systems are decidable. We next prove that termination for almost orthogonal growing term rewriting systems is decidable. Our proof use Gramlich's theorem that a weakly innermost normalizing rewriting system $\mathcal{R}$ is terminating if every critical pair of $\mathcal{R}$ is trivial and overlay. We show that the set of all ground term being reachable a normal form by innermost reduction is recognized by a tree automaton for left-linear growing systems. By basic property of tree automata, the decidability result is obtained.

In Chapter 6, we study the evaluation strategy (E-strategy) [9, 11, 20, 24]. The E-strategy is a reduction strategy adopted by the OBJ algebraic specification languages such that OBJ2 [9], OBJ3 [11] and CafeOBJ [24]. The outermost strategy has a better termination behavior than the innermost strategy although the outermost strategy can not be implemented as efficiently as the innermost strategy. The E-strategy is a compromise between the outermost and the innermost strategies. Each function symbol is given a list of natural numbers which is called local strategy. By local strategies, it is determined which redex is contracted. The result of the reduction by the E-strategy is not always a normal form. We first consider a restriction for local strategies to avoid this problem. Next we present the class of term rewriting systems for which the E-strategy is normalizing. Because our normalizability proof relies on Huet and Lévy's theorem, this class is undecidable. In Sections 6.3 and 6.4, we give a sufficient condition for normalizability of the E-strategy and we explain how to give local strategies to function symbols for term rewriting systems satisfying this condition.

Chapter 2

Preliminaries

In this chapter, we present the basic concept of term rewriting which is used in this thesis. Following Klop [17], we first introduce abstract reduction systems. More details on term rewriting can be found in [2, 7, 17]. In Section 2.3, we explain the landmark theorem of Huet and Lévy [13] and give the notions of index and sequentiality. In Section 2.4, tree automata are introduced. Several decidability results in this thesis are obtained by using tree automata techniques.

2.1 Abstract Reduction Systems

In this section, we define abstract reduction systems which are set equipped with a binary relation. Most properties of term rewriting systems are described on this abstract level. We can avoid repeating similar definitions and properties by defining them.

Definition 2.1.1 An *abstract reduction system* (ARS) is a structure $A = \langle D, \to \rangle$ consisting of a set D and a binary relation $\to$ on D, called a reduction relation. We write $a \to b$ if $(a, b) \in \to$

Definition 2.1.2 Let $A = \langle D, \to \rangle$ be an ARS.

1. The identity of elements of D is denoted by $\equiv$.

2. The transitive-reflexive closure of $\to$ is denoted by $\to^*$. The transitive closure of $\to$ is denoted by $\to^+$ and $\to^\equiv$ denotes the reflexive closure of $\to$.

3. The set of natural numbers is denoted by $\mathcal{N}$. Let $k \in \mathcal{N}$. Then $\to^k$ denotes the k-steps reduction.

4. The symmetric closure of $\to$ is denoted by $\leftrightarrow$. The transitive-reflexive closure of $\leftrightarrow$ is denoted by $=$.

5. We write $a \leftarrow b$ if $b \to a$.

6. An element $a \in D$ is a *normal form* if there exists no $b \in D$ such that $a \to b$. The set of normal forms is denoted by NF_A. An element *a has a normal form* if $a \to^* b$ for some normal form b.

Definition 2.1.3 Let $A = \langle D, \to \rangle$ be an ARS.

1. A (or $\to$) is *strongly normalizing* or *terminating* if there are no infinite reduction sequences $x_0 \to x_1 \to x_2 \to \cdots$.

2. A (or $\to$) is *Church-Rosser* or *confluent* if $\forall a_1, a_2, a_3 \in D$, $a_1 \to^* a_2$ and $a_1 \to^* a_3$ imply $a_2 \to^* b$ and $a_3 \to^* b$ for some $b \in D$.

3. A (or $\to$) *has the normal form property* if $\forall a \in D$, $\forall b \in \mathrm{NF}_A$, $a = b$ implies $a \to^* b$.

Definition 2.1.4 Let $A = \langle D, \to \rangle$ be an ARS.

1. A relation $\to_s$ on D is a *reduction strategy* for A (or $\to$) if $\to_s \subseteq \to^+$ and every normal form with respect to $\to_s$ is also a normal form with respect to $\to$. If $\to_s$ is a subrelation of $\to$ then it is called a *one-step* reduction strategy. Otherwise, $\to_s$ is called a *many-step* reduction strategy.

2. A reduction strategy $\to_s$ for A is *normalizing* if for each a having a normal form with respect to $\to$, there are no infinite sequences $a \equiv a_0 \to_s a_1 \to_s a_2 \to_s \cdots$.

2.2 Term Rewriting Systems

Definition 2.2.1 A *signature* $\mathcal{F}$ is a finite set of *function symbols* denoted by $f, g, h, \ldots$. Every $f \in \mathcal{F}$ is associated with a natural number denoting its *arity*. Function symbols of arity 0 are called *constant*. $\mathcal{F}_n$ denotes the set of all n-ary function symbols. Hence $\mathcal{F} = \bigcup_{n \geq 0} \mathcal{F}_n$.

Definition 2.2.2 Let $\mathcal{F}$ be a signature and let $\mathcal{V}$ be an enumerable set of *variables* denoted by $x, y, z, \ldots$ where $\mathcal{F} \cap \mathcal{V} = \phi$. The set $\mathcal{T}(\mathcal{F}, \mathcal{V})$ of all *terms* built from $\mathcal{F}$ and $\mathcal{V}$ is the smallest set such that

- $\mathcal{V} \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V})$,
- if $f \in \mathcal{F}_n$ and $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ then $f(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{F}, \mathcal{V})$.

The set $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is sometimes denoted by $\mathcal{T}$. Terms not containing variables are called *ground* terms. The set of all ground terms built from $\mathcal{F}$ is denoted by $\mathcal{T}(\mathcal{F})$. A term t is *linear* if every variable in t occurs only once.

Definition 2.2.3 Let $\Box$ be an extra constant. A *context* $C[, \ldots,]$ is a term in $\mathcal{T}(\mathcal{F} \cup \{\Box\}, \mathcal{V})$. If $C[, \ldots,]$ is a context with n occurrences of $\Box$ and $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ then $C[t_1, \ldots, t_n]$ is the result of replacing from left to right the occurrences of $\Box$ by $t_1, \ldots, t_n$. A context containing precisely one occurrence of $\Box$ is denoted by $C[\ ]$. If t has an occurrence of some (function or variable) symbol e then we write $e \in t$. The variable occurrence z of $C[z]$ is *fresh* if $z \notin C[\ ]$.

Definition 2.2.4 Let $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$.

1. The height $\rho(t)$ of t is defined by

$$\rho(t) = \begin{cases} 1 + \max\{\rho(t_1), \ldots, \rho(t_n)\} & \text{if } t \equiv f(t_1, \ldots, t_n) \text{ and } n > 0, \\ 1 & \text{otherwise.} \end{cases}$$

2. Let $\mathcal{N}_+$ be the set of positive integers. A *position* (or *occurrence*) is a element of $\mathcal{N}_+^*$, i.e., a finite sequence of positive integers. The empty position is denoted by ε and the concatenation of positions p and q is denoted by $p.q$. The set $\mathcal{P}os(t)$ of positions in t is defined as follows:

$$\mathcal{P}os(t) = \begin{cases} \{\varepsilon\} & \text{if } t \in \mathcal{V}, \\ \{\varepsilon\} \cup \{\, i.p \mid 1 \le i \le n\,,\ p \in \mathcal{P}os(t_i)\,\} & \text{if } t \equiv f(t_1, \ldots, t_n). \end{cases}$$

Positions are partially ordered by the prefix ordering $\le$, i.e., $p \le q$ if there exists r such that $p.r = q$. In this case we define q/p as r. If $p \not\le q$ and $q \not\le p$ then we say that p and q are *disjoint*, and write $p \perp q$. The depth $|\,p\,|$ of a position p is defined by

$$|\,p\,| = \begin{cases} 0 & \text{if } p = \varepsilon, \\ 1 + |\,q\,| & \text{if } p = i.q. \end{cases}$$

3. If $p \in \mathcal{P}os(t)$ then the *subterm* $t|_p$ of t at a position p is defined by

$$t|_p \equiv \begin{cases} t & \text{if } p = \varepsilon, \\ t_i|_q & \text{if } t \equiv f(t_1, \ldots, t_n) \text{ and } p = i.q. \end{cases}$$

If s is a subterm of t then we write $s \subseteq t$. A subterm s of t is *proper* if $s \not\equiv t$. We write $s \subset t$ to indicate that s is a proper subterm of t.

4. If $p \in \mathcal{P}os(t)$ then the symbol $t(p)$ at p of t is defined as follows:

$$t(p) = \begin{cases} t|_p & \text{if } t|_p \in \mathcal{V}, \\ f & \text{if } t|_p \equiv f(t_1, \ldots, t_n). \end{cases}$$

The set of variable positions in t is denoted by $\mathcal{P}os_{\mathcal{V}}(t)$, i.e., $\mathcal{P}os_{\mathcal{V}}(t) = \{\, p \in \mathcal{P}os(t) \mid t(p) \in \mathcal{V}\,\}$. We define $\mathcal{P}os_{\mathcal{F}}(t)$ as $\mathcal{P}os(t) \backslash \mathcal{P}os_{\mathcal{V}}(t)$. Hence $\mathcal{P}os_{\mathcal{F}}(t) = \{\, p \in \mathcal{P}os(t) \mid t(p) \in \mathcal{F}\,\}$.

5. If $p \in \mathcal{P}os(t)$ and $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ then the term $t[s]_p$ obtained from t by replacing the subterm $t|_p$ with s is defined as follows:

$$t[s]_p \equiv \begin{cases} s & \text{if } p = \varepsilon, \\ f(t_1, \ldots, t_i[s]_q, \ldots, t_n) & \text{if } t \equiv f(t_1, \ldots, t_n) \text{ and } p = i.q. \end{cases}$$

If $p_1, \ldots, p_n \in \mathcal{P}os(t)$ are pairwise disjoint then we write $t[s_1, \ldots, s_n]_{p_1,\ldots,p_n}$ instead of $t[s_1]_{p_1} \cdots [s_n]_{p_n}$.

Example 2.2.5 Let $\mathcal{F} = \{\, f, g, a, b\,\}$. Consider the linear term $t \equiv f(g(x), f(g(a), y))$. We have $\rho(t) = 4$, $\mathcal{P}os(t) = \{\, \varepsilon,\ 1,\ 2,\ 1.1,\ 2.1,\ 2.2,\ 2.1.1\,\}$ and $\mathcal{P}os_{\mathcal{V}}(t) = \{\, 1.1,\ 2.2\,\}$. Then $t|_{2.1} \equiv g(a)$, $t(1) = g$ and $t[g(b)]_2 \equiv f(g(x), g(b))$.

Definition 2.2.6 A *substitution* θ is a mapping from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$. Every substitution θ is extended to a homomorphism from $\mathcal{T}(\mathcal{F}, \mathcal{V})$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$, i.e., $\theta(f(t_1, \ldots, t_n)) \equiv f(\theta(t_1), \ldots, \theta(t_n))$ for each n-ary function symbol f and terms $t_1, \ldots, t_n$. A *variable renaming* is a bijective substitution. A term s is an *instance* of a term t if there exists a substitution θ such that $s \equiv \theta(t)$. We write $t\theta$ instead of $\theta(t)$.

Definition 2.2.7 A *term rewriting system* (TRS) is a pair $(\mathcal{F}, \mathcal{R})$ consisting of a signature $\mathcal{F}$ and a finite set $\mathcal{R}$ of *rewrite rules.* A rewrite rule is a pair $\langle l, r\rangle$ of terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$ such that:

(1) $l \notin \mathcal{V}$,

(2) any variable in r also occurs in l.

We write $l \to r$ for $\langle l, r\rangle$. An instance of the left-hand side of a rewrite rule is a *redex.* The rewrite rules of a term rewriting system $(\mathcal{F}, \mathcal{R})$ define a reduction relation $\to_{\mathcal{R}}$ on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ as follows: $t \to_{\mathcal{R}} s$ iff there exist a rewrite rule $l \to r \in \mathcal{R}$, a position $p \in \mathcal{P}os(t)$ and a substitution θ such that $t|_p \equiv l\theta$ and $s \equiv t[r\theta]_p$. We call $r\theta$ the *contractum* of $l\theta$. We may write $t \xrightarrow{p}_{\mathcal{R}} s$ or $t \xrightarrow{\Delta}_{\mathcal{R}} s$ to specify the redex position p or the redex occurrence $\Delta \equiv l\theta$ of t in this reduction. When no confusion can arise, we omit the subscript $\mathcal{R}$.

Example 2.2.8 Let $\mathcal{F} = \{\, add,\ mult,\ s,\ 0 \,\}$ and

$$\mathcal{R} = \begin{cases} add(x, 0) \to x \\ add(x, s(y)) \to s(add(x, y)) \\ mult(x, 0) \to 0 \\ mult(x, s(y)) \to add(mult(x, y), x). \end{cases}$$

We have the following reduction sequence (at each step the underlined redex is contracted):

$$\begin{aligned}
\underline{mult(add(s(0),0), s(s(0)))} &\to_{\mathcal{R}} add(mult(\underline{add(s(0),0)}, s(0)), add(s(0),0)) \\
&\to_{\mathcal{R}} add(\underline{mult(s(0), s(0))}, add(s(0),0)) \\
&\to_{\mathcal{R}} add(add(\underline{mult(s(0),0)}, s(0)), add(s(0),0)) \\
&\to_{\mathcal{R}} add(\underline{add(0, s(0))}, add(s(0),0)) \\
&\to_{\mathcal{R}} add(s(\underline{add(0,0)}), add(s(0),0)) \\
&\to_{\mathcal{R}} add(s(0)), \underline{add(s(0),0)}) \\
&\to_{\mathcal{R}} \underline{add(s(0), s(0))} \\
&\to_{\mathcal{R}} s(\underline{add(s(0),0)}) \\
&\to_{\mathcal{R}} s(s(0)).
\end{aligned}$$

All notions defined in the previous section for abstract reduction systems carry over to term rewriting systems by associating the ARS $\langle \mathcal{T}(\mathcal{F}, \mathcal{V}), \to_{\mathcal{R}}\rangle$ with the TRS $(\mathcal{F}, \mathcal{R})$. We sometimes write $\mathcal{R}$ instead of $(\mathcal{F}, \mathcal{R})$ if the signature is clear from the context.

Definition 2.2.9 Let $\mathcal{R}$ be a TRS.

1. $\mathcal{R}$ is *ground* (*linear*) if for every $l \to r \in \mathcal{R}$, l and r are ground (linear).

2. $\mathcal{R}$ is *left-linear* (*right-linear*) if for every $l \to r \in \mathcal{R}$, l (r) is linear.

Example 2.2.10 Consider the TRS $\mathcal{R}$ of Example 2.2.8. $\mathcal{R}$ is left-linear. But $\mathcal{R}$ is not right-linear (linear) because the right-hand side of the fourth rewrite rule is non-linear.

Definition 2.2.11 Let $l \to r$ and $l' \to r'$ be two rewrite rules of a TRS $\mathcal{R}$. We assume that they are renamed to have no common variables. Suppose that p is a position in $\mathcal{P}os_{\mathcal{F}}(l)$ such that $l|_p$ and l' are unifiable with a most general unifier σ. Then we say that $l \to r$ and $l' \to r'$ are overlapping and the pair $\langle l[r']_p\sigma, r'\sigma\rangle$ is called a *critical pair* of $\mathcal{R}$. If $l \to r$ and $l' \to r'$ are same rule, then we do not consider the case $p = \varepsilon$. A critical pair $\langle l[r']_p\sigma, r'\sigma\rangle$ with $p = \varepsilon$ is an *overlay*. A critical pair $\langle t, s\rangle$ is *trivial* if $t \equiv s$.

Example 2.2.12 Let

$$\mathcal{R} = \begin{cases} f(g(x), y) \to f(x, x) \\ f(x, a) \to g(x) \\ g(b) \to b. \end{cases}$$

Then $\mathcal{R}$ has three critical pairs $\langle f(x,x), g(g(x))\rangle$, $\langle g(g(x)), f(x,x)\rangle$ and $\langle f(b,y), f(b,b)\rangle$. The critical pairs $\langle f(x,x), g(g(x))\rangle$ and $\langle g(g(x)), f(x,x)$ are overlays.

Definition 2.2.13 Let $\mathcal{R}$ be a TRS.

1. $\mathcal{R}$ is *non-overlapping* if $\mathcal{R}$ has no critical pair.
2. $\mathcal{R}$ is *orthogonal* if $\mathcal{R}$ is left-linear and non-overlapping.
3. $\mathcal{R}$ is *almost orthogonal* if $\mathcal{R}$ is left-linear and all critical pairs of $\mathcal{R}$ are trivial overlays.

Theorem 2.2.14 ([29]) Every orthogonal TRS is Church-Rosser. □

2.3 Sequential TRSs

2.3.1 Sequentiality

Huet and Lévy [13] investigated normalizing one-step reduction strategies for orthogonal TRSs. They proved that every orthogonal TRS has a normalizing call-by-need strategy. We first explain this theorem.

Let $A : t_0 \to t_1 \to \cdots \to t_n$ be a reduction sequence. We denote the first i steps of A by $A[i]$ and denote the rest of A by $A[i, n]$. We may write $A : t_0 \to^* t_n$ instead of $A : t_0 \to t_1 \to \cdots \to t_n$.

Definition 2.3.1 Let $A : t \to s$ be a reduction step contracting the redex at $p \in \mathcal{P}os(t)$ by the rewrite rule $l \to r \in \mathcal{R}$. Let $q \in \mathcal{P}os(t)$. The set $q\backslash A$ of *descendants* of q in s by A is defined as follows:

$$q\backslash A = \begin{cases} \{\, q \,\} & \text{if } q < p \ \text{ or } \ q \perp p, \\ \{\, p.p_3.p_2 \mid r|_{p_3} \equiv l|_{p_1} \,\} & \text{if } q = p.p_1.p_2 \text{ with } p_1 \in \mathcal{P}os_{\mathcal{V}}(l) \\ \phi & \text{otherwise.} \end{cases}$$

If $Q \subseteq \mathcal{P}os(t)$ then $Q\backslash A$ denotes the set $\bigcup_{q \in Q} q\backslash A$. The notion of descendant extends to reduction sequences as follows. Let $A : t_0 \to t_1 \to \cdots \to t_n$. The set $q\backslash A$ is defined by $q\backslash A = \{\, q \,\}$ if $n = 0$ and $q\backslash A = (q\backslash A[1])\backslash A[1, n]$ if $n > 0$.

Example 2.3.2 Let

$$\mathcal{R} = \begin{cases} f(g(x), y) \to f(x, f(x, a)) \\ h(x) \to g(x) \end{cases}$$

and $A : t \equiv f(h(a), a) \to f(g(a), a) \to f(a, f(a, a)) \equiv t'$. Then position 1.1 has two descendants 1 and 2.1 in t'. All positions in t except 1.1 have no descendants in t'.

Definition 2.3.3 Let $\mathcal{R}$ be a TRS.

1. A redex position p in a term t is *needed* if in every reduction sequence from t to a normal form a redex at some descendant of p is contracted. In this case we also say that the redex at position p is *needed.*

2. The *needed reduction* $\to_{\mathcal{N}}$ is defined on $\mathcal{T}$ as follows: $t \to_{\mathcal{N}} s$ iff $t \xrightarrow{p} s$ and p is needed in t.

Note that if a term t does not have a normal form then all redexes in t are needed.

Example 2.3.4 Consider the TRS $\mathcal{R}$ of Example 2.3.2 and the term $t \equiv f(h(a), h(a))$. The redex $h(a)$ in t at position 1 is needed. However, the redex $h(a)$ in t at position 2 is not needed because we have $f(h(a), h(a)) \to f(g(a), h(a)) \to f(a, f(a, a))$, which is a needed reduction sequence.

Theorem 2.3.5 ([13]) Let $\mathcal{R}$ be an orthogonal TRS. The needed reduction $\to_{\mathcal{N}}$ is a normalizing reduction strategy for $\mathcal{R}$. □

The theorem proved in [13] is actually stronger: if a term t has a normal form then there exists no infinite reduction sequence starting with t in which infinitely many needed redexes are contracted. Middeldorp [19] generalized this theorem to computations to root-stable term.

Definition 2.3.6 Let $\mathcal{R}$ be a TRS.

1. A term t is *root-stable* if there exists no redex s such that $t \to^* s$.

2. A redex position p (or a redex $t|_p$) in a term t is *root-needed* if in every reduction sequence from t to a root-stable term a redex at some descendant of p is contracted.

Theorem 2.3.7 ([19]) Let $\mathcal{R}$ be an orthogonal TRS.

(1) Every non-root-stable term has a root-needed redex.

(2) If a term t is reducible to some root-stable term then every infinite reduction sequence starting with t in which infinitely many root-needed redexes are contracted contains a root-stable term. □

The above theorems give us a normalizing reduction strategy. However, needed redexes are defined as redexes which contracted in all reduction to the normal form. Hence, in order to decide which are needed redexes, we have search all reduction to the normal form i.e., we require look-ahead. Huet and Lévy introduced the class of sequential TRSs in which call-by-need computations are possible without look-ahead.

Definition 2.3.8 Let $(\mathcal{F}, \mathcal{R})$ be a TRS. We add a new constant Ω to $\mathcal{F}$. Elements of $\mathcal{T}(\mathcal{F} \cup \{\Omega\}, \mathcal{V})$ are called *Ω-terms*. The set $\mathcal{T}(\mathcal{F} \cup \{\Omega\}, \mathcal{V})$ is abbreviated to $\mathcal{T}_\Omega$. An *Ω-normal form* is an Ω-term without redexes, containing at least one occurrence of Ω. Only terms containing neither redexes nor Ω's are called *normal forms*. The set of all normal forms is denoted by $\mathrm{NF}_\mathcal{R}$. t_Ω denotes the Ω-term obtained from t by replacing all variables in t with Ω. The set Red is defined by $Red = \{ \, l_\Omega \mid l \rightarrow r \in \mathcal{R} \, \}$.

Definition 2.3.9

1. The *prefix ordering* $\leq$ on $\mathcal{T}_\Omega$ is defined as follows:

   - $\Omega \leq t$ for all $t \in \mathcal{T}_\Omega$,
   - $f(s_1, \ldots, s_n) \leq f(t_1, \ldots, t_n)$ if $s_i \leq t_i$ for any $1 \leq i \leq n$,
   - $x \leq x$ for all $x \in \mathcal{V}$.

   We write $t < s$ if $t \leq s$ and $t \not\equiv s$.

2. Two Ω-terms t and s are *compatible*, written by $t \uparrow s$, if there exists an Ω-term r such that $t \leq r$ and $s \leq r$; otherwise, t and s are *incompatible* which is indicated by $t \# s$. The least upper bound of two Ω-terms t and s is denoted by $t \sqcup s$ if $t \uparrow s$. Let $S \subseteq \mathcal{T}_\Omega$. We write $t \uparrow S$ if there exists some $s \in S$ such that $t \uparrow s$; otherwise, $t \# S$.

Example 2.3.10 Let $\mathcal{R}$ be the TRS of Example 2.3.2. Then $Red = \{ \, f(g(\Omega), \Omega), \; h(\Omega) \, \}$. We have $f(\Omega, f(\Omega, \Omega)) \leq f(g(\Omega), f(\Omega, a))$ and $f(\Omega, f(g(a), \Omega)) \uparrow f(h(\Omega), f(\Omega, x))$. We obtain $f(\Omega, f(g(a), \Omega)) \sqcup f(h(\Omega), f(\Omega, x)) \equiv f(h(\Omega), f(g(a), x))$.

Definition 2.3.11 Let P be a predicate on $\mathcal{T}_\Omega$. An Ω-position p of an Ω-term t is an *index* with respect to P if for every Ω-term s with $t \leq s$, $P(s) = true$ implies $s|_p \not\equiv \Omega$. The set of indices of t with respect to P is denoted by $I_P(t)$.

Let $t \in \mathcal{T}$ and $p \in \mathcal{P}os(t)$. Then we can see that $p \in I_P(t[\Omega]_p)$ iff $P(t[\Omega]_p) = false$.

Definition 2.3.12 Let $\mathcal{R}$ be a TRS. We define the predicate nf on $\mathcal{T}_\Omega$ as follows: $nf(t) = true$ iff $t \rightarrow^*_\mathcal{R} s$ for some normal form s.

Note that if $\mathcal{R}$ is a left-linear TRS then nf is a monotonic predicate, i.e., $nf(t) = true$ implies $nf(s) = true$ whenever $t \leq s$. The following lemma can be easily proven.

Lemma 2.3.13 Let $\mathcal{R}$ be an orthogonal TRS. Let $t \in \mathcal{T}$. A redex position p of t is needed iff $p \in I_{nf}(t[\Omega]_p)$. $\square$

Definition 2.3.14 A left-linear TRS is *sequential* if every Ω-normal form has an index w.r.t. nf

Example 2.3.15 Let $\mathcal{R}$ be Berry's TRS, i.e.,

$$\mathcal{R} = \begin{cases} f(a, b, x) \rightarrow c \\ f(b, x, a) \rightarrow c \\ f(x, a, b) \rightarrow c. \end{cases}$$

Consider the Ω-normal form $t \equiv f(\Omega, \Omega, \Omega)$. Position 1 is not an index of t w.r.t. nf because we have the Ω-term $s \equiv f(\Omega, a, b)$ with $t \leq s$ and $nf(s) = true$. Similaly, neither position 2 nor 3 is an index of t w.r.t. nf. Thus $\mathcal{R}$ is not sequential.

Unfortunately, in general neither indices w.r.t. nf nor sequentiality is decidable.

2.3.2 Strong Sequentiality

Huet and Lévy [13] formalized strong sequentiality which is a sufficient condition for sequentiality. Strong sequentiality is a property based on the left-hand sides of the rewrite rules of TRSs alone. They introduced the arbitrary reduction in order to forget the right-hand sides of the rewrite rules.

Definition 2.3.16 Let $\mathcal{R}$ be a TRS.

1. The *arbitrary reduction* $\rightarrow_?$ on $\mathcal{T}_\Omega$ is defined as follows: $t \rightarrow_? s$ iff $s \equiv t[s']_p$ for some redex position p in t and $s' \in \mathcal{T}_\Omega$.

2. The predicate $nf_?$ on $\mathcal{T}_\Omega$ is defined as follows: $nf_?(t) = \mathit{true}$ iff $t \rightarrow_?^* s$ for some normal form s.

Definition 2.3.17 A left-linear TRS is *strongly sequential* if every Ω-normal form has an index w.r.t. $nf_?$.

Indices of a term t w.r.t. $nf_?$ are indices of t w.r.t. nf because $\rightarrow_\mathcal{R} \subseteq \rightarrow_?$. Thus every strongly sequential TRS is sequential.

Huet and Lévy [13] gave a procedure to compute the indices with respect to $nf_?$.

Definition 2.3.18 The Ω-*reduction* $\rightarrow_\Omega$ is defined on $\mathcal{T}_\Omega$ as follows: $t \rightarrow_\Omega s$ iff $s \equiv t[\Omega]_p$ for some $p \in \mathcal{P}os(t)$ such that $t|_p \uparrow \mathit{Red}$ and $t|_p \not\equiv \Omega$. The set of normal forms with respect to Ω-reduction is denoted by NF_Ω.

Example 2.3.19 Let

$$\mathcal{R} = \begin{cases} f(x, f(y, a)) \rightarrow x \\ f(a, b) \rightarrow a \end{cases}$$

and $t \equiv f(f(\Omega, b), f(a, \Omega))$. Then $\mathit{Red} = \{ f(\Omega, f(\Omega, a)),\ f(a, b) \}$ and we have the following five Ω-reduction sequences from t to the normal form of t w.r.t. Ω-reduction:

$$\begin{array}{l} t \rightarrow_\Omega \Omega, \\ t \rightarrow_\Omega f(\Omega, f(a, \Omega)) \rightarrow_\Omega \Omega, \\ t \rightarrow_\Omega f(\Omega, f(a, \Omega)) \rightarrow_\Omega f(\Omega, \Omega) \rightarrow_\Omega \Omega, \\ t \rightarrow_\Omega f(f(\Omega, b), \Omega) \rightarrow_\Omega \Omega, \\ t \rightarrow_\Omega f(f(\Omega, b), \Omega) \rightarrow_\Omega f(\Omega, \Omega) \rightarrow_\Omega \Omega. \end{array}$$

The following lemma holds for Ω-reduction.

Lemma 2.3.20 ([18]) Ω-reduction is Church-Rosser and strongly normalizing. □

Definition 2.3.21 Let t be an Ω-term. The normal form of t with respect to Ω-reduction is denoted by $\omega(t)$.

Note that $\omega(t)$ is well-defined according to the previous lemma. We write $e \in \omega(t)$ if the normal form of t with respect to Ω-reduction has an occurrence of some symbol e.

Theorem 2.3.22 ([13]) Let t be an Ω-term and let p be an Ω-position in t. Then p is an index of t w.r.t. $nf_?$ iff $z \in \omega(t[z]_p)$ where z is fresh. □

Example 2.3.23 Consider the TRS $\mathcal{R}$ of Example 2.3.19. Using Theorem 2.3.22, we obtain $I_{nf_?}(f(\Omega, \Omega)) = \{2\}$ because $\omega(f(z, \Omega)) \equiv \Omega$ and $\omega(f(\Omega, z)) \equiv f(\Omega, z)$.

The decidability of strong sequentiality for orthogonal TRSs was first shown by Huet and Lévy [13] and then simplified proofs were presented by Klop and Middeldorp [18]. Jouannaud and Sadfi [15] proved the decidability of strong sequentiality assuming left-linearity instead of orthogonality. Also this result was proven by Comon [3].

Theorem 2.3.24 Strong sequentiality of left-linear TRSs is decidable. $\square$

We can obtain the following decidable reduction strategy for strongly sequential TRSs.

Definition 2.3.25 The *index reduction* $\rightarrow_I$ is defined on $\mathcal{T}$ as follows: $t \rightarrow_I s$ iff $t \xrightarrow{p} s$ for some p with $p \in I_{nf_?}(t[\Omega]_p)$.

Huet and Lévy [13] showed that index reduction is a normalizing strategy for every orthogonal strongly sequential TRSs. Toyama [30] generalized this result to the class of root balanced joinable strongly sequential TRSs. The *root reduction* $t \rightarrow_r s$ is defined by $t \xrightarrow{p} s$ and $p = \varepsilon$.

Definition 2.3.26 A TRS $\mathcal{R}$ is *root balanced joinable* if for any critical pair $\langle p, q\rangle$ of $\mathcal{R}$, there exist a term t and $k \geq 0$ such that $p \rightarrow_r^k t$ and $q \rightarrow_r^k t$.

Theorem 2.3.27 ([30]) Let $\mathcal{R}$ be a left-linear TRS. If $\mathcal{R}$ is root balanced joinable and strongly sequential then $\mathcal{R}$ has the normal form property and index reduction is a normalizing strategy for $\mathcal{R}$. $\square$

Huet and Lévy gave a syntactic characterization, which is called left-normal [25], for orthogonal strongly sequential TRSs in [13]. Toyama [30] removed the non-overlapping condition.

Definition 2.3.28 A TRS $\mathcal{R}$ is *left-normal* if in every rewrite rule $l \rightarrow r \in \mathcal{R}$ the function symbols in l precede the variable in l.

Example 2.3.29 The TRS of Example 2.3.2 is left-normal. The TRS of Example 2.3.19 is not left-normal since the variables x and y precede the constant a in the left-hand side $f(x, f(y, a))$.

Theorem 2.3.30 ([30]) Let $\mathcal{R}$ be a left-linear left-normal TRS. Then $\mathcal{R}$ is strongly sequential. Furthermore, if p is the leftmost-outermost redex position of a term t then p is an index of $t[\Omega]_p$ w.r.t. $nf_?$. $\square$

2.3.3 NV-sequentiality

Oyamaguchi [28] introduced a more general sufficient condition for sequentiality, which is called NV-sequentiality. NV-sequentiality is not only based on the analysis of the left-hand sides of the rewrite rules of TRSs but also on the non-variable parts of the right-hand sides.

Definition 2.3.31 Let $\mathcal{R}$ be a TRS.

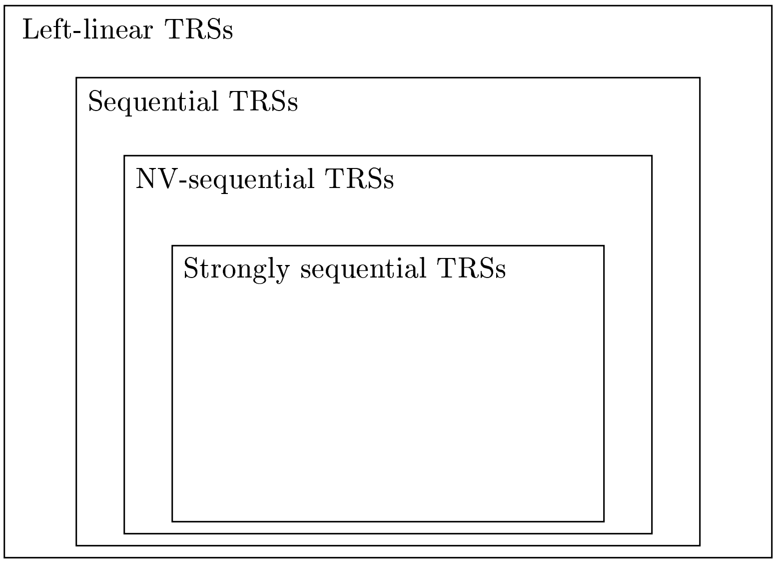

Figure 2.1.

1. The reduction relation $\to_{nv}$ on $\mathcal{T}_\Omega$ is defined as follows: $t \to_{nv} s$ iff there exist a rewrite rule $l \to r \in \mathcal{R}$, a position $p \in \mathcal{P}os(t)$ and a substitution θ such that $t|_p \equiv l\theta$ and $s \equiv t[s']_p$ for some $s' \geq r_\Omega$.

2. The predicate *term* on $\mathcal{T}_\Omega$ is defined as follows: $term(t) = true$ iff $t \to_{nv}^{*} s$ for some $s \in \mathcal{T}$.

Example 2.3.32 Let

$$\mathcal{R} = \begin{cases} f(g(a), x) \to x \\ f(a, x) \to g(f(x, x)) \\ g(x) \to g(x) \end{cases}$$

and $t \equiv f(f(a,a), \Omega)$. We have $term(t) = true$ because $t \to_{nv} f(g(f(a,a)), \Omega) \to_{nv} f(g(a), \Omega) \to_{nv} a$. Note that $nf(t) = false$.

Definition 2.3.33 A left-linear TRS is *NV-sequential* if every Ω-normal form has an index w.r.t. *term*.

Oyamaguchi [28] showed that every NV-sequential TRS is sequential and the class of NV-sequential TRSs properly includes the class of strongly sequential TRSs.

Theorem 2.3.34 ([28]) Let $\mathcal{R}$ be a left-linear TRS. Let $t \in \mathcal{T}_\Omega$ and $p \in \mathcal{P}os(t)$ with $t|_p \equiv \Omega$. It is decidable whether p is an index of t w.r.t. *term* in polynomial time. $\square$

Oyamaguchi [28] also showed that NV-sequentiality of orthogonal TRSs is decidable. This result was generalized to left-linear TRSs by Comon [3].

Theorem 2.3.35 NV-sequentiality is a decidable property of left-linear TRSs. $\square$

2.4 Tree Automata

Tree automata are generalization of sequential automata. Tree automata are useful for the decision problems in term rewriting [3, 5, 6, 8, 14]. Following Comon et al. [4], we adopt the definition of tree automata which is based on rewrite rules. More information on tree automata can be found in [4, 10].

Definition 2.4.1 A *tree automaton* is a tuple $\mathcal{A} = (\mathcal{F}, Q, Q^f, \Delta)$ where $\mathcal{F}$ is a signature, Q is a finite set of *states*, $Q^f \subseteq Q$ is a set of *final* states and Δ is a set of ground rewrite rules of the form $f(q_1, \ldots, q_n) \to q$ or $q \to q'$ where $f \in \mathcal{F}$, $q_1, \ldots, q_n, q, q' \in Q$. The latter rules are called *ϵ-rules*.

We use $\to_{\mathcal{A}}$ for the reduction relation $\to_\Delta$ on $\mathcal{T}(\mathcal{F} \cup Q)$.

Definition 2.4.2

1. A term $t \in \mathcal{T}(\mathcal{F})$ is *accepted* by $\mathcal{A}$ if $t \to_{\mathcal{A}}^* q$ for some $q \in Q_f$.
2. The tree language $L(\mathcal{A})$ recognized by $\mathcal{A}$ is the set of all terms accepted by $\mathcal{A}$.
3. A set $L \subseteq \mathcal{T}(\mathcal{F})$ is *recognizable* if there exists a tree automaton $\mathcal{A}$ such that $L = L(\mathcal{A})$.

Definition 2.4.3

1. A tree automaton $\mathcal{A}$ is *deterministic* if there are neither ϵ-rules nor different rules with the same left-hand side.
2. A tree automaton $\mathcal{A}$ is *complete* if there is at least one rule $f(q_1, \ldots, q_n) \to q$ in Δ for all $f \in \mathcal{F}$ and $q_1, \ldots, q_n \in Q$.

The following properties of tree automata are well-known [4, 10].

Lemma 2.4.4 Let L be a recognizable set. Then there exists a complete and deterministic tree automaton recognizing L. □

Lemma 2.4.5 The class of recognizable tree languages is closed under union, intersection and complementation. □

Lemma 2.4.6 The emptiness problem for tree automata is decidable. □

Chapter 3

NVNF-Sequentiality of Left-Linear TRSs

In this Chapter, we introduce an extension of NV-sequentiality [28]. This sequentiality is called NVNF-sequentiality. Like NV-sequentiality, NVNF-sequentiality is based on the analysis of left-hand sides and the non-variable parts of the right-hand side of rewrite rules. However, the reachability to a normal form is considered in NVNF-sequentiality. We first show that the class of NVNF-sequential TRSs properly includes the class of NV-sequential TRSs. Next we prove the decidability of indices with respect to NVNF-sequentiality. This implies that every orthogonal NVNF-sequential TRS has a decidable normalizing call-by-need strategy.

3.1 NVNF-Sequentiality

In this section we explain the notion of NVNF-sequentiality. NVNF-sequentiality is defined by using the reduction $\to_{nv}$ like NV-sequentiality. But indices w.r.t. NVNF-sequentiality are determined by the reachability to normal forms. The following predicate was given in [28].

Definition 3.1.1 Let $\mathcal{R}$ be a TRS. The predicate $nvnf$ on $\mathcal{T}_\Omega$ is defined as follows: $nvnf(t) = true$ iff $t \to_{nv}^{*} s$ for some normal form s.

Note that for every Ω-term t, $nvnf(t) = true$ implies $term(t) = true$.

Example 3.1.2 Let $\mathcal{F}_1 = \{\, f, a, b, c \,\}$ and

$$\mathcal{R}_1 = \begin{cases} f(a,b,x) \to a \\ f(b,x,a) \to b \\ f(x,a,b) \to c \\ c \to c. \end{cases}$$

Consider the Ω-term $t \equiv f(\Omega,\Omega,\Omega)$. Position 1 is an index of t w.r.t. $nvnf$. But the position 1 is not an index of t w.r.t. $term$ because we have $term(f(\Omega,a,b)) = true$.

Definition 3.1.3 A left-linear TRS is *NVNF-sequential* if every Ω-normal form has an index with respect to $nvnf$.

The decidability of NVNF-sequentiality was proven by Comon [3].

Theorem 3.1.4 NVNF-sequentiality of left-linear TRSs is decidable. □

In the remainder of this section we discuss the relationship between sequentiality, NVNF-sequentiality and NV-sequentiality.

Lemma 3.1.5

(i) Every NV-sequential TRS is NVNF-sequential.

(ii) Every NVNF-sequential TRS is sequential.

Proof.

(i) Suppose that $\mathcal{R}$ is NV-sequential. Let t be an Ω-normal form. Then t has an index p w.r.t. $term$. We will show that p is an index w.r.t. $nvnf$. Let s be an Ω-term such that $t \leq s$ and $nvnf(s) = ture$. Since $term(s) = ture$ and p is an index of t w.r.t. $term$, we obtain $s|_p \not\equiv \Omega$. Thus p is an index of t w.r.t. $nvnf$.

(ii) Similar to (i). □

We now prove that NVNF-sequentiality is a proper extension of NV-sequentiality.

Lemma 3.1.6 The TRS $(\mathcal{F}_1, \mathcal{R}_1)$ of Example 3.1.2 is NVNF-sequential but not NV-sequential.

Proof. Because the Ω-normal form $f(\Omega, \Omega, \Omega)$ has no indices w.r.t. $term$, $\mathcal{R}_1$ is not NV-sequential. In order to show that $\mathcal{R}_1$ is NVNF-sequential, we first prove the claim: for every Ω-term t, if $p \in I_{nvnf}(t[\Omega]_p)$ and $q \in I_{nvnf}(t|_p)$ then $p.q \in I_{nvnf}(t)$.

Proof of the claim. Because $\rightarrow_{nv} = \rightarrow_{\mathcal{R}_1}$, $nvnf(t) = true$ iff $nf(t) = true$ for every Ω-term t. Thus it suffices to show that if $p \in I_{nf}(t[\Omega]_p)$ and $q \in I_{nf}(t|_p)$ then $p.q \in I_{nf}(t)$. This follows from Theorem 6.4.10 in Chapter 6 because every variable in the left-hand side of the rewrite rule occurs at depth one.

We now prove that every Ω-normal form t has an index w.r.t. $nvnf$. The proof is by induction on the size of t. The case $t \equiv \Omega$ is trivial. Let $t \equiv f(t_1, t_2, t_3)$. We have the following four cases.

Case 1. t_1 is an Ω-normal form. Then by induction hypothesis, t_1 has an index w.r.t. $nvnf$. Since $1 \in I_{nvnf}(f(\Omega, t_2, t_3))$, it follows from the claim that t has an index w.r.t. $nvnf$.

Case 2. $t_1 \equiv a$. If t_2 contains Ω's then t_2 has an index w.r.t. $nvnf$ by induction hypothesis. Since we have $2 \in I_{nvnf}(f(a, \Omega, t_3))$, it follows from the claim that t has an index w.r.t. $nvnf$. Otherwise, t_3 is an Ω-normal form. From induction hypothesis, t_3 has an index w.r.t. $nvnf$. We can obtain $3 \in I_{nvnf}(f(a, t_2, \Omega))$ because t_2 is a normal form and $t_2 \not\equiv b$. Therefore from the claim, t has an index w.r.t. $nvnf$.

Case 3. $t_1 \equiv b$. Similar to Case 2.

Case 4. Otherwise, t_2 or t_3 is an Ω-normal form. Thus from induction hypothesis, t_2 or t_3 has an index w.r.t. $nvnf$. Because we can obtain $2 \in I_{nvnv}(f(t_1, \Omega, t_3))$ and $3 \in I_{nvnv}(f(t_1, t_2, \Omega))$, it follows from the claim that t has an index w.r.t. $nvnf$. □

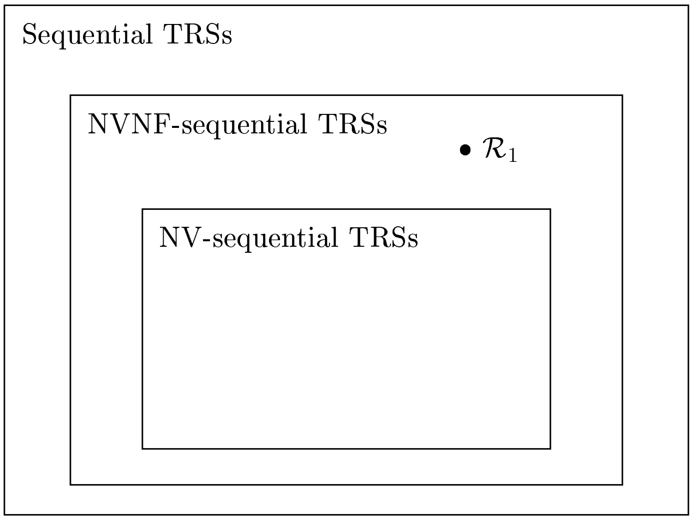

Figure 3.1.

Remark. The claim in the proof of Lemma 3.1.6 does not hold for arbitrary left-linear TRSs. Let $\mathcal{R} = \{ f(g(x), a) \to a \}$. Consider the Ω-normal form $f(g(\Omega), \Omega)$. We have $1 \in I_{nvnf}(f(\Omega, \Omega))$ and $1 \in I_{nvnf}(g(\Omega))$. However, $1.1 \notin I_{nvnf}(f(g(\Omega), \Omega))$.

From Lemmas 3.1.5 and 3.1.6, we obtain the following theorem.

Theorem 3.1.7 The class of NVNF-sequential TRSs properly includes the class of NV-sequential TRSs. □

3.2 Decidability of Indices with respect to NVNF-Sequentiality

In this section we show that for a given Ω-term t, it is decidable whether an Ω-position is an index of t w.r.t. $nvnf$. Throughout this section we assume that we are dealing with left-linear TRSs.

We first give a characterization of indices w.r.t. $nvnf$. For this purpose, we introduce the Ω_V-reduction [28].

Definition 3.2.1 The Ω_V-reduction is defined on $\mathcal{T}_\Omega$ as follows: $t \to_{\Omega_V} s$ iff there exist $l \to r \in \mathcal{R}$ and $p \in \mathcal{P}os(t)$ such that $t|_p \uparrow l_\Omega$, $t|_p \not\equiv \Omega$ and $s \equiv t[r_\Omega]_p$.

Example 3.2.2 Let

$$\mathcal{R} = \begin{cases} f(x, f(a, y)) \to g(y) \\ f(g(x), b) \to f(x, a). \end{cases}$$

We have the Ω_V-reduction $t \equiv f(\Omega, f(g(a), \Omega)) \to_{\Omega_V} f(\Omega, g(\Omega))$. We have also the Ω_V-reduction sequence $t \to_{\Omega_V} f(\Omega, f(\Omega, a)) \to_{\Omega_V} g(\Omega)$.

The following lemma expresses a relationship between Ω_V-reduction $\to_{\Omega_V}$ and $\to_{nv}$.

Lemma 3.2.3

(i) If $t \to_{nv}^* s$ and $t' \leq t$ then $t' \to_{\Omega_V}^* s'$ for some $s' \leq s$.

(ii) If $t \to_{\Omega_V}^* s$ then $t' \to_{nv}^* s$ for some $t' \geq t$.

Proof.

(i) We will prove the claim that if $t \to_{nv} s$ and $t' \leq t$ then $t' \to_{\Omega_V}^{\equiv} s'$ for some $s' \leq s$. Let $t \to_{nv} s$. Then there exist $l \to r \in \mathcal{R}$, $p \in \mathcal{P}os(t)$ and a substitution θ such that $t|_p \geq l\theta$ and $s \equiv t[s_1]_p$ for some $s_1 \geq r_\Omega$. We first consider the case $p \notin \mathcal{P}os(t')$. Clearly $t' \leq s$. Thus the claim holds. Next we consider the case $p \in \mathcal{P}os(t')$. If $t'|_p \equiv \Omega$ then $t' \leq s$ and therefore the claim holds. Otherwise, we can obtain $t' \to_{\Omega_V} t'[r_\Omega]_p$ because $t'|_p \uparrow l_\Omega$. We have $t'[r_\Omega]_p \leq t[s_1]_p \equiv s$. Hence the claim holds. Using the claim, we can prove (i) by induction on the length of $t \to_{nv}^* s$.

(ii) This is proven by induction on the length of $t \to_{\Omega_V}^* s$. The case of zero length is trivial. Assume that $t \to_{\Omega_V} s_1 \to_{\Omega_V}^* s$ where $t|_p \uparrow l_\Omega$, $t|_p \not\equiv \Omega$ and $s_1 \equiv t[r_\Omega]_p$ for $l \to r \in \mathcal{R}$ and $p \in \mathcal{P}os(t)$. From induction hypothesis, there exists an Ω-term s_2 such that $s_2 \to_{nv}^* s$ and $s_2 \geq s_1$. Let $t' \equiv s_2[t|_p \sqcup l_\Omega]_p$. Because $s_2|_p \geq r_\Omega$, we have $t' \to_{nv} s_2$ and thus $t' \to_{nv}^* s$. Since $s_2 \geq t[r_\Omega]_p$ and $t|_p \sqcup l_\Omega \geq t|_p$, we obtain $t' \equiv s_2[t|_p \sqcup l_\Omega]_p \geq t[t|_p]_p \equiv t$. □

We use t_x to denote the term obtained from $t \in \mathcal{T}_\Omega$ by replacing all Ω's with x.

Lemma 3.2.4 Let t be an Ω-term and let p be an Ω-position in t. Let z be a variable such that $z \notin t$. Then p is not an index of t w.r.t. $nvnf$ iff $t[z]_p \to_{\Omega_V}^* s$ for some s containing neither redexes nor z's.

Proof.

($\Rightarrow$) Suppose that p is not an index of t w.r.t. $nvnf$. Then there exists an Ω-term t' such that $t' \geq t$, $t'|_p \equiv \Omega$ and $nvnf(t') = true$. Because t does not contain z's, we can assume w.l.o.g. that $t' \to_{nv}^* s$ for some normal from s with $z \notin s$. From the left-linearity of $\mathcal{R}$, we obtain $t'[z]_p \to_{nv}^* s$. According to Lemma 3.2.3 (i), $t'[z]_p \to_{\Omega_V}^* s'$ for some $s' \leq s$. Becase s contains neither redexes nor z's, neither does s'.

($\Leftarrow$) We assume that $t[z]_p \to_{\Omega_V}^* s$ for s containing neither redexes nor z's. Then from by Lemma 3.2.3 (ii), $t' \to_{nv}^* s$ for some $t' \geq t[z]_p$. Let $t'' \equiv t'_x[\Omega]_p$ and $s' \equiv s_x$. We can easily show $t'' \to_{nv}^* s'$. Because s does not cotain redexes, s' is a normal form and hence $nvnf(t'') = true$. Clearly $t'' \geq t$ and $t''|_p \equiv \Omega$. Therefore p is not an index of t w.r.t. $nvnf$. □

We next show that if there exists an Ω-term s containing neither redexes nor z's such that $t[z]_p \to_{\Omega_V}^* s$ then we have an upper bound of the least hight of such Ω-terms.

Definition 3.2.5 Let $\mathcal{R}$ be a TRS. The set $\mathrm{RH}_\mathcal{R}$ is defined by $\mathrm{RH}_\mathcal{R} = \{\, r_\Omega \,|\, l \to r \in \mathcal{R}\}$. $\mathrm{RH}^*_\mathcal{R}$ is the smallest set such that $\mathrm{RH}_\mathcal{R} \subseteq \mathrm{RH}^*_\mathcal{R}$ and if $t \in \mathrm{RH}^*_\mathcal{R}$, $p \in \mathcal{P}os(t)$ and $r \in \mathrm{RH}_\mathcal{R}$ then $t[r]_p \in \mathrm{RH}^*_\mathcal{R}$.

It is clear that if $r \in \mathrm{RH}_{\mathcal{R}}$ and $r \to^*_{\Omega_V} t$ then $t \in \mathrm{RH}^*_{\mathcal{R}}$.

Lemma 3.2.6 If $t \to^+_{\Omega_V} s$ then there exist $p_1, \ldots, p_n \in \mathcal{P}os(t)$ which are pairwise disjoint and satisfy the following conditions.

1. $s \equiv t[s|_{p_1}, \ldots, s|_{p_n}]_{p_1,\ldots,p_n}$,
2. for each $1 \leq i \leq n$, there exists $r_i \in \mathrm{RH}_{\mathcal{R}}$ such that $t|_{p_i} \to^+_{\Omega_V} r_i \to^*_{\Omega_V} s|_{p_i}$.

Proof. We assume the following Ω_V-reduction sequence:

$$t \equiv t_0 \xrightarrow{q_0}_{\Omega_V} t_1 \xrightarrow{q_1}_{\Omega_V} \cdots \xrightarrow{q_n}_{\Omega_V} t_{n+1} \equiv s$$

where $n \geq 0$. Let $q_{i_1}, \ldots, q_{i_k}$ be the minimal positions in $\{q_0, q_1, \ldots, q_n\}$ w.r.t. $\leq$. Then $q_{i_1}, \ldots, q_{i_k} \in \mathcal{P}os(t)$ and they are pairwise disjoint. By the minimality of $q_{i_1}, \ldots, q_{i_k}$, they satisfy the conditions in the lemma. □

Definition 3.2.7 Let $\mathcal{R}$ be a TRS. The maximum hight of the left-hand sides and the right-hand sides of rewrite rules in $\mathcal{R}$ is denoted by $\rho_{\mathcal{R}}$.

Lemma 3.2.8 Let $r \in \mathrm{RH}_{\mathcal{R}}$. Let $r \to^*_{\Omega_V} s$ and $\rho(s) > \rho_{\mathcal{R}} \times n$ for some $n \geq 0$. Then there exist $p_0, p_1, \ldots, p_n \in \mathcal{P}os(s)$ such that:

1. $p_0 < p_1 < \cdots < p_n$,
2. for each $0 \leq i \leq n$, there exists $r_i \in \mathrm{RH}_{\mathcal{R}}$ such that $r \to^+_{\Omega_V} s[r_i]_{p_i}$ and $r_i \to^*_{\Omega_V} s|_{p_i}$.

Proof. We prove the lemma by induction on n. *Base step.* The case $n = 0$ is trivial because we can take ε as p_0. *Induction step.* We use induction on the length m of $r \to^*_{\Omega_V} s$. Assume that

$$r \equiv t_0 \xrightarrow{q_0}_{\Omega_V} t_1 \xrightarrow{q_1}_{\Omega_V} \cdots\cdots \xrightarrow{q_{m-1}}_{\Omega_V} t_m \equiv s.$$

Let S be the set of minimal positions in $\{q_0, q_1, \ldots, q_{m-1}\}$ w.r.t. $\leq$. Because $\rho(s) > \rho_{\mathcal{R}}$, S is not empty and for any $q \in S$, $q \in \mathcal{P}os(r)$ and $q \in \mathcal{P}os(s)$.

Case 1. $S = \{\varepsilon\}$. Then we have $t_j \in \mathrm{RH}_{\mathcal{R}}$ for some $j \geq 1$. Applying induction hypothesis on m to $t_j \to^*_{\Omega_V} s$, we obtain $p_0, p_1, \cdots, p_n \in \mathcal{P}os(s)$ such that: 1. $p_0 < p_1 < \cdots < p_n$, 2. for each $0 \leq i \leq n$, there exists $r_i \in \mathrm{RH}_{\mathcal{R}}$ such that $t_j \to^+_{\Omega_V} s[r_i]_{p_i}$ and $r_i \to^*_{\Omega_V} s|_{p_i}$. Clearly $r \to^+_{\Omega_V} s[r_i]_{p_i}$ for each $0 \leq i \leq n$. Thus the lemma holds.

Case 2. $S \neq \{\varepsilon\}$. Let $S = \{q'_1, \ldots, q'_{m'}\}$ with $m' > 0$. Then from the minimality of $q'_1, \ldots, q'_{m'}$, $s \equiv r[s|_{q'_1}, \ldots, s|_{q'_{m'}}]_{q'_1,\ldots,q'_{m'}}$ and for each $1 \leq i \leq m'$ there exist $r'_i \in \mathrm{RH}_{\mathcal{R}}$ such that $r|_{q'_i} \to^+_{\Omega_V} r'_i \to^*_{\Omega_V} s|_{q'_i}$. Because $\rho(s) > \rho_{\mathcal{R}} \times n$, we have j such that $\rho(s|_{q'_j}) > \rho_{\mathcal{R}} \times (n-1)$. Applying induction hypothesis on n to $r'_j \to^*_{\Omega_V} s|_{q'_j}$, we obtain $p_0, \cdots, p_{n-1} \in \mathcal{P}os(s|_{q'_j})$ such that: 1. $p_0 < \cdots < p_{n-1}$, 2. for each $0 \leq i \leq n-1$, there exists $r_i \in \mathrm{RH}_{\mathcal{R}}$ such that $r'_j \to^*_{\Omega_V} s|_{q'_j}[r_i]_{p_i}$ and $r_i \to^*_{\Omega_V} s|_{q'_j.p_i}$. Let $p'_0 = \varepsilon$ and $p'_i = q'_j.p_{i-1}$ for each $1 \leq i \leq n$. Then $p'_0 < p'_1 < \cdots < p'_n \in \mathcal{P}os(s)$ and we have $r \to^*_{\Omega_V} s[r]_{p_0}$ and $r \to^*_{\Omega_V} s|_{p_0}$. Because $r \to^*_{\Omega_V} s[r'_j]_{q'_j}$, we obtain $r \to^*_{\Omega_V} s[s|_{q'_j}[r_{i-1}]_{p_{i-1}}]_{q'_j} \equiv s[r_{i-1}]_{p'_i}$ and $r_{i-1} \to^*_{\Omega_V} s|_{p'_i}$ for each $1 \leq i \leq n$. Therefore the lemma holds. □

Definition 3.2.9 Let $\mathcal{R}$ be a TRS. Let t be an Ω-term. The *prefix* Ω-term $\mathsf{pref}_{\mathcal{R}}(t)$ of t is defined by $\mathsf{pref}_{\mathcal{R}}(t) \equiv t[\Omega, \ldots, \Omega]_{p_1,\ldots,p_n}$ where $\{p_1, \ldots, p_n\} = \{\, p \in \mathcal{P}os(t) \mid |\, p\,| = \rho_{\mathcal{R}}\}$.

Lemma 3.2.10 Let t and s be Ω-terms without redexes. Let $p \in \mathcal{P}os(t)$. If $\mathsf{pref}_{\mathcal{R}}(t|_p) \equiv \mathsf{pref}_{\mathcal{R}}(s)$ then $t[s]_p$ does not contain redexes.

Proof. From the left-linearity of $\mathcal{R}$. □

Definition 3.2.11 Let $\mathcal{R}$ be a TRS. The constant $\mu_{\mathcal{R}}$ is defined as follows:

$$\mu_{\mathcal{R}} = \rho_{\mathcal{R}} \times (\,|\{\, \mathsf{pref}_{\mathcal{R}}(t) \mid t \in \mathrm{RH}^*_{\mathcal{R}} \,\}| \times |\mathcal{R}| + 1)$$

where $|A|$ denotes the number of elements in a set A.

Lemma 3.2.12 Let t be an Ω-term and let p be an Ω-position in t. Let z be a variable with $z \notin t$. Then $p \notin I_{nvnf}(t)$ iff there exists an Ω-term s containing neither redexes nor z's such that $t[z]_p \to^*_{\Omega_V} s$ and $\rho(s) \leq \rho(t) + \mu_{\mathcal{R}}$.

Proof.

($\Rightarrow$) Assume that $p \notin I_{nvnf}(t)$. Using Lemma 3.2.4, we can obtain the minimal Ω-term s containing neither redexes nor z's such that $t[z]_p \to^*_{\Omega_V} s$. Suppose $\rho(s) > \rho(t) + \mu_{\mathcal{R}}$. Since s does not contain z's, $t[z]_p \to^+_{\Omega_V} s$. From Lemma 3.2.6, there exist $p_1, \ldots, p_n \in \mathcal{P}os(s)$ such that: 1. $s \equiv t[z]_p[s|_{p_1}, \ldots, s|_{p_n}]_{p_1,\ldots,p_n}$, 2. for each $1 \leq i \leq n$, there exists $r_i \in \mathrm{RH}_{\mathcal{R}}$ such that $t[z]_p|p_i \to^+_{\Omega_V} r_i \to^*_{\Omega_V} s|p_i$. By the assumption that $\rho(s) > \rho(t) + k_{\mathcal{R}}$, $\rho(s|_{p_j}) > \mu_{\mathcal{R}}$ for some j. From Lemma 3.2.8 and the definition of $\mu_{\mathcal{R}}$, we can obtain $r \in \mathrm{RH}_{\mathcal{R}}$ and $q_1, q_2 \in \mathcal{P}os(s|_{p_j})$ with $q_1 < q_2$ such that $\mathsf{pref}_{\mathcal{R}}(s|_{p_j.q_1}) \equiv \mathsf{pref}_{\mathcal{R}}(s|_{p_j.q_2})$ and for $i = 1, 2$, $r_j \to^*_{\Omega_V} s|_{p_j}[r]_{q_i}$ and $r \to^*_{\Omega_V} s|_{p_j.q_i}$. Let $s' \equiv s[s|_{p_j.q_2}]_{p_j.q_1}$, see Figure 3.2. Then $z \notin s'$ and it follows from Lemma 3.2.10 that s' does not contain redexes. Because $r_j \to^*_{\Omega_V} s|_{p_j}[r]_{q_1}$ and $r \to^*_{\Omega_V} s|_{p_j.q_2}$, we have $t[z]_p \to^*_{\Omega_V} s[r_j]_{p_j} \to^*_{\Omega_V} s[s|_{p_j}[r]_{q_1}]_{p_j} \equiv s[r]_{p_j.q_1} \to^*_{\Omega_V} s[s|_{p_j.q_2}]_{p_j.q_1} \equiv s'$. However, this contradicts the minimality of s.

($\Leftarrow$) From Lemma 3.2.4. □

By Lemma 3.2.12, in order to determine whether an Ω-position p in an Ω-term t is an index w.r.t. $nvnf$, we need to check the reachability from $t[z]_p$ to a finite number of Ω-terms by Ω_V-reduction. It was shown by Oyamaguchi [28] that Ω-reduction is simulated by the usual reduction of some TRS.

Definition 3.2.13 Let $\mathcal{R}$ be a TRS. The TRS $\mathcal{R}_\Omega$ is defined as follows:

$$\mathcal{R}_\Omega = \{\, l \to r_\Omega \mid l \to r \in \mathcal{R} \,\} \cup \{\, \Omega \to t \mid t \subseteq l_\Omega,\, l \to r \in \mathcal{R} \,\}.$$

From the assumption that $\mathcal{R}$ is left-linear, $\mathcal{R}_\Omega$ is left-linear and right-ground (i.e., all the right-hand side of its rewrite rules is ground).

Lemma 3.2.14 ([28]) Let $\mathcal{R}$ be a left-linear TRS.

(i) If $t \to^*_{\Omega_V} s$ then $t \to^*_{\mathcal{R}_\Omega} s$.

(ii) If $t \to^*_{\mathcal{R}_\Omega} s$ and $t' \leq t$ then $t' \to^*_{\Omega_V} s'$ for some $s' \leq s$. □

We can replace $\to^*_{\Omega_V}$ with $\to^*_{\mathcal{R}_\Omega}$ in Lemma 3.2.12.

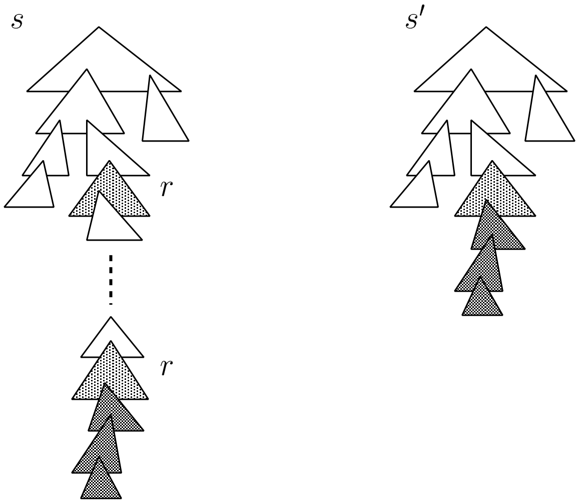

Figure 3.2.

Lemma 3.2.15 Let t be an Ω-term and let p be an Ω-position in t. Let z be a variable with $z \notin t$. Then $p \notin I_{nvnf}(t)$ iff there exists an Ω-term s containing neither redexes nor z's such that $t[z]_p \rightarrow^*_{\mathcal{R}_\Omega} s$ and $\rho(s) \leq \rho(t) + \mu_{\mathcal{R}}$.

Proof. From Lemmas 3.2.12 and 3.2.14. $\square$

It has been shown that the reachability problem is decidable for left-linear and right-ground TRSs [5, 26]. Thus we obtain the following theorem.

Theorem 3.2.16 Let $\mathcal{R}$ be a left-linear TRS. Let $t \in \mathcal{T}_\Omega$ and $p \in \mathcal{P}os(t)$ with $t|_p \equiv \Omega$. It is decidable whether p is an index of t w.r.t $nvnf$. $\square$

Chapter 4

Index Reduction of Overlapping TRSs

In this chapter, we investigate normalizing strategies for left-linear overlapping TRSs. Huet and Lévy [13] showed that every orthogonal strongly sequential TRS has a decidable normalizing strategy which is called index reduction. Toyama [30] extended this result to root balanced joinable strongly sequential TRSs. In Section 4.1, we prove that index reduction is normalizing for stable balanced joinable strongly sequential TRSs. This class properly includes the class of root balanced joinable strongly sequential TRSs. In Section 4.2, we discuss reduction strategies for NV-sequential TRSs which were introduced by Oyamaguchi [28]. We introduce the notion of NV-stable balanced joinability and prove that every NV-stable balanced joinable NV-sequential TRS has a decidable normalizing strategy.

In this chapter we are dealing with left-linear TRSs only.

4.1 A Normalizing Strategy for Stable Balanced Joinable TRSs

4.1.1 Stable Balanced Joinability

In this subsection, we define stable balanced joinable TRSs. For that purpose, we need the notions of transitivity, which was introduced by Toyama et al. [31], and stability for indices w.r.t $nf_?$. In the following we will refer to an index w.r.t. $nf_?$ as an *index* for short. We write $C[\Omega_I]$ if the displayed occurrence of Ω in $C[\Omega]$ is an index. Thus by Theorem 2.3.22, $C[\Omega_I]$ iff $z \in \omega(C[z])$ where z is fresh. Let $C[\Omega_I]$ and let Δ be a redex. Then Δ is also called an index of $C[\Delta]$ and we write $C[\Delta_I]$.

Definition 4.1.1 The displayed index in $C[\Omega_I]$ is *transitive* if $C'[C[\Omega_I]]$ for any $C'[\Omega_I]$. The transitive index is denoted by $C[\Omega_T]$.

Example 4.1.2 Let $Red = \{ f(g(\Omega)) \}$. The Ω-occurrence in $g(\Omega)$ is an index. However, this index in $g(\Omega)$ is not transitive because the Ω-occurrence in $f(g(\Omega))$ is not an index.

We recall properties of indices and transitive indices [13, 15, 18, 31].

Lemma 4.1.3

(i) If $C[\Omega_I]$ and $C[z] \leq C'[z]$ where z is fresh, then $C'[\Omega_I]$.

(ii) If $C[C'[\Omega_I]]$ then $C'[\Omega_I]$. □

Lemma 4.1.4 If $C[\Omega_T]$ and $C[z] \leq C'[z]$ where z is fresh, then $C'[\Omega_T]$.

Proof. Let $C'''[\Omega_I]$. Since $C[\Omega_T]$, we have $C'''[C[\Omega_I]]$. Clearly $C'''[C[z]] \leq C'''[C'[z]]$. By Lemma 4.1.3 (i), it is obtained that $C'''[C'[\Omega_I]]$. Thus $C'[\Omega_T]$. □

Definition 4.1.5

- The displayed transitive index in $C[\Omega_T]$ is *stable*, which is denoted by $C[\Omega_S]$, if $C\theta[\Omega_T]$ for any substitution θ.
- The *stable reduction* $\to_S$ is defined as $C[l\theta] \to_S C[r\theta]$ where $C[\Omega_S]$ and $l \to r \in \mathcal{R}$.

Lemma 4.1.6 If $t \to_S s$ and $C[\Omega_I]$ then $C[t\theta] \to_I C[s\theta]$ for any θ.

Proof. Let $t \equiv C'[l\theta'] \to_S C'[r\theta'] \equiv s$. From $C'[\Omega_S]$, it follows that $C'\theta[\Omega_T]$ for any θ. By the definition of transitivity, we have $C[C'\theta[\Omega_I]]$. Thus $C[t\theta] \equiv C[C'\theta[l\theta'\theta]] \to_I C[C'\theta[r\theta'\theta]] \equiv C[s\theta]$. □

Definition 4.1.7 A critical pair $\langle p, q\rangle$ is *stable balanced joinable* if $p \to_S^k t$ and $q \to_S^k t$ for some t and $k \geq 0$. A TRS $\mathcal{R}$ is *stable balanced joinable* if every critical pair is stable balanced joinable.

Note that every root balanced joinable TRS is stable balanced joinable because $\to_r \subseteq \to_S$.

4.1.2 Normalizability of Index Reduction

In this subsection, we show that index reduction is normalizing for every stable balanced joinable strongly sequential TRS. Our proof uses the theorem of Toyama [30] concerning reduction strategies. We first explain this theorem.

Definition 4.1.8 Let $A = \langle D, \to\rangle$ be an ARS. We write $a \longleftrightarrow\!\!\!\!\to b$ if there exists a connection $a \to^{m_1} \cdot \leftarrow^{n_1} \cdot \to^{m_2} \cdot \leftarrow^{n_2} \cdots \to^{m_p} \cdot \leftarrow^{n_p} b$ with $\sum m_i > \sum n_i$. We write $a \twoheadleftarrow\!\!\!\!\longrightarrow b$ if $b \longleftrightarrow\!\!\!\!\to a$.

Definition 4.1.9 Let $A = \langle D, \to\rangle$ be an ARS. A reduction relation $\to$ on D is *balanced weakly Church-Rosser* if $\forall a_1, a_2, a_3 \in D$, $a_1 \to a_2$ and $a_1 \to a_3$ imply $a_2 \to^k b$ and $a_3 \to^k b$ for some $b \in D$ and $k \geq 0$.

Theorem 4.1.10 ([30]) Let $A = \langle D, \to\rangle$ be an ARS. Let $\to_s$ be a reduction strategy for $\to$ such that:

(i) $\to_s$ is balanced weakly Church-Rosser,

(ii) If $a \to b$ then $a =_s b$ or $a \longleftrightarrow\!\!\!\!\to_s \cdot \leftrightarrow \cdot \twoheadleftarrow\!\!\!\!\longrightarrow_s b$.

Then $\to$ has the normal form property and $\to_s$ is a normalizing strategy. □

Let Δ and Δ' be two redex occurrences of $t \in \mathcal{T}$. Let $\Delta \equiv C[s_1, \ldots, s_n]$ and $C[\Omega, \ldots, \Omega] \in \mathit{Red}$. We say that Δ and Δ' (or Δ' and Δ) are *overlapping* if $\Delta' \subseteq \Delta$ and $\Delta' \not\subseteq s_i$ for any $1 \leq i \leq n$.

Lemma 4.1.11 Let $\mathcal{R}$ be stable balanced joinable. Let $t \stackrel{\Delta}{\rightarrow}_I t'$ and $t \stackrel{\Delta'}{\rightarrow} t''$, where $\Delta' \subseteq \Delta$ and Δ and Δ' are overlapping. Then $t' \rightarrow_I^k s$ and $t'' \rightarrow_I^k s$ for some s and $k \geq 0$.

Proof. Let $t \equiv C[\Delta] \equiv C[C'[\Delta']]$. Then $t' \equiv C[q\theta]$ and $t'' \equiv C[p\theta]$ for some critical pair $\langle p, q \rangle$ and θ. Since $\mathcal{R}$ is stable balanced joinable, we have $p \rightarrow_S^k s'$ and $q \rightarrow_S^k s'$ for some s'. Thus, from Lemma 4.1.6 and $C[\Omega_I]$, we obtain $t' \equiv C[q\theta] \rightarrow_I^k C[s'\theta]$ and $t'' \equiv C[p\theta] \rightarrow_I^k C[s'\theta]$. □

Lemma 4.1.12 ([30]) Let $C[\Delta_I, \Delta']$. Then $C[\Delta_I, t]$ for any t. □

Lemma 4.1.13 Let $\mathcal{R}$ be stable balanced joinable. If $t \rightarrow_I t'$ and $t \rightarrow_I t''$ then $t' \rightarrow_I^k s$ and $t'' \rightarrow_I^k s$ for some s and $k \geq 0$.

Proof. Let $t \stackrel{\Delta}{\rightarrow}_I t'$ and $t \stackrel{\Delta'}{\rightarrow}_I t''$. If Δ and Δ' are disjoint then from Lemma 4.1.12 the lemma follows. If Δ and Δ' are not disjoint, then by Theorem 2.3.22, Δ and Δ' must be overlapping. Thus the lemma holds by Lemma 4.1.11. □

The *parallel reduction* $t \mathrel{+\!\!\!\!\twoheadrightarrow} s$ is defined as $t \equiv C[\Delta_1, \ldots, \Delta_n] \stackrel{\Delta_1}{\rightarrow} \cdots \stackrel{\Delta_n}{\rightarrow} s$ $(n \geq 0)$. We write $t \mathrel{+\!\!\!\!\twoheadrightarrow}' s$ if $t \stackrel{\Delta_1 \cdots \Delta_n}{\mathrel{+\!\!\!\!\twoheadrightarrow}} s$ and $n > 0$.

Lemma 4.1.14 Let $\mathcal{R}$ be strongly sequential and stable balanced joinable and $t \mathrel{+\!\!\!\!\twoheadrightarrow} s$. Then $t =_I s$ or $t \longleftrightarrow\!\!\!\!\!\twoheadrightarrow_I \cdot \mathrel{+\!\!\!\!\twoheadrightarrow} \cdot \twoheadleftarrow\!\!\!\!\!\longrightarrow_I s$.

Proof. Let $t \stackrel{\Delta_1 \cdots \Delta_n}{\mathrel{+\!\!\!\!\twoheadrightarrow}} s$. We prove the lemma by induction on n. The case $n = 0$ is trivial. Let $t \stackrel{\Delta_1 \cdots \Delta_n}{\mathrel{+\!\!\!\!\twoheadrightarrow}} s$ $(n > 0)$. There are two cases.

(1) Some Δ_i, say Δ_1, is an index. Let $t \stackrel{\Delta_1}{\rightarrow}_I t' \stackrel{\Delta_2 \cdots \Delta_n}{\mathrel{+\!\!\!\!\twoheadrightarrow}} s$. Applying induction hypothesis to $t' \stackrel{\Delta_2 \cdots \Delta_n}{\mathrel{+\!\!\!\!\twoheadrightarrow}} s$, we obtain the lemma.

(2) No Δ_i is an index. Since $\mathcal{R}$ is strongly sequential, t has an index. Let Δ be an index of t and $t \stackrel{\Delta}{\rightarrow}_I t''$. Furthermore, consider the following two cases.

(2-1) Δ and Δ_i are non-overlapping for any i. Using the left-linearity of $\mathcal{R}$ and Lemma 4.1.12, we can easily show that $t'' \mathrel{+\!\!\!\!\twoheadrightarrow} s'$ and $s \rightarrow_I s'$ for some s'. Thus we have $t \longleftrightarrow\!\!\!\!\!\twoheadrightarrow_I \cdot \mathrel{+\!\!\!\!\twoheadrightarrow} \cdot \twoheadleftarrow\!\!\!\!\!\longrightarrow_I s$.

(2-2) Δ and some Δ_i, say Δ_1, are overlapping. Let $t \stackrel{\Delta_1}{\rightarrow} t' \stackrel{\Delta_2 \cdots \Delta_n}{\mathrel{+\!\!\!\!\twoheadrightarrow}} s$. By Theorem 2.3.22, we have $\Delta_1 \subseteq \Delta$. From Lemma 4.1.11, it follows that $t'' \rightarrow_I^k s'$ and $t' \rightarrow_I^k s'$ for some s' and $k \geq 0$. Thus we have $t \longleftrightarrow\!\!\!\!\!\twoheadrightarrow_I t'$. Applying induction hypothsis to $t' \stackrel{\Delta_2 \cdots \Delta_n}{\mathrel{+\!\!\!\!\twoheadrightarrow}} s$, we obtain the lemma. □

Theorem 4.1.15 Let $\mathcal{R}$ be strongly sequential and stable balanced joinable. Then $\mathcal{R}$ has the normal form property, and index reduction $\rightarrow_I$ is a normalizing strategy for $\mathcal{R}$.

Proof. It is obvious that $\to_I$ is a reduction strategy for $\twoheadrightarrow\!\!\!\!\!\!+\;'$. Take $\to_I$ as $\to_s$ and $\twoheadrightarrow\!\!\!\!\!\!+\;'$ as $\to$ in Theorem 4.1.10. From Lemmas 4.1.13 and 4.1.14, the conditions (i) and (ii) in Theorem 4.1.10 are satisfied. Thus, from $\to \subseteq \twoheadrightarrow\!\!\!\!\!\!+\;' \subseteq \to^*$, the theorem follows. $\square$

Definition 4.1.16 The *Quasi-index reduction* (or *hyper-index reduction*) is defined as $\to^* \cdot \to_I$.

In Theorem 4.1.15 index reduction can be relaxed into quasi-index reduction.

Theorem 4.1.17 Let $\mathcal{R}$ be strongly sequential and stable balanced joinable. Then quasi-index reduction $\to^* \cdot \to_I$ is a normalizing strategy for $\mathcal{R}$.

Proof. Similar to Theorem 7.2 in [30]. $\square$

4.1.3 Decidability of Stable Transitive Indices

Stable balanced joinability is an undecidable property for left-linear TRSs. Because the halting problem for Turing machines is reducible to this problem by the construction of a left-linear TRS which can simulate the computations of a Turing machine. (For a construction, see [17].) In this subsection, we show that for a given $C[\Omega]$ we can determine whether the displayed occurrence of Ω in $C[\Omega]$ is a stable transitive index. Then we have the semi-decidablity of stable balanced joinability for left-linear TRSs as follows. Let $\mathcal{R}$ be a left-linear TRS. Let $\langle p, q\rangle$ be a critical pair of $\mathcal{R}$. We first generate all 1-step stable reductions from p and q. In next step we generate all 2-step stable reductions from p and q, then all 3-step stable reductions, ... If there exist $t \in \mathcal{T}$ and $k \in \mathcal{N}$ such that $p \to_S^k t$ and $q \to_S^k t$ then we can find such t and k. Thus, stable balanced joinability of $\langle p, q\rangle$ is semi-decidable. Since a number of critical pairs of $\mathcal{R}$ is finite, stable balanced joinability of $\mathcal{R}$ is semi-decidable.

Lemma 4.1.18 Let $C[t, \Omega_I]$. Then $C[x, \Omega_I]$ where x is a fresh variable.

Proof. Suppose that the displayed occurrence of Ω in $C[x, \Omega]$ is not an index. Thus $z \notin \omega(C[x, z])$ where z is fresh. Let θ be a substitution such that $x\theta \equiv t$ and $y\theta \equiv y$ for any $y \not\equiv x$. Then $C[t, z] \equiv C[x, z]\theta$. Because Ω-reduction is closed under substitutions, $C[t, z] \equiv C[x, z]\theta \to_\Omega^* \omega(C[x, z])\theta$. Since $\omega(C[t, z]) \leq \omega(C[x, z])\theta$ and $z \notin \omega(C[x, z])\theta$, we obtain $z \notin \omega(C[t, z])$. However, this is contradictory to $C[t, \Omega_I]$. $\square$

Definition 4.1.19 The set *Red** is defined as follows:

$$Red^* = \{\, t_\Omega \mid l \equiv C[t],\ C[\Omega_I],\ l \to r \in \mathcal{R}\,\}.$$

Note that the above definition of *Red** is different from the original one by Toyama et al. [31]. In fact, our *Red** is a subset of theirs, and these two sets are equal if $\mathcal{R}$ is orthogonal.

Example 4.1.20 Let

$$\mathcal{R}_1 = \begin{cases} f(a, x) \to a \\ f(b, g(x)) \to g(b) \\ b \to b. \end{cases}$$

Then $Red^* = \{\, f(a, \Omega),\ f(b, g(\Omega)),\ a,\ b\,\}$.

Lemma 4.1.21 Let $C[\Omega_I]$ and $C[t] \uparrow Red$. Then $t \uparrow Red^*$.

Proof. Since $C[t] \uparrow Red$, there exists a left-hand side l of $\mathcal{R}$ such that $C[t] \uparrow l_\Omega$. Because $C[\Omega_I]$, we have $l \equiv C'[s]$ for some s and $C'[\;]$ such that $t \uparrow s_\Omega$ and $C[z] \uparrow C'_\Omega[z]$ where z is fresh. Now we show that $s_\Omega \in Red^*$. Without loss of generality, we may state that $C[z] \equiv C''[s_1, \ldots, s_n, z, \Omega, \ldots, \Omega]$ and $C'[z] \equiv C''[x_1, \ldots, x_n, z, t_1, \ldots, t_m]$ where $C''[, \ldots ,]$ does not contain variables and $\Omega < t_{i\Omega}$ for $i = 1, \ldots, m$. Repeated application of Lemma 4.1.18 yields $C''[x_1, \ldots, x_n, \Omega_I, \Omega, \ldots, \Omega]$. Since $C''[x_1, \ldots, x_n, z, \Omega, \ldots, \Omega] \leq C'[z]$, it follows from Lemma 4.1.3 (i) that $C'[\Omega_I]$. Thus $s_\Omega \in Red^*$. □

Lemma 4.1.22 Let $C[\Omega] \in \mathcal{T}_\Omega$. Then $C[\Omega_T]$ iff $z \in \omega(C[z])$ and $\omega(C[z]) \# Red^*$ where z is fresh.

Proof.

($\Rightarrow$) Since we have $C[\Omega_I]$, $z \in \omega(C[z])$. Let $C'[z] \equiv \omega(C[z])$. Suppose $C'[z] \uparrow s$ for some $s \in Red^*$. Then there exists $C''[\;]$ such that $C''[\Omega_I]$ and $C''_\Omega[s] \in Red$. Since $C''[C'[z]] \uparrow Red$, $\omega(C''[C[z]]) \equiv \omega(C''[C'[z]]) \equiv \Omega$. But this contradicts $C[\Omega_T]$. Hence $\omega(C[z]) \# Red^*$.

($\Leftarrow$) We obtain $C[\Omega_I]$ because $z \in \omega(C[z])$. We will prove $C'[C[\Omega_I]]$ for any $C'[\Omega_I]$. Let $\omega(C[z]) \equiv C_1[z]$ and $\omega(C'[z]) \equiv C'_1[z]$. It suffices to show that $C'_1[C_1[z]] \in \mathrm{NF}_\Omega$. Suppose $C'_1[C_1[z]] \notin \mathrm{NF}_\Omega$. Since $C_1[z] \in \mathrm{NF}_\Omega$ and $C'_1[z] \in \mathrm{NF}_\Omega$, there exists $C''[C_1[z]] \subseteq C'_1[C_1[z]]$ such that $C''[C_1[z]] \uparrow Red$. From $C'_1[\Omega_I]$ and Lemma 4.1.3 (ii), $C''[\Omega_I]$. By using Lemma 4.1.21 we obtain $C_1[z] \uparrow Red^*$. But this contradicts $\omega(C[z]) \# Red^*$. □

Lemma 4.1.23 Let $C[\Omega] \in \mathcal{T}_\Omega$. Then $C[\Omega_S]$ iff $C_\Omega[\Omega_T]$.

Proof.

($\Rightarrow$) Let θ be a substitution such that $x\theta$ is a redex for any $x \in C[\;]$. Note that $C\theta[\Omega_T]$ and $\omega(C\theta[z]) \equiv \omega(C_\Omega[z])$. We will show that $C'[C_\Omega[\Omega_I]]$ for any $C'[\Omega_I]$. Because $C'[C\theta[\Omega_I]]$, we have $z \in \omega(C'[C\theta[z]]) \equiv \omega(C'[\omega(C\theta[z])]) \equiv \omega(C'[\omega(C_\Omega[z])]) \equiv \omega(C'[C_\Omega[z]])$. Thus $C'[C_\Omega[\Omega_I]]$.

($\Leftarrow$) Clearly $C_\Omega[z] \leq C\theta[z]$ for any θ. From Lemma 4.1.4 and $C_\Omega[\Omega_T]$, it follows that $C\theta[\Omega_T]$ for any θ. Therefore we obtain $C[\Omega_S]$. □

Lemma 4.1.24 Let $C[\Omega] \in \mathcal{T}_\Omega$. Then $C[\Omega_S]$ iff $z \in \omega(C_\Omega[z])$ and $\omega(C_\Omega[z]) \# Red^*$ where z is fresh.

Proof. It is trivial from Lemmas 4.1.22 and 4.1.23. □

Therefore, by the previous lemma, we can decide whether $C[\Omega_S]$ for a given $C[\Omega]$.

Example 4.1.25 Let

$$\mathcal{R}_2 = \begin{cases} f(g(x), y) \to h(g(x)) \\ g(a) \to g(b) \\ c \to g(c). \end{cases}$$

$\mathcal{R}_2$ has only one critical pair $\langle f(g(b), y),\, h(g(a)) \rangle$. $Red^* = \{\, f(g(\Omega),\, \Omega),\, g(\Omega),\, g(a),\, a,\, c \,\}$. Because $\omega(h(z)) \equiv h(z) \# Red^*$, it follows from Lemma 4.1.24 that $h(\Omega_S)$. Since $f(g(b), y) \to_S h(g(b)) \leftarrow_S h(g(a))$, $\mathcal{R}_2$ is stable balanced joinable. Note that $\mathcal{R}_2$ is not root balanced joinable. $\mathcal{R}_2$ is strongly sequential from Theorem 2.3.30 since $\mathcal{R}_2$ is left-normal. Thus, from Theorem 4.1.15, index reduction is a normalizing strategy for $\mathcal{R}_2$.

4.2 A Normalizing Strategy for NV-Stable Balanced Joinable TRSs

4.2.1 NV-Stable Balanced Joinability and a Normalizing Strategy

In this subsection, similar to Subsection 4.1.1, we define NV-stable balanced joinability for left-linear TRSs. We prove that NV-index reduction is normalizing for NV-stable balanced joinable NV-sequential TRSs. In the following indices w.r.t. *term* are called NV-indices. If the displayed occurrence of Ω in $C[\Omega]$ is an NV-index then we write $C[\Omega_{I_V}]$; otherwise $C[\Omega_{NI_V}]$. If $C[\Omega_{I_V}]$ then a redex occurrence Δ in $C[\Delta]$ is also called an NV-index. If Δ is an NV-index of $C[\Delta]$ then we write $C[\Delta_{I_V}]$; otherwise $C[\Delta_{NI_V}]$. The following lemma is used later.

Lemma 4.2.1 ([28])

(i) If $C[\Omega_{I_V}]$ and $C[z] \leq C'[z]$ where z is fresh, then $C'[\Omega_{I_V}]$.

(ii) If $C[C'[\Omega_{I_V}]]$ then $C'[\Omega_{I_V}]$. □

Definition 4.2.2 The displayed NV-index in $C[\Omega_{I_V}]$ is *transitive* if $C'[C[\Omega_{I_V}]]$ for any Ω-term $C'[\Omega_{I_V}]$. If the displayed occurrence of Ω in $C[\Omega]$ is a transitive NV-index then we write $C[\Omega_{T_V}]$; otherwise $C[\Omega_{NT_V}]$.

The following example shows that a transitive index is not always a transitive NV-index.

Example 4.2.3 Consider $\mathcal{R}_1$ of Example 4.1.20. We can show $g(\Omega_T)$ by using Lemma 4.1.22. However, we have $g(\Omega_{NT_V})$ because $f(b, g(\Omega_{NI_V}))$ for $f(b, \Omega_{I_V})$.

Definition 4.2.4

- The displayed transitive NV-index in $C[\Omega_{T_V}]$ is *stable* if $C\theta[\Omega_{T_V}]$ for any θ. If the displayed occurrence of Ω in $C[\Omega]$ is a stable transitive NV-index then we write $C[\Omega_{S_V}]$; otherwise $C[\Omega_{NS_V}]$.
- The *NV-stable reduction* $\rightarrow_{S_V}$ is defined as $C[l\theta] \rightarrow_{S_V} C[r\theta]$ where $C[\Omega_{S_V}]$ and $l \rightarrow r \in \mathcal{R}$.

Lemma 4.2.5 If $t \rightarrow_{S_V} s$ and $C[\Omega_{I_V}]$ then $C[t\theta] \rightarrow_{I_V} C[s\theta]$ for any θ.

Proof. Similar to Lemma 4.1.6. □

Definition 4.2.6 A critical pair $\langle p, q\rangle$ is *NV-stable balanced joinable* if $p \rightarrow^k_{S_V} t$ and $q \rightarrow^k_{S_V} t$ for some t and $k \geq 0$. A TRS $\mathcal{R}$ is *NV-stable balanced joinable* if every critical pair is NV-stable balanced joinable.

Note that the class of NV-stable balanced joinable TRSs includes all root balanced joinable TRSs. However, this class does not include all stable balanced joinable TRSs. Consider $\mathcal{R}_1$ of Example 4.1.20 which is stable balanced joinable. $\mathcal{R}_1$ has only one critical pair $\langle f(b, g(x)), g(b)\rangle$. Because $g(\Omega_{NT_V})$, $g(b)$ cannot be reduced by $\rightarrow_{S_V}$. Thus, $\mathcal{R}_1$ is not NV-stable balanced joinable. Figure 4.1 shows the relationship between these classes. Areas (1), (2) and (3) denote the class of root balanced joinable, stable balanced joinable and stable balanced joinable TRSs, respectively. Note that stable balanced joinable TRS $\mathcal{R}_2$ of Example 4.1.25 is also NV-stable balanced joinable. In Example 4.2.25, we will give NV-stable balanced joinable TRS $\mathcal{R}_3$ which is not stable balanced joinable.

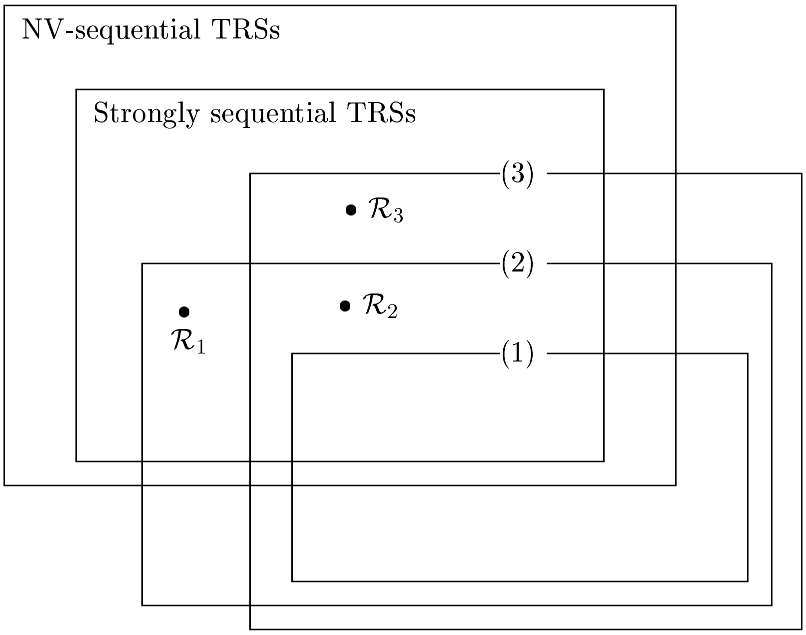

Figure 4.1.

We now define NV-index reduction as follows.

Definition 4.2.7 The *NV-index reduction* $\rightarrow_{I_V}$ is defined on $\mathcal{T}$ as follows: $t \rightarrow_{I_V} s$ iff $t \stackrel{\Delta}{\rightarrow} s$ for some NV-index Δ.

We can easily show that if $\mathcal{R}$ is NV-sequential then NV-index reduction is a reduction strategy for $\mathcal{R}$. We can prove the following theorems by an argument similar to that in Subsection 4.1.2.

Theorem 4.2.8 Let $\mathcal{R}$ be NV-stable balanced joinable NV-sequential TRS. Then $\mathcal{R}$ has the normal form property, and NV-index reduction $\rightarrow_{I_V}$ is a normalizing strategy for $\mathcal{R}$. $\square$

Definition 4.2.9 The *Quasi-NV-index reduction* (or *hyper-NV-index reduction*) is defined as $\rightarrow^* \cdot \rightarrow_{I_V}$.

Theorem 4.2.10 Let $\mathcal{R}$ be NV-sequential and NV-stable balanced joinable. Then quasi-NV-index reduction $\to^* \cdot \to_{I_V}$ is a normalizing strategy for $\mathcal{R}$. $\square$

Since $\to_I \subseteq \to_{I_V}$, we obtain the following corollary. The calculating on index is much easier than NV-index.

Corollary 4.2.11 Let $\mathcal{R}$ be strongly sequential and NV-stable balanced joinable. Then index reduction $\to_I$ is a normalizing strategy for $\mathcal{R}$. $\square$

4.2.2 Decidability of Stable Transitive NV-Indices

NV-stable balanced joinability is also an undecidable property of left-liner TRSs. In this subsection, we show that for a given $C[\Omega]$ it is decidable whether $C[\Omega_{S_V}]$. By a similar method to stable balanced joinability, we have the semi-decidability of NV-stable balanced joinability for left-linear TRSs. The next two lemmas express properties of NV-indices.

Lemma 4.2.12 ([28]) Let $C[z] \in \mathcal{T}_\Omega$ where z is a fresh variable. $C[\Omega_{NI_V}]$ iff there exist $C'[z] \subseteq C[z]$ and t such that $C'[z] \to^*_{\Omega_V} t$, $t \uparrow \mathit{Red}$ and $z \in t$. $\square$

Lemma 4.2.13 Let $C[t, \Omega_{I_V}]$. Then $C[x, \Omega_{I_V}]$ where x is fresh.

Proof. Similar to Lemma 4.1.18. $\square$

Definition 4.2.14 The set Red^*_V is defined as follows:

$$\mathit{Red}^*_V = \{\, t_\Omega \mid l \equiv C[t],\ C[\Omega_{I_V}],\ l \to r \in \mathcal{R}\,\}.$$

Lemma 4.2.15 Let $C[\Omega_{I_V}]$ and $C[t] \uparrow \mathit{Red}$. Then $t \uparrow \mathit{Red}^*_V$.

Proof. Similar to Lemma 4.1.21. $\square$

Lemma 4.2.16 Let $C[\Omega_{I_V}]$. Then $C[\Omega_{NT_V}]$ iff there exists t such that $C[z] \to^*_{\Omega_V} t$ and $t \uparrow \mathit{Red}^*_V$ where z is fresh.

Proof.

($\Rightarrow$) Let $C'[C[\Omega_{NI_V}]]$ for $C'[\Omega_{I_V}]$. Then by Lemma 4.2.12 and $C[\Omega_{I_V}]$, there exist $C''[C[z]] \subseteq C'[C[z]]$ and s such that $C''[C[z]] \to^*_{\Omega_V} s$, $s \uparrow \mathit{Red}$ and $z \in s$. We have $s \equiv C''_1[C_1[z]]$ for some $C''_1[\ ]$ and $C_1[\ ]$ such that $C''[z] \to^*_{\Omega_V} C''_1[z]$ and $C[z] \to^*_{\Omega_V} C_1[z]$. By Lemma 4.2.1 (ii) and $C'[\Omega_{I_V}]$, $C''[\Omega_{I_V}]$ and therefore $C''_1[\Omega_{I_V}]$. From Lemma 4.2.15, it is follows that $C_1[z] \uparrow \mathit{Red}^*_V$.

($\Leftarrow$) Let s be an Ω-term such that $t \uparrow s$ and $s \in \mathit{Red}^*_V$. Then by the definition of Red^*_V there exists $C'[\Omega_{I_V}]$ such that $C'_\Omega[s] \in \mathit{Red}$. It is clear that $C'[C[z]] \to^*_{\Omega_V} C'[t]$ and $C'[t] \uparrow \mathit{Red}$. Since $C[\Omega_{I_V}]$, $z \in t$ and therefore $z \in C'[t]$. From Lemma 4.2.12, it follows that $C'[C[\Omega_{NI_V}]]$. Thus $C[\Omega_{NT_V}]$. $\square$

We use tree automata techniques in our proof.

Definition 4.2.17 ([5]) A *ground tree transducer* $\mathcal{G}$ over a signature $\mathcal{F}$ is a pair $(\mathcal{A}_1, \mathcal{A}_2)$ where $\mathcal{A}_1 = (\mathcal{F}, Q_1, Q_I, \Delta_1)$ and $\mathcal{A}_2 = (\mathcal{F}, Q_2, Q_I, \Delta_2)$ are tree automata.

The relation $\to_{\mathcal{G}}$ associated with $\mathcal{G}$ is defined on $\mathcal{T}(\mathcal{F})$ by $t \to_{\mathcal{G}} t'$ iff there exists $s \in \mathcal{T}(\mathcal{F} \cup Q_I)$ such that $t \to^*_{\mathcal{A}_1} s \xleftarrow{*}_{\mathcal{A}_2} t'$. A relation associated a ground tree transducer is called a GTT-relation.

Lemma 4.2.18 Let $C[\Omega_{I_V}]$. Then $C[\Omega_{NS_V}]$ iff $C\theta[\Omega_{NT_V}]$ for some θ such that $y\theta \in \mathcal{T}(\mathcal{F}, \{x\})$ for any $y \in C[\ ]$.

Proof.

($\Rightarrow$) If $C[\Omega_{NS_V}]$ then $C\theta[\Omega_{NT_V}]$ for some θ. From Lemma 4.2.16, there exists t such that $C\theta[z] \to^*_{\Omega_V} t$ and $t \uparrow \mathit{Red}^*_V$ where z is fresh. Let θ' be a substitution such that $z\theta' \equiv z$ and $y\theta' \equiv x$ for any $y \in C\theta[\ ]$. Because Ω_V-reduction is closed under substitutions, $C\theta[z]\theta' \to^*_{\Omega_V} t\theta'$. Since Ω-terms in Red^*_V do not contain variables, $t\theta' \uparrow \mathit{Red}^*_V$. Let θ'' be a substitution such that $y\theta'' \equiv y\theta\theta'$ for any y. Then $y\theta'' \in \mathcal{T}(\mathcal{F}, \{x\})$ for any $y \in C[\ ]$. From $C\theta''[z] \equiv C\theta[z]\theta'$ and Lemma 4.2.16, $C\theta''[\Omega_{NT_V}]$.

($\Leftarrow$) Trivial. $\square$

By the previous lemma, stability of transitive NV-indices in $t \in \mathcal{T}_\Omega$ only depends on instances of t in $\mathcal{T}(\mathcal{F} \cup \{\Omega\}, \{x\})$. In the Ω_V-reduction, every variable can be considered as constant. Thus we fix $\mathcal{F}' = \mathcal{F} \cup \{\Omega, x, z\}$ and after this we restrict the Ω_V-reduction to $\mathcal{T}(\mathcal{F}') \times \mathcal{T}(\mathcal{F}')$. Let $T_T = \{\, t \in \mathcal{T}(\mathcal{F}') \mid t \to^*_{\Omega_V} s,\ s \uparrow \mathit{Red}^*_V \,\}$. We will show that T_T is recognizable.

Lemma 4.2.19 ([6]) Let L be a recognizable set and let $\to_{\mathcal{G}}$ be a GTT-relation. Then the set $\{\, t \mid t \to_{\mathcal{G}} s,\ s \in L \,\}$ is recognizable.

Let $T_R = \{\, s \in \mathcal{T}(\mathcal{F}') \mid s \uparrow \mathit{Red}^*_V \,\}$. According to the previous lemma, it suffices to show that T_R is recognizable and $\to^*_{\Omega_V}$ is a GTT-relation. t^x denotes the term obtained from t by replacing all variables and Ω's in t with x.

Lemma 4.2.20 T_R is a recognizable set.

Proof. Let $\mathcal{A} = (\mathcal{F}', Q, Q_f, \Delta)$, where $Q = \{\, q_t \mid t \subseteq s^x,\ s \in \mathit{Red}^*_V \,\} \cup \{\, q_x, q_\Omega \,\}$, $Q_f = \{\, q_t \mid t \equiv s^x,\ s \in \mathit{Red}^*_V \,\} \cup \{\, q_\Omega \,\}$ and Δ consists of the following rules:

(i) $f(q_{t_1}, \ldots, q_{t_n}) \to q_t$ where $f \in \mathcal{F}$, $f(t_1, \ldots, t_n) \uparrow t_\Omega$ and $t \not\equiv \Omega$,

(ii) $\Omega \to q_\Omega$, $x \to q_x$, $z \to q_x$.

We show that $L(\mathcal{A}) = T_R$.

($\subseteq$) We first prove the following claim: if $s \in \mathcal{T}(\mathcal{F}')$ and $s \to^*_{\mathcal{A}} q_t$ then $s \uparrow t_\Omega$. The proof is by induction on the size of s. *Base step*: Trivial. *Induction step*: Let $s \equiv f(s_1, \ldots, s_n)$. Then there exists a rule $f(q_{t_1}, \ldots, q_{t_n}) \to q_t$ in Δ such that $s_i \to^*_{\mathcal{A}} q_{t_i}$ for any i. Note that $f(t_1, \ldots, t_n) \uparrow t_\Omega$. By induction hypothesis, we have $s_i \uparrow t_{i\Omega}$ for any i. If $t_\Omega \equiv \Omega$ then trivially $s \uparrow t_\Omega$. Otherwise, $t_\Omega \equiv f(t'_1, \ldots, t'_n)$ and $t_i \uparrow t'_i$ for any i. We now show that $s_i \uparrow t'_i$ for any i. If $t_i \equiv \Omega$ then $s_i \equiv \Omega$ by construction of $\mathcal{A}$. Therefore $s_i \uparrow t'_i$. If $t_i \not\equiv \Omega$ then we obtain $t_{i\Omega} \geq t'_i$ from $t_i \uparrow t'_i$ because $\Omega \notin t_i$ and t'_i does not contain variables. Hence $s_i \uparrow t'_i$. Thus the claim follows. Assume $s \in \mathcal{T}(\mathcal{F}')$ and $s \to^*_{\mathcal{A}} q_t$ with $q_t \in Q_f$. If $t \equiv \Omega$ then $s \equiv \Omega$ and therefore $s \in T_R$. Otherwise, from the claim, it follows that $s \uparrow t_\Omega$, i.e., $s \uparrow \mathit{Red}^*_V$. Thus $s \in T_R$.

($\supseteq$) It is clear that $\Omega \in T_R$ is accepted by $\mathcal{A}$. If $s \in T_R$ and $s \not\equiv \Omega$ then $s \uparrow t_\Omega$ for some $q_t \in Q_f$ with $t \not\equiv \Omega$. Hence we prove that for any $s \not\equiv \Omega$, if $s \uparrow t_\Omega$ and $q_t \in Q$ with $t \not\equiv \Omega$ then $s \to_{\mathcal{A}}^* q_t$. The proof is by induction on the size of s. *Base step*: Trivial. *Induction step*: Let $s \equiv f(s_1, \ldots, s_n)$. *Case 1.* $t \equiv x$. Let $t'_i \equiv \Omega$ if $s_i \equiv \Omega$; otherwise, let $t'_i \equiv x$. From induction hypothesis, it follows that $s_i \to_{\mathcal{A}}^* q_{t'_i}$ for any i. Since $f(q_{t'_1}, \ldots, q_{t'_n}) \to q_x \in \Delta$, $s \equiv f(s_1, \ldots, s_n) \to_{\mathcal{A}}^* q_x$. *Case 2.* $t \equiv f(t_1, \ldots, t_n)$. Note that $s_i \uparrow t_{i\Omega}$, $q_{t_i} \in Q$ and $t_i \not\equiv \Omega$ for any i. Let $t'_i \equiv \Omega$ if $s_i \equiv \Omega$; otherwise, let $t'_i \equiv t_i$. From induction hypothesis and the rule $\Omega \to q_\Omega$, we have $s_i \to_{\mathcal{A}}^* q_{t'_i}$ for any i. Because $f(t'_1, \ldots, t'_n) \uparrow t_\Omega$, there exists $f(q_{t'_1}, \ldots, q_{t'_n}) \to q_t$ in Δ. Thus $s \equiv f(s_1, \ldots, s_n) \to_{\mathcal{A}}^* q_t$. □

Lemma 4.2.21 $\to_{\Omega_V}^*$ is a GTT-relation.

Proof. We define tree automata $\mathcal{A}_1$ and $\mathcal{A}_2$ as follows. $\mathcal{A}_1 = (\mathcal{F}', Q_1, Q_I, \Delta_1)$, where $Q_1 = \{\, q_t \mid t \subseteq s^x,\ s \in \mathit{Red} \,\} \cup \{\, q_x, q_\Omega \,\}$, $Q_I = \{\, q_t \mid t \equiv s^x,\ s \in \mathit{Red} \,\}$ and Δ_1 consists of the following rules:

(i) $f(q_{t_1}, \ldots, q_{t_n}) \to q_t$ where $f \in \mathcal{F}$, $f(t_1, \ldots, t_n) \uparrow t_\Omega$ and $t \not\equiv \Omega$,

(ii) $\Omega \to q_\Omega,\ \ x \to q_x,\ \ z \to q_x$.

$\mathcal{A}_2 = (\mathcal{F}', Q_2, Q_I, \Delta_2)$ where $Q_2 = Q_I \cup \{\, q'_t \mid t \subseteq r_\Omega,\ l \to r \in \mathcal{R} \,\}$ and Δ_2 consists of the following rules:

(i)$'$ $f(q'_{t_1}, \ldots, q'_{t_n}) \to q'_t$ where $f(t_1, \ldots, t_n) \equiv t$,

(ii)$'$ $q'_s \to q_t$ where $t \equiv l^x$ and $s \equiv r_\Omega$ for some $l \to r \in \mathcal{R}$.

We can prove the following claims by a argument similar to that in Lemma 4.2.20.

(1) Let $s \in \mathcal{T}(\mathcal{F}')$ and $q_t \in Q_I$. Then $s \to_{\mathcal{A}_1}^* q_t$ iff $s \uparrow t_\Omega$ and $s \not\equiv \Omega$.

(2) Let $s \in \mathcal{T}(\mathcal{F}')$ and $q_t \in Q_I$. Then $s \to_{\mathcal{A}_2}^* q_t$ iff $s \equiv r_\Omega$ and $t \equiv l^x$ for some $l \to r \in \mathcal{R}$.

Let $\mathcal{G} = (\mathcal{A}_1, \mathcal{A}_2)$. Then it follows from the above claims that $\to_{\Omega_V} \subseteq \to_{\mathcal{G}} \subseteq \to_{\Omega_V}^*$. Because the transitive-reflexive closure of a GTT-relation is a GTT-relation [5], $\to_{\Omega_V}^*$ is a GTT relation. □

Lemma 4.2.22 T_T is a recognizable set.

Proof. From Lemmas 4.2.19, 4.2.20 and 4.2.21. □

By Lemmas 4.2.22 and 2.4.4, there exists a complete and deterministic automaton $\mathcal{A}_T$ such that $L(\mathcal{A}_T) = T_T$. The number of states in $\mathcal{A}_T$ is denoted by $|Q_T|$.

Lemma 4.2.23 Let $C[\Omega_{I_V}]$. Then $C[\Omega_{NS_V}]$ iff $C\theta[\Omega_{NT_V}]$ for some θ such that $\rho(y\theta) \leq |Q_T|$ and $y\theta \in \mathcal{T}(\mathcal{F}, \{x\})$ for any $y \in C[\ ]$.

Proof.

($\Rightarrow$) From Lemmas 4.2.16 and 4.2.18, $C\theta'[z]$ is accepted by $\mathcal{A}_T$ for some θ' such that $y\theta' \in \mathcal{T}(\mathcal{F}, \{x\})$ for any $y \in C[\ ]$. Because $\mathcal{A}_T$ is complete and deterministic, for any $y \in C[\ ]$ there is exactly one state q of $\mathcal{A}_T$ such that $y\theta' \rightarrow^*_{\mathcal{A}_T} q$. Since there exists $s \in \mathcal{T}(\mathcal{F}, \{x\})$ such that $\rho(s) \leq |Q_T|$ and $s \rightarrow^*_{\mathcal{A}_T} q$ by pumping lemma [10], we define θ'' by $y\theta'' \equiv s$. Then it is obvious that $C\theta''[z]$ is accepted by $\mathcal{A}_T$. Thus, from Lemma 4.2.16, $C\theta''[\Omega_{NT_V}]$.

($\Leftarrow$) Trivial. $\square$

Theorem 4.2.24 It is decidable whether $C[\Omega_{S_V}]$ for a given $C[\Omega]$.

Proof. It is decidable whether $C[\Omega_{I_V}]$ [28, 3]. If $C[\Omega_{NI_V}]$ then $C[\Omega_{NS_V}]$. Otherwise, by Lemma 4.2.23, it is suffices to check whether $C\theta[\Omega_{T_V}]$ for any θ such that $\rho(y\theta) \leq |Q_T|$ and $y\theta \in \mathcal{T}(\mathcal{F}, \{x\})$ for any $y \in C[\ ]$, which is also decidable. $\square$

Example 4.2.25 Let

$$\mathcal{R}_3 = \begin{cases} f(a, h(x), y) \rightarrow g(h(y), h(x)) \\ g(a, x) \rightarrow a \\ h(a) \rightarrow h(b) \\ b \rightarrow b. \end{cases}$$

The critical pair is only $\langle f(a, h(b), y),\ g(h(y), h(a))\rangle$. $\mathcal{R}_3$ is NV-stable balanced joinable because we can show that $f(a, h(b), y) \rightarrow_{S_V} g(h(y), h(b)) \leftarrow_{S_V} g(h(y), h(a))$. Note that $\mathcal{R}_3$ is not stable balanced joinable. $\mathcal{R}_3$ is strongly sequential by Theorem 2.3.30 since $\mathcal{R}_3$ is a left-normal TRS. Thus, from Corollary 4.2.11, index reduction $\rightarrow_I$ is a normalizing strategy for $\mathcal{R}_3$.

4.3 Remarks

It is not easy to generalize our results to more general sequential TRSs (NVNF- , shallow [3] or growing [14] sequential TRSs). Because the reduction contracting index w.r.t. NVNF-, shallow or growing sequentiality does not have the balanced weakly Church-Rosser property. For example, consider the following TRS:

$$\mathcal{R} = \begin{cases} f(x) \rightarrow b \\ b \rightarrow g(b) \\ h(a) \rightarrow a. \end{cases}$$

We can show that $\mathcal{R}$ is NVNF-sequential. Since redexes $f(b)$ and b in $f(b)$ are indices w.r.t. NVNF-sequentiality, we have two reductions $f(b) \rightarrow b$ and $f(b) \rightarrow f(g(b))$. However b and $f(g(b))$ are not balanced joinable. Thus the reduction contracting index w.r.t. NVNF-sequentiality is not balanced weakly Church-Rosser. Two indices of a term w.r.t. NVNF-(shallow or growing) sequentiality not being disjoint are not necessarily overlapping.

Chapter 5

Growing Term Rewriting Systems

In this chapter we investigate properties of growing TRSs. A TRS is called growing if for every its rewrite rule variables occurring both the left-hand side and the right-hand side occur at depth zero or one in the left-hand side. Jacquemard [14] showed that the set of ground terms having a normal form is recognized by a tree automaton if a TRS is linear growing. In Section 5.1, we generalize Jacquemard's result to left-linear growing TRSs. This implies the decidability of reachability and joinability for subclasses of TRSs. Moreover, this gives us decidable better approximations of TRSs. These approximations extend the class of left-linear term rewriting systems having a decidable call-by-need strategy. In Section 5.2, we prove that termination is decidable for almost orthogonal growing TRSs.

5.1 Left-Linear Growing TRSs

In this section, we regard pairs of terms as rewrite rules without restrictions. Hence the left-hand side of a rewrite rule may be a variable and the right-hand side of a rewrite rule can have variables not occurring in the left-hand side. This is convenient for approximations of TRSs. Moreover, we consider rewriting on ground terms only. This entails no loss of generality and would simplify matters.

The definition of growing was given by Jacquemard in [14]. Unlike Jacquemard, we do not assume linearity for growing TRSs.

Definition 5.1.1 A rewrite rule $l \to r$ is *growing* if all variables in $\mathcal{V}(l) \cap \mathcal{V}(r)$ occur at depth 0 or 1 in l. A TRS $\mathcal{R}$ is *growing* if every rewrite rule in $\mathcal{R}$ is growing.

Example 5.1.2 Let

$$\mathcal{R} = \begin{cases} f(f(x,y),z) \to f(z,g(z)) \\ g(x) \to f(g(y),z). \end{cases}$$

Then $\mathcal{R}$ is growing. But the following $\mathcal{R}'$ is not growing.

$$\mathcal{R}' = \begin{cases} f(f(x,y),z) \to f(x,g(z)) \\ g(x) \to f(g(y),z). \end{cases}$$

5.1.1 Recognizability

In this subsection, we show that if $\mathcal{R}$ is a left-linear growing TRS then the set $(\rightarrow^*_{\mathcal{R}})[L] = \{ t \in \mathcal{T}(\mathcal{F}) \mid \exists s \in L \;\; t \rightarrow^*_{\mathcal{R}} s \}$ is recognizable for every recognizable tree language L.

Let $\mathcal{R}$ be a left-linear growing TRS and let L be a tree language recognized by $\mathcal{A}_L = (\mathcal{F}, Q_L, Q^f_L, \Delta_L)$. We now construct a tree automaton recognizing $(\rightarrow^*_{\mathcal{R}})[L]$ from $\mathcal{R}$ and $\mathcal{A}_L$. Let $\mathcal{L} = \{ l \in \mathcal{T}(\mathcal{F}, \mathcal{V}) \mid l \notin \mathcal{V},\ f(\ldots, l, \ldots) \rightarrow r \in \mathcal{R} \}$. Since the set of all ground instances of a linear term is recognizable, we assume that for each $l \in \mathcal{L}$, $\mathcal{A}_l = (\mathcal{F}, Q_l, Q^f_l, \Delta_l)$ is an automaton such that $L(\mathcal{A}_l) = \{ l\sigma \mid \sigma : \mathcal{V} \rightarrow \mathcal{T}(\mathcal{F}) \}$. Without loss of generality, we assume that $Q_a \cap Q_b = \phi$ for any $a, b \in \{L\} \cup \mathcal{L}$ with $a \not\equiv b$. A tree automaton $\mathcal{A}_\cup = (\mathcal{F}, Q_\cup, Q^f_\cup, \Delta_\cup)$ is defined by $Q_\cup = \bigcup_{l \in \mathcal{L}} Q_l \cup Q_L$, $Q^f_\cup = Q^f_L$ and $\Delta_\cup = \bigcup_{l \in \mathcal{L}} \Delta_l \cup \Delta_L$.

Then we construct tree automata $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_k$ as follows. Let $\mathcal{A}_0 = (\mathcal{F}, Q_0, Q^f_0, \Delta_0)$ where $Q_0 = 2^{Q_\cup}$, $Q^f_0 = \{ A \in Q_0 \mid \exists q \in A,\ q \in Q^f_\cup \}$ and Δ_0 contains the following rules:

$$f(A_1, \ldots, A_n) \rightarrow A$$
$$\text{if } A = \{ q \in Q_\cup \mid \exists q_1 \in A_1, \ldots, \exists q_n \in A_n,\ f(q_1, \ldots, q_n) \ \rightarrow^*_{\mathcal{A}_\cup} q \}.$$

$\mathcal{A}_{i+1} = (\mathcal{F}, Q_{i+1}, Q^f_{i+1}, \Delta_{i+1})$ (or $\mathcal{A}_k = (\mathcal{F}, Q_k, Q^f_k, \Delta_k)$) is obtained from $\mathcal{A}_i$ as follows:

- If there exist $f(A_1, \ldots, A_n) \rightarrow A \in \Delta_i$, $l \rightarrow r \in \mathcal{R}$ and $A' \in Q_i$ satisfying the following **Condition 1** or **2**:

  - **Condition 1:**
    1. $l \equiv f(l_1, \ldots, l_n)$,
    2. for each $1 \leq j \leq n$, $l_j \notin \mathcal{V}$ implies $q \in Q^f_{l_j}$ for some $q \in A_j$,
    3. there exists $\theta : \mathcal{V} \rightarrow Q_i$ such that:
       (a) $r\theta \rightarrow^*_{\mathcal{A}_i} A'$,
       (b) for each $x \in r$, if $x \equiv l_j$ for some j then $x\theta = A_j$,
       otherwise $t \rightarrow^*_{\mathcal{A}_i} x\theta$ for some $t \in \mathcal{T}(\mathcal{F})$,
    4. $A \subset A \cup A'$,
  - **Condition 2:**
    $1'$. $l \in \mathcal{V}$,
    $2'$. there exists $\theta : \mathcal{V} \rightarrow Q_i$ such that:
       (a') $r\theta \rightarrow^*_{\mathcal{A}_i} A'$,
       (b') for each $x \in r$, if $x \equiv l$ then $x\theta = A$,
       otherwise $t \rightarrow^*_{\mathcal{A}_i} x\theta$ for some $t \in \mathcal{T}(\mathcal{F})$,
    $3'$. $A \subset A \cup A'$,

  then $Q_{i+1} = Q_i$, $Q^f_{i+1} = Q^f_i$ and
  $\Delta_{i+1} = (\Delta_i \backslash \{f(A_1, \ldots, A_n) \rightarrow A\}) \cup \{f(A_1, \ldots, A_n) \rightarrow A \cup A'\}$.

- Otherwise, $\mathcal{A}_k = \mathcal{A}_i$.

From 4 of **Condition 1** and $3'$ of **Condition 2**, it is clear that the process of construction terminates. Note that $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_k$ are deterministic and complete.

Example 5.1.3 Let $\mathcal{F} = \{ f, g, a, b \}$ and

$$\mathcal{R} = \begin{cases} f(g(x), y) \to y \\ g(x) \to f(x, x) \\ a \to g(a). \end{cases}$$

Let $L = \{ a \}$ and $\mathcal{A}_L = (\mathcal{F}, \{ q_a \}, \{ q_a \}, \{ a \to q_a \})$. Then $\mathcal{L} = \{ g(x) \}$ and we assume that the automaton $\mathcal{A}_{g(x)} = (\mathcal{F}, Q_{g(x)}, Q^f_{g(x)}, \Delta_{g(x)})$ is defined by $Q_{g(x)} = \{ q_x, q_{g(x)} \}$, $Q^f_{g(x)} = \{ q_{g(x)} \}$ and $\Delta_{g(x)} = \{ a \to q_x,\ b \to q_x,\ f(q_x, q_x) \to q_x,\ g(q_x) \to q_x,\ g(q_x) \to q_{g(x)} \}$. We have the automaton $\mathcal{A}_0 = (\mathcal{F}, Q_0, Q^f_0, \Delta_0)$ where $Q_0 = 2^{\{q_a, q_x, q_{g(x)}\}}$, $Q^f_0 = \{ \{q_a\}, \{q_a, q_x\}, \{q_a, q_{g(x)}\}, \{q_a, q_x, q_{g(x)}\} \}$ and Δ_0 is the following set of rules:

$$\Delta_0 = \begin{cases} a \to \{q_a, q_x\} & \\ b \to \{q_x\} & \\ f(A_1, A_2) \to \{q_x\} & \text{if } q_x \in A_1 \text{ and } q_x \in A_2 \\ f(A_1, A_2) \to \phi & \text{if } q_x \notin A_1 \text{ or } q_x \notin A_2 \\ g(A) \to \{q_x, q_{g(x)}\} & \text{if } q_x \in A \\ g(A) \to \phi & \text{if } q_x \notin A. \end{cases}$$

We can see that $f(\{q_{g(x)}\}, \{q_{g(x)}\}) \to \phi \in \Delta_0$, $f(g(x), y) \to y \in \mathcal{R}$ and $\{q_{g(x)}\} \in Q_0$ satisfy **Condition 1**. Thus we first replace the right-hand side of the rule $f(\{q_{g(x)}\}, \{q_{g(x)}\}) \to \phi \in \Delta_0$ with $\{q_{g(x)}\}$. Then the right-hand side of the rule $g(\{q_{g(x)}\}) \to \phi \in \Delta_1$ can be replaced with $\{q_{g(x)}\}$ because we have $f(\{q_{g(x)}\}, \{q_{g(x)}\}) \to_{\mathcal{A}_1} \{q_{g(x)}\}$. Consequently, Δ_k includes the following new rules:

$$\begin{cases} a \to \{q_a, q_x, q_{g(x)}\} & \\ f(A_1, A_2) \to A_2 & \text{if } A_1 \in \{ \{q_{g(x)}\}, \{q_a, q_{g(x)}\} \} \text{ and} \\ & \quad A_2 \neq \phi \\ f(A_1, A_2) \to A_2 & \text{if } A_1 \in \{ \{q_x, q_{g(x)}\}, \{q_a, q_x, q_{g(x)}\} \} \text{ and} \\ & \quad A_2 \notin \{ \phi, \{q_x\} \} \\ g(\{q_{g(x)}\}) \to \{q_{g(x)}\} & \\ g(\{q_a, q_{g(x)}\}) \to \{q_a, q_{g(x)}\} & \\ g(\{q_a, q_x, q_{g(x)}\}) \to \{q_a, q_x, q_{g(x)}\}. & \end{cases}$$

Consider two terms $f(g(b), g(a)) \in (\to^*_{\mathcal{R}})[L]$ and $f(g(a), g(b)) \notin (\to^*_{\mathcal{R}})[L]$. We have

$$\begin{aligned} f(g(b), g(a)) &\to^*_{\mathcal{A}_k} f(g(\{q_x\}), g(\{q_a, q_x, q_{g(x)}\})) \\ &\to^*_{\mathcal{A}_k} f(\{q_x, q_{g(x)}\}, \{q_a, q_x, q_{g(x)}\}) \\ &\to_{\mathcal{A}_k} \{q_a, q_x, q_{g(x)}\} \in Q^f_k. \end{aligned}$$

Hence $f(g(b), g(a))$ is accepted by $\mathcal{A}_k$. The term $f(g(a), g(b))$ is not accepted by $\mathcal{A}_k$ because

$$\begin{aligned} f(g(a), g(b)) &\to^*_{\mathcal{A}_k} f(g(\{q_a, q_x, q_{g(x)}\}), g(\{q_x\})) \\ &\to^*_{\mathcal{A}_k} f(\{q_a, q_x, q_{g(x)}\}, \{q_x, q_{g(x)}\}) \\ &\to_{\mathcal{A}_k} \{q_x, q_{g(x)}\} \notin Q^f_k. \end{aligned}$$

Remark. Jacquemard's construction in [14] does not necessarily generate a tree automaton $\mathcal{A}$ such that $L(\mathcal{A}) = (\to_{\mathcal{R}}^*)[L]$ for a *non-right-linear* TRS $\mathcal{R}$. Consider the left-linear non-right-linear growing TRS: $\mathcal{F} = \{ a, b, f, g \}$ and

$$\mathcal{R} = \begin{cases} g(x) \to f(x,x) \\ a \to b. \end{cases}$$

Let $L = \{ f(a,b) \}$ and $\mathcal{A}_L = (\mathcal{F}, Q_L, Q_L^f, \Delta_L)$ where $Q_L = \{ q_a, q_b, q_f \}$, $Q_L^f = \{ q_f \}$ and $\Delta_L = \{ a \to q_a, \ b \to q_b, \ f(q_a, q_b) \to q_f \}$. We add only the rule $a \to q_b$ to Δ_L at Jacquemard's construction process and hence we obtain the tree automaton $\mathcal{A} = (\mathcal{F}, Q_L, Q_L^f, \Delta_L \cup \{ a \to q_b \})$. Note that the rule $g(q_a) \to q_f$ is not added to Δ_L because we do not have $f(q_a, q_a) \to_{\mathcal{A}}^* q_f$. Although we have $g(a) \to_{\mathcal{R}}^* f(a,b) \in L$, $g(a)$ is not accepted by $\mathcal{A}$. In order to accept $g(a)$, the automaton needs the information that a can be reduced to both of q_a and q_b.

In the following we prove that $L(\mathcal{A}_k) = (\to_{\mathcal{R}}^*)[L]$. We may omit the subscript i of Q_i and Q_i^f.

Lemma 5.1.4 Let $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, $\theta : \mathcal{V} \to Q$ and $\sigma : \mathcal{V} \to \mathcal{T}(\mathcal{F})$ such that $x\sigma \to_{\mathcal{R}}^* \cdot \to_{\mathcal{A}_\cup}^* q$ for any $x \in t$ and $q \in x\theta$. For any $0 \leq i \leq k$, if $t\theta \to_{\mathcal{A}_i}^* A \in Q$ then $t\sigma \to_{\mathcal{R}}^* \cdot \to_{\mathcal{A}_\cup}^* q$ for any $q \in A$.

Proof. We prove the lemma by induction on i.

Base step. We use induction on the structure of t. The case of $t \equiv x$ is trivial. Let $t \equiv f(t_1, \ldots, t_n)$. We assume that $t\theta \equiv f(t_1, \ldots, t_n)\theta \to_{\mathcal{A}_0}^* f(A_1, \ldots, A_n) \to_{\mathcal{A}_0} A$. Let $q \in A$. Then there exist $q_1 \in A_1, \ldots, q_n \in A_n$ such that $f(q_1, \ldots, q_n) \to_{\mathcal{A}_\cup}^* q$. By induction hypothesis, for each $1 \leq j \leq n$ there exists s_j such that $t_j\sigma \to_{\mathcal{R}}^* s_j \to_{\mathcal{A}_\cup}^* q_j$. Thus we have $t\sigma \equiv f(t_1\sigma, \ldots, t_n\sigma) \to_{\mathcal{R}}^* f(s_1, \ldots, s_n) \to_{\mathcal{A}_\cup}^* f(q_1, \ldots, q_n) \to_{\mathcal{A}_\cup}^* q$.

Induction step. We use induction on the number m of application of the rule that Δ_{i-1} does not have in the reduction $t\theta \to_{\mathcal{A}_i}^* A$. If $m = 0$ then $t\theta \to_{\mathcal{A}_{i-1}}^* A$. Thus it follows from induction hypothesis on i that $t\sigma \to_{\mathcal{R}}^* \cdot \to_{\mathcal{A}_\cup}^* q$ for any $q \in A$. Let $m > 0$. Suppose that

$$t\theta \equiv t\theta[f(t_1, \ldots, t_n)\theta]_p \to_{\mathcal{A}_{i-1}}^* t\theta[f(A_1, \ldots, A_n)]_p \to_{\mathcal{A}_i} t\theta[A']_p \to_{\mathcal{A}_i}^* A$$

with $f(A_1, \ldots, A_n) \to A' \notin \Delta_{i-1}$. Let $\tilde{t} \equiv t[z]_p$ where $z \notin t$. We define $\tilde{\theta} : \mathcal{V} \to Q$ and $\tilde{\sigma} : \mathcal{V} \to \mathcal{T}(\mathcal{F})$ as follows: if $x \equiv z$ then $x\tilde{\theta} = A'$ and $x\tilde{\sigma} \equiv f(t_1, \ldots, t_n)\sigma$, otherwise $x\tilde{\theta} = x\theta$ and $x\tilde{\sigma} = x\sigma$. Clearly $\tilde{t}\tilde{\theta} \equiv t\theta[A']_p$ and $\tilde{t}\tilde{\sigma} \equiv t\sigma$. We will show the following claim:

$$x\tilde{\sigma} \to_{\mathcal{R}}^* \cdot \to_{\mathcal{A}_\cup}^* q \ \text{ for any } x \in \tilde{t} \text{ and } q \in x\tilde{\theta}.$$

Then by applying induction hypothesis on m to $\tilde{t}\tilde{\theta} \equiv t\theta[A']_p \to_{\mathcal{A}_i}^* A$, we can obtain $\tilde{t}\tilde{\sigma} \equiv t\sigma \to_{\mathcal{R}}^* \cdot \to_{\mathcal{A}_\cup}^* q$ for any $q \in A$. Thus the lemma holds.

Proof of the claim. Let $x \in \tilde{t}$. If $x \not\equiv z$ then it follows from the assumption of the lemma that $x\tilde{\sigma} \to_{\mathcal{R}}^* \cdot \to_{\mathcal{A}_\cup}^* q$ for any $q \in x\tilde{\theta}$. We consider the case $x \equiv z$. Assume that $f(A_1, \ldots, A_n) \to A_1' \in \Delta_{i-1}$, $l \to r \in \mathcal{R}$ and $A_2' \in Q_{i-1}$ satisfy **Condition 1** or **2** and $A' = A_1' \cup A_2'$. Since $f(t_1, \ldots, t_n)\theta \to_{\mathcal{A}_{i-1}}^* f(A_1, \ldots, A_n) \to_{\mathcal{A}_{i-1}} A_1'$, it follows from induction hypothesis on i that

$$f(t_1, \ldots, t_n)\sigma \to_{\mathcal{R}}^* \cdot \to_{\mathcal{A}_\cup}^* q \ \text{ for any } q \in A_1'. \tag{5.1}$$

We consider the following two cases.

Case 1. **Condition 1** is satisfied. Let $l \equiv f(l_1, \ldots, l_n)$. By applying induction hypothesis on i to $t_j\theta \to^*_{\mathcal{A}_{i-1}} A_j$, we obtain $t_j\sigma \to^*_{\mathcal{R}} \cdot \to^*_{\mathcal{A}_\cup} q$ for any $q \in A_j$. For each $1 \leq j \leq n$, let s_j be a term such that if $l_j \in \mathcal{V}$ then $s_j \equiv t_j\sigma$, otherwise $t_j\sigma \to^*_{\mathcal{R}} s_j \to^*_{\mathcal{A}_\cup} q \in Q^f_{l_j}$. From disjointness of the sets of states, $s_j \to^*_{\mathcal{A}_\cup} q \in Q^f_{l_j}$ implies $s_j \to^*_{\mathcal{A}_{l_j}} q \in Q^f_{l_j}$. Hence $f(s_1, \ldots, s_n)$ is an instance of $f(l_1, \ldots, l_n)$. Let $\theta' \colon \mathcal{V} \to Q$ be a substitution defined by 3 of **Condition 1**. Let $\sigma' : \mathcal{V} \to \mathcal{T}(\mathcal{F})$ be a substitution such that for any $y \in r$ if $y \equiv l_j$ then $y\sigma' \equiv s_j$, otherwise $y\sigma' \to^*_{\mathcal{A}_{i-1}} y\theta'$. Then we have the following reduction:

$$f(t_1, \ldots, t_n)\sigma \to^*_{\mathcal{R}} f(s_1, \ldots, s_n) \to_{\mathcal{R}} r\sigma'$$

and by using induction hypothesis on i, we can show that $y\sigma' \to^*_{\mathcal{R}} \cdot \to^*_{\mathcal{A}_\cup} q$ for any $y \in r$ and $q \in y\theta'$. Applying induction hypothesis on i to $r\theta' \to^*_{\mathcal{A}_{i-1}} A'_2$, it is obtained that $r\sigma' \to^*_{\mathcal{R}} \cdot \to^*_{\mathcal{A}_\cup} q$ for any $q \in A'_2$. Thus we have

$$f(t_1, \ldots, t_n)\sigma \to^*_{\mathcal{R}} \cdot \to^*_{\mathcal{A}_\cup} q \;\text{ for any } q \in A'_2. \tag{5.2}$$

Because $z\tilde{\theta} = A' = A'_1 \cup A'_2$ and $z\tilde{\sigma} \equiv f(t_1, \ldots, t_n)\sigma$, it follows from (5.1) and (5.2) that $z\tilde{\sigma} \to^*_{\mathcal{R}} \cdot \to^*_{\mathcal{A}_\cup} q$ for any $q \in z\tilde{\theta}$. Therefore the claim holds.

Case 2. **Condition 2** is satisfied. Let $\theta' \colon \mathcal{V} \to Q$ be a substitution defined by $2'$ of **Condition 2**. Let $\sigma' : \mathcal{V} \to \mathcal{T}(\mathcal{F})$ be a substitution such that for any $y \in r$ if $y \equiv l$ then $y\sigma' \equiv f(t_1, \ldots, t_n)\sigma$, otherwise $y\sigma' \to^*_{\mathcal{A}_{i-1}} y\theta'$. Using (5.1) and induction hypothesis on i, we can show that $y\sigma' \to^*_{\mathcal{R}} \cdot \to^*_{\mathcal{A}_\cup} q$ for any $y \in r$ and $q \in y\theta'$. Applying induction hypothesis on i to $r\theta' \to^*_{\mathcal{A}_{i-1}} A'_2$, it is obtained that $r\sigma' \to^*_{\mathcal{R}} \cdot \to^*_{\mathcal{A}_\cup} q$ for any $q \in A'_2$. Since $f(t_1, \ldots, t_n)\sigma \to_{\mathcal{R}} r\sigma'$,

$$f(t_1, \ldots, t_n)\sigma \to^*_{\mathcal{R}} \cdot \to^*_{\mathcal{A}_\cup} q \;\text{ for any } q \in A'_2. \tag{5.3}$$

Therefore, it follows from (5.1) and (5.3) that $z\tilde{\sigma} \to^*_{\mathcal{R}} \cdot \to^*_{\mathcal{A}_\cup} q$ for any $q \in z\tilde{\theta}$. Hence the claim holds. $\square$

Lemma 5.1.5 $L(\mathcal{A}_k) \subseteq (\to^*_{\mathcal{R}})[L]$.

Proof. Let $t \in L(\mathcal{A}_k)$ i.e., $t \to^*_{\mathcal{A}_k} A$ for some $A \in Q^f$. By the definition of Q^f, A has a final state q of $\mathcal{A}_L$. From Lemma 5.1.4, there exists $s \in \mathcal{T}(\mathcal{F})$ such that $t \to^*_{\mathcal{R}} s \to^*_{\mathcal{A}_\cup} q$. By disjointness of the sets of states, we have $s \to^*_{\mathcal{A}_L} q \in Q^f_L$. Thus $t \in (\to^*_{\mathcal{R}})[L]$. $\square$

Lemma 5.1.6 Let $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. Let $\theta, \theta' : \mathcal{V} \to Q$ with $x\theta \subseteq x\theta'$ for any $x \in t$. If $t\theta \to^*_{\mathcal{A}_i} A \in Q$ then $t\theta' \to^*_{\mathcal{A}_k} A'$ for some $A' \in Q$ with $A \subseteq A'$.

Proof. We prove the lemma by induction on i.

Base step. We use the induction on the structure of t. The case of $t \equiv x$ is trivial. Let $t \equiv f(t_1, \ldots, t_n)$. Then we suppose that $f(t_1, \ldots, t_n)\theta \to^*_{\mathcal{A}_0} f(A_1, \ldots, A_n) \to_{\mathcal{A}_0} A \in Q$. By induction hypothesis, for each $1 \leq j \leq n$ there exists $A'_j \in Q$ such that $t_j\theta' \to^*_{\mathcal{A}_k} A'_j$ and $A_j \subseteq A'_j$. By the definition of $\mathcal{A}_0$, Δ_0 has a rule $f(A'_1, \ldots, A'_n) \to A'$ with $A \subseteq A'$. Then by the construction of $\mathcal{A}_k$, Δ_k has a rule $f(A'_1, \ldots, A'_n) \to A''$ with $A' \subseteq A''$. Thus we obtain $f(t_1, \ldots, t_n)\theta' \to^*_{\mathcal{A}_k} f(A'_1, \ldots, A'_n) \to_{\mathcal{A}_k} A''$ and $A \subseteq A''$.

Induction step. We use the induction on the structure of t. The case of $t \equiv x$ is trivial. Let $t \equiv f(t_1, \ldots, t_n)$. Suppose that $f(t_1, \ldots, t_n)\theta \to^*_{\mathcal{A}_i} f(A_1, \ldots, A_n) \to_{\mathcal{A}_i} A \in Q$.

By induction hypothesis on the structure of t, for each $1 \leq j \leq n$ there exists $A'_j \in Q$ such that $t_j\theta' \to^*_{\mathcal{A}_k} A'_j$ and $A_j \subseteq A'_j$. Since $\mathcal{A}_k$ is deterministic and complete, there exists exactly one $A' \in Q$ such that $f(A'_1, \ldots, A'_n) \to A' \in \Delta_k$. We will show that $A \subseteq A'$. If $f(A_1, \ldots, A_n) \to A \in \Delta_{i-1}$ then from induction hypothesis on i it follows that $A \subseteq A'$. Otherwise, we assume that $f(A_1, \ldots, A_n) \to B_1 \in \Delta_{i-1}$, $l \to r \in \mathcal{R}$ and $B_2 \in Q$ satisfy **Condition 1** or **2** and $A = B_1 \cup B_2$. From induction hypothesis on i, we get $B_1 \subseteq A'$. We consider the following two cases.

Case 1. **Condition 1** is satisfied. Let $l \equiv f(l_1, \ldots, l_n)$ and let $\theta_1 : \mathcal{V} \to Q$ be a substitution defined by 3 of **Condition 1**. Then let θ_2 be a substitution from $\mathcal{V}$ to Q such that for every $x \in r$ if $x \equiv l_j$ then $x\theta_2 = A'_j$, otherwise $t \to^*_{\mathcal{A}_k} x\theta_2$ for some $t \in \mathcal{T}(\mathcal{F})$ with $t \to^*_{\mathcal{A}_{i-1}} x\theta_1$. Using induction hypothesis on i, we can show that $x\theta_1 \subseteq x\theta_2$ for every $x \in r$. Applying induction hypothesis on i to $r\theta_1 \to^*_{\mathcal{A}_{i-1}} B_2$, we obtain $r\theta_2 \to^*_{\mathcal{A}_k} B'_2$ for some $B'_2 \in Q$ with $B_2 \subseteq B'_2$. Therefore $f(A'_1, \ldots, A'_n) \to A' \in \Delta_k$, $l \to r \in \mathcal{R}$ and $B'_2 \in Q$ satisfy $1, 2$ and 3 of **Condition 1**. By the construction of $\mathcal{A}_k$, they must not satisfy 4 of **Condition 1**. Thus we have $A' = A' \cup B'_2$. Hence $A = B_1 \cup B_2 \subseteq A' \cup B'_2 = A'$.

Case 2. **Condition 2** is satisfied. Let $\theta_1 : \mathcal{V} \to Q$ be a substitution defined by $2'$ of **Condition 2**. Then let $\theta_2 : \mathcal{V} \to Q$ be a substitution such that for every $x \in r$ if $x \equiv l$ then $x\theta_2 = A'$, otherwise $t \to^*_{\mathcal{A}_k} x\theta_2$ for some $t \in \mathcal{T}(\mathcal{F})$ with $t \to^*_{\mathcal{A}_{i-1}} x\theta_1$. Using induction hypothesis on i, we can show that $y\theta_1 \subseteq y\theta_2$ for every $y \in r$. Applying induction hypothesis on i to $r\theta_1 \to^*_{\mathcal{A}_{i-1}} B_2$, we obtain $r\theta_2 \to^*_{\mathcal{A}_k} B'_2$ for some $B'_2 \in Q$ with $B_2 \subseteq B'_2$. Thus $f(A'_1, \ldots, A'_n) \to A' \in \Delta_k$, $l \to r \in \mathcal{R}$ and $B'_2 \in Q$ satisfy $1'$ and $2'$ of **Condition 2**. By the construction of $\mathcal{A}_k$ they must not satisfy $3'$ of **Condition 2**, i.e., $A' = A' \cup B'_2$. Hence $A = B_1 \cup B_2 \subseteq A' \cup B'_2 = A'$. □

Lemma 5.1.7 Let $t \in \mathcal{T}(\mathcal{F})$ and $t \to^*_{\mathcal{A}_k} A \in Q$. If $t \to^*_{\mathcal{A}_\cup} q \in Q_\cup$ then $q \in A$.

Proof. Since $\mathcal{A}_0$ is complete, there exists $A' \in Q$ such that $t \to^*_{\mathcal{A}_0} A'$. By induction of the structure of t, we can show that $A' = \{ q \in Q_\cup \mid t \to^*_{\mathcal{A}_\cup} q \}$. Thus, if $t \to^*_{\mathcal{A}_\cup} q \in Q_\cup$ then $q \in A'$. Because $\mathcal{A}_k$ is deterministic, we get $A' \subseteq A$ by Lemma 5.1.6. Hence $q \in A$. □

Lemma 5.1.8 $L(\mathcal{A}_k) \supseteq (\to^*_{\mathcal{R}})[L]$.

Proof. Assume that $t \to^*_{\mathcal{R}} s$ for some $s \in L$. We show that $t \in L(\mathcal{A}_k)$ by induction on the length m of this reduction. If $m = 0$ then $t \in L$. Thus $t \to^*_{\mathcal{A}_\cup} q$ for some $q \in Q^f_L$. Since $\mathcal{A}_k$ is complete, there exists $A \in Q$ such that $t \to^*_{\mathcal{A}_k} A$. According to Lemma 5.1.7, $q \in A$ and therefore $A \in Q^f$. Hence $t \in L(\mathcal{A}_k)$. Let $m > 0$. Then we assume that

$$t \equiv t[l\sigma]_p \to_{\mathcal{R}} t[r\sigma]_p \to^*_{\mathcal{R}} s \in L$$

with $l \to r \in \mathcal{R}$. By induction hypothesis, $t[r\sigma]_p$ is accepted by $\mathcal{A}_k$. Since $\mathcal{A}_k$ is deterministic, there exists $\theta : \mathcal{V} \to Q$ such that

$$t[r\sigma]_p \to^*_{\mathcal{A}_k} t[r\theta]_p \to^*_{\mathcal{A}_k} t[A]_p \to^*_{\mathcal{A}_k} B \in Q^f$$

where $A \in Q$. By completeness of $\mathcal{A}_k$, we assume that

$$t \equiv t[f(t_1, \ldots, t_n)]_p \to^*_{\mathcal{A}_k} t[f(A_1, \ldots, A_n)]_p \to_{\mathcal{A}_k} t[A']_p \to^*_{\mathcal{A}_k} B' \in Q$$

where $f(A_1, \ldots, A_n) \to A' \in \Delta_k$ and $n \geq 0$. We consider the following two cases.

Case 1. $l \equiv f(l_1, \ldots, l_n)$. If $l_j \notin \mathcal{V}$ then t_j is accepted by $\mathcal{A}_{l_j}$ and thus A_j has $q \in Q^f_{l_j}$ by Lemma 5.1.7. Because $\mathcal{A}_k$ is deterministic, for any $x \in r$, $x \equiv l_j$ implies $x\theta \equiv A_j$. Therefore $f(A_1, \ldots, A_n) \to A' \in \Delta_k$, $l \to r \in \mathcal{R}$ and $A \in Q$ fulfill 1, 2 and 3 of **Condition 1**. By the construction of $\mathcal{A}_k$, they must not satisfy 4 of **Condition 1**. Thus $A \subseteq A'$. Since Lemma 5.1.6 yields $B \subseteq B'$, we obtain $B' \in Q^f$. Therefore $t \in L(\mathcal{A}_k)$.

Case 2. $l \equiv x$ for some $x \in \mathcal{V}$. Because $\mathcal{A}_k$ is deterministic, if $x \in r$ then $x\theta \equiv A'$. Therefore $f(A_1, \ldots, A_n) \to A' \in \Delta_k$, $l \to r \in \mathcal{R}$ and $A \in Q$ fulfill $1'$ and $2'$ of **Condition 2**. By the construction of $\mathcal{A}_k$, they must not satisfy $3'$ of **Condition 2** and thus $A \subseteq A'$. According to Lemma 5.1.6, $B \subseteq B'$ and therefore $B' \in Q^f$. Hence $t \in L(\mathcal{A}_k)$. □

Lemma 5.1.9 $L(\mathcal{A}_k) = (\to^*_{\mathcal{R}})[L]$.

Proof. From Lemmas 5.1.5 and 5.1.8. □

Thus we obtain the following theorem.

Theorem 5.1.10 Let $\mathcal{R}$ be a left-linear growing TRS and let L be a recognizable tree language. Then the set $(\to^*_{\mathcal{R}})[L]$ is recognized by a tree automaton. □

5.1.2 Reachability and Joinability

The reachability problem for $\mathcal{R}$ is the problem of deciding whether $t \to^*_{\mathcal{R}} s$ for given two terms t and s. It is well-known that this problem is undecidable for general TRSs. Oyamaguchi [27] has shown that this problem is decidable for right-ground TRSs. Decidability for linear growing TRSs was shown by Jacquemard [14]. Since a singleton set of a term is recognizable, we can extend these results by using Theorem 5.1.10.

Theorem 5.1.11 The reachability problem for left-linear growing TRSs is decidable. □

For a TRS $\mathcal{R}$, we define $\mathcal{R}^{-1}$ by $\mathcal{R}^{-1} = \{ r \to l \mid l \to r \in \mathcal{R} \}$. Clearly $t \to^*_{\mathcal{R}} s$ iff $s \to^*_{\mathcal{R}^{-1}} t$. By Theorem 5.1.11, we obtain the following theorem.

Theorem 5.1.12 Let $\mathcal{R}$ be a TRS such that $\mathcal{R}^{-1}$ is left-linear and growing . The reachability problem for $\mathcal{R}$ is decidable. □

If $\mathcal{R}$ is right-ground TRS then $\mathcal{R}^{-1}$ is left-linear and growing. Thus, the above theorem is a generalization of Oyamaguchi's result.

The joinability problem for $\mathcal{R}$ is the problem of deciding for given finite number of terms $t_1, \ldots, t_n$, whether there exists a term s such that $t_i \to^* s$ for any $1 \leq i \leq n$. Oyamaguchi [27] has shown that this problem is decidable for right-ground TRSs. This result is extended as follows.

Theorem 5.1.13 Let $\mathcal{R}$ be a TRS such that $\mathcal{R}^{-1}$ is left-linear and growing. The joinability problem for $\mathcal{R}$ is decidable.

Proof. Let $t_1, \ldots, t_n$ be terms. Then $t_1, \ldots, t_n$ are joinable iff

$$(\to^*_{\mathcal{R}^{-1}})[\{t_1\}] \cap \cdots \cap (\to^*_{\mathcal{R}^{-1}})[\{t_n\}] \neq \phi.$$

By Theorem 5.1.10, $(\to^*_{\mathcal{R}^{-1}})[\{t_i\}]$ is recognizable for any $1 \leq i \leq n$. Thus from Lemmas 2.4.5 and 2.4.6 the theorem follows. □

5.1.3 Decidable Approximations

Durand and Middeldorp [8] studied approximations of TRSs to call-by-need computations. They presented the framework for decidable call-by-need computations without notions of index and sequentiality.

Definition 5.1.14 A TRS $\mathcal{R}'$ is an *approximation* of a TRS $\mathcal{R}$ if $\rightarrow^*_{\mathcal{R}} \subseteq \rightarrow^*_{\mathcal{R}'}$. An *approximation mapping* τ is a mapping from TRSs to TRSs such that $\tau(\mathcal{R})$ is an approximation of $\mathcal{R}$ for every $\mathcal{R}$.

Jacquemard introduced right-linear growing approximations in [14]. A *right-linear growing approximation* of $\mathcal{R} = \{\ l_i \rightarrow r_i \mid 1 \leq i \leq n\ \}$ is a right-linear growing TRS $\{\ l_i \rightarrow r'_i \mid 1 \leq i \leq n\ \}$ where for any $1 \leq i \leq n$, r'_i is obtained from r_i by replacing variables which do not satisfy the right-linearity or growing condition with fresh variables. We say that an approximation mapping τ is *right-linear growing* if $\tau(\mathcal{R})$ is a right-linear growing approximation of $\mathcal{R}$ for every $\mathcal{R}$. We now give better approximations than Jacquemard's ones based on the result of Subsection 5.1.1.

Definition 5.1.15 Let $\mathcal{R} = \{\ l_i \rightarrow r_i \mid 1 \leq i \leq n\ \}$. The *growing approximation* of $\mathcal{R}$ is defined as a growing TRS $\{\ l_i \rightarrow r_i\sigma_i \mid 1 \leq i \leq n\ \}$ where σ_i is a variable renaming such that for every variable x, if x occurs at depth more than 1 in l_i then $x\sigma_i \notin \mathcal{V}(l_i)$, otherwise $x\sigma_i \equiv x$. An approximation mapping τ is *growing* if $\tau(\mathcal{R})$ is a growing approximation of $\mathcal{R}$ for every $\mathcal{R}$.

If $\mathcal{R}$ is a growing TRS then the growing approximation of $\mathcal{R}$ is $\mathcal{R}$ itself. If $\mathcal{R}$ is a left-linear TRS then the growing approximation of $\mathcal{R}$ is also left-linear.

Example 5.1.16 Let

$$\mathcal{R}_1 = \begin{cases} f(g(x), y) \rightarrow f(x, f(y, x)) \\ g(x) \rightarrow f(x, x). \end{cases}$$

Then the growing approximation of $\mathcal{R}_1$ is

$$\mathcal{R}'_1 = \begin{cases} f(g(x), y) \rightarrow f(z, f(y, z)) \\ g(x) \rightarrow f(x, x). \end{cases}$$

The following definition gives sufficient conditions for neededness.

Definition 5.1.17 Let $\mathcal{R}$ be a TRS. Let τ be an approximation mapping. The redex at a position p in $t \in \mathcal{T}(\mathcal{F})$ is τ*-needed* if there exists no $s \in \mathrm{NF}_{\mathcal{R}}$ such that $t[\Omega]_p \rightarrow^*_{\tau(\mathcal{R})} s$.

Lemma 5.1.18 Let $\mathcal{R}$ be an orthogonal TRS whose rewrite rules satisfy the restrictions in Definition 2.2.7. Let τ be an approximation mapping. If a redex in a term is τ-needed then it is needed.

Proof. By using Lemma 2.3.13. $\square$

In order to obtain a decidable call-by-need strategy, every term that is not a normal form has a decidable needed redex. Thus the following classes of TRSs are formulated.

Definition 5.1.19 Let τ be an approximation mapping. The class $\mathcal{C}_\tau$ of TRSs is defined as follows: $\mathcal{R} \in \mathcal{C}_\tau$ iff every term that is not a normal form has a τ-needed redex.

Lemma 5.1.20 Let τ be a growing approximation mapping. Let $\mathcal{R}$ be an orthogonal growing TRS whose rewrite rules satisfy the restrictions in Definition 2.2.7. Then $\mathcal{R} \in \mathcal{C}_\tau$.

Proof. Since $\mathcal{R}$ is a growing TRS, $\tau(\mathcal{R}) = \mathcal{R}$. Using Lemma 2.3.13, we can show that a redex is needed iff it is τ-needed. Thus it follows Lemma 2.3.5 that $\mathcal{R} \in \mathcal{C}_\tau$. □

Theorem 5.1.21 Let τ be a growing approximation mapping and let τ' be a right-linear growing approximation mapping. Then $\mathcal{C}_{\tau'} \subset \mathcal{C}_\tau$ even if these classes are restricted to orthogonal TRSs.

Proof. Because $\to^*_{\tau(\mathcal{R})} \subseteq \to^*_{\tau'(\mathcal{R})}$, τ'-needed redexes are τ-needed. Thus $\mathcal{C}_{\tau'} \subseteq \mathcal{C}_\tau$. Let

$$\mathcal{R}'_2 = \begin{cases} f(a,b,x) \to a \\ f(b,x,a) \to b \\ f(x,a,b) \to g(a) \\ h(a,b) \to a \\ h(b,a) \to a \\ h(a,c) \to b \\ h(c,a) \to b, \end{cases}$$

and let $\mathcal{R}_2 = \mathcal{R}'_2 \cup \{ g(x) \to h(x,x) \}$. By Lemma 5.1.20, $\mathcal{R}_2 \in \mathcal{C}_\tau$. We will show that $\mathcal{R}_2 \notin \mathcal{C}_{\tau'}$. Let $r \equiv f(a,a,b)$. A right-linear growing approximation $\tau'(\mathcal{R}_2)$ of $\mathcal{R}_2$ is one of $\mathcal{R}_3 = \mathcal{R}'_2 \cup \{ g(x) \to h(x,y) \}$, $\mathcal{R}_4 = \mathcal{R}'_2 \cup \{ g(x) \to h(y,x) \}$ and $\mathcal{R}_5 = \mathcal{R}'_2 \cup \{ g(x) \to h(y,z) \}$. In any case, we can show that $f(r,r,r)$ does not have τ'-needed redexes. Thus $\mathcal{R}_2 \notin \mathcal{C}_{\tau'}$. □

Durand and Middeldorp gave a sufficient condition for the decidability of τ-neededness and membership of $\mathcal{C}_\tau$.

Theorem 5.1.22 ([8]) Let $\mathcal{R}$ be a TRS. Let τ be an approximation mapping. If the set $\{ t \in \mathcal{T}(\mathcal{F} \cup \{\Omega\}) \mid \exists s \in \mathrm{NF}_\mathcal{R}\ t \to^*_{\tau(\mathcal{R})} s \}$ is recognizable then

(1) it is decidable whether a redex in a term is τ-needed,

(2) it is decidable whether $\mathcal{R} \in \mathcal{C}_\tau$. □

If $\mathcal{R}$ is left-linear then the set $\mathrm{NF}_\mathcal{R}$ is recognizable. Hence we have the following decidability result from Theorems 5.1.10 and 5.1.22.

Theorem 5.1.23 Let $\mathcal{R}$ be a left-linear TRS. Let τ be a growing approximation mapping.

(1) It is decidable whether a redex in a term is τ-needed.

(2) It is decidable whether $\mathcal{R} \in \mathcal{C}_\tau$. □

Corollary 5.1.24 Let $\mathcal{R}$ be an orthogonal growing TRS whose rewrite rules satisfy the restrictions in Definition 2.2.7. It is decidable whether a redex in a term is needed. □

5.2 Termination of Almost Orthogonal Growing TRSs

In this section, we show that termination of almost orthogonal growing TRSs is decidable. We assume that every rewrite rule $l \to r$ satisfy the restrictions that l is not a variable and any variable in r also occurs in l. If a TRS $\mathcal{R}$ contains a rewrite rule which does not satisfy either of these restrictions then $\mathcal{R}$ is not strongly normalizing. We first explain the theorem of Gramlich [12], which is used in our proof.

We say that a term t is *strongly normalizing* if there exists no infinite reduction sequence starting with t. A reduction $t \xrightarrow{p}_{\mathcal{R}} s$ is *innermost* if every proper subterm of $t|_p$ is a normal form. The innermost reduction is denoted by $\to_{\mathcal{I}}$. A term t is *weakly innermost normalizing* if $t \to_{\mathcal{I}}^* s$ for some normal form s. A TRS is *weakly innermost normalizing* if every term is weakly innermost normalizing.

Theorem 5.2.1 ([12]) Let $\mathcal{R}$ be a TRS such that all critical pairs of $\mathcal{R}$ are trivial overlays.

(a) $\mathcal{R}$ is strongly normalizing iff $\mathcal{R}$ is weakly innermost normalizing.

(b) For any term t, t is strongly normalizing iff t is weakly innermost normalizing. $\square$

According to Theorem 5.2.1, if we can prove the decidability of weakly innermost normalizability then termination is decidable. The following lemma shows that it is sufficient to consider rewriting on ground terms only.

Lemma 5.2.2 Let $\mathcal{R}$ be a TRS. $\mathcal{R}$ is strongly normalizing iff every ground term is strongly normalizing. $\square$

We will show that for every left-linear growing TRS, the set of ground terms being weakly innermost normalizing is recognizable. From here on we assume that $\mathcal{R}$ is left-linear growing TRS.

We must construct a tree automaton which recognizes the set of ground terms being weakly innermost normalizing. We start with the deterministic and complete tree automaton $\mathcal{A}_{NF}$ by Comon [3] which accepts ground normal forms. The set $\mathcal{S}_{\mathcal{R}}$ is defined as follows: $\mathcal{S}_{\mathcal{R}} = \{\, t \in \mathcal{T}_\Omega \mid t \subset l_\Omega,\ l \to r \in \mathcal{R} \,\}$. $\mathcal{S}_{\mathcal{R}}^*$ is the smallest set such that $\mathcal{S}_{\mathcal{R}} \subseteq \mathcal{S}_{\mathcal{R}}^*$ and if $t, s \in \mathcal{S}_{\mathcal{R}}^*$ and $t \uparrow s$ then $t \sqcup s \in \mathcal{S}_{\mathcal{R}}^*$. $\mathcal{A}_{NF} = (\mathcal{F}, Q_{NF}, Q_{NF}^f, \Delta_{NF})$ is defined by $Q_{NF} = \{\, q_t \mid t \in \mathcal{S}_{\mathcal{R}}^*$ and t does not contain redexes $\} \cup \{\, q_\Omega, q_{\mathrm{red}} \,\}$ where $\mathrm{red} \notin \mathcal{F}$, $Q_{NF}^f = Q_{NF} \backslash \{\, q_{\mathrm{red}} \,\}$ and Δ_{NF} consists of the following rules:

- $f(q_{t_1}, \ldots, q_{t_n}) \to q_t$
  if $f(t_1, \ldots, t_n)$ is not a redex and
  t is a maximal Ω-term w.r.t. $\leq$ such that $t \leq f(t_1, \ldots, t_n)$ and $q_t \in Q_{NF}^f$,
- $f(q_{t_1}, \ldots, q_{t_n}) \to q_{\mathrm{red}}$ if $f(t_1, \ldots, t_n)$ is a redex,
- $f(q_1, \ldots, q_n) \to q_{\mathrm{red}}$ if $q_{\mathrm{red}} \in \{\, q_1, \ldots, q_n \,\}$.

The following lemma shows that $\mathcal{A}_{NF}$ recognizes the set of ground normal forms.

Lemma 5.2.3 (**[3]**) Let $t \in \mathcal{T}(\mathcal{F})$.

(i) $\mathcal{A}_{NF}$ is deterministic and complete.

(ii) If $t \to^*_{\mathcal{A}_{NF}} q_s$ where $q_s \neq q_{\text{red}}$ then t is a normal form, $s \leq t$ and $u \leq s$ for any $q_u \in Q^f_{NF}$ with $u \leq t$.

(iii) If $t \to^*_{\mathcal{A}_{NF}} q_{\text{red}}$ then t is not a normal form. $\square$

We construct tree automata $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_k$ as follows. Let $\mathcal{A}_0 = (\mathcal{F}, Q_0, Q^f_0, \Delta_0) = \mathcal{A}_{NF}$. $\mathcal{A}_{i+1} = (\mathcal{F}, Q_{i+1}, Q^f_{i+1}, \Delta_{i+1})$ (or $\mathcal{A}_k = (\mathcal{F}, Q_k, Q^f_k, \Delta_k)$) is obtained from $\mathcal{A}_i$ as follows:

- If there exist $q_{t_1} \in Q^f_i, \ldots, q_{t_n} \in Q^f_i$, $f(l_1, \ldots, l_n) \to r \in \mathcal{R}$ and $q \in Q_i$ such that

  (1) $f(l_1, \ldots, l_n)_\Omega \leq f(t_1, \ldots, t_n)$

  (2) there exists $\theta : \mathcal{V} \to Q_i$ such that $r\theta \to^*_{\mathcal{A}_i} q$ and $x \equiv l_j$ implies $x\theta = q_{t_j}$ for every $x \in r$ and $1 \leq j \leq n$,

  (3) $f(q_{t_1}, \ldots, q_{t_n}) \to q \notin \Delta_i$,

  then $Q_{i+1} = Q_i$, $Q^f_{i+1} = Q^f_i$ and $\Delta_{i+1} = \Delta_i \cup \{f(q_{t_1}, \ldots, q_{t_n}) \to q\}$.

- Otherwise, $\mathcal{A}_k = \mathcal{A}_i$.

The process of construction terminates by the condition (3). Note that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are non-deterministic. In the following we prove that

$$L(\mathcal{A}_k) = \{\ t \in \mathcal{T}(\mathcal{F}) \mid t \text{ is weakly innermost normalizing }\}.$$

We may omit the subscript i of Q_i and Q^f_i.

Lemma 5.2.4 Let $t \in \mathcal{T}(\mathcal{F})$. For any $0 \leq i \leq k$, if $t \to^*_{\mathcal{A}_i} q \in Q$ then $t \to^*_{\mathcal{I}} s \to^*_{\mathcal{A}_{NF}} q$ for some $s \in \mathcal{T}(\mathcal{F})$.

Proof. We prove the lemma by induction on i. *Base step.* Trivial. *Induction step.* Assume that $q_{s_1} \in Q^f, \ldots, q_{s_n} \in Q^f$, $f(l_1, \ldots, l_n) \to r \in \mathcal{R}$ and $q_1 \in Q$ are satisfy the conditions of construction and Δ_i is obtained by adding the rule $f(q_{s_1}, \ldots, q_{s_n}) \to q_1$ to Δ_{i-1}. We use induction on the number m of application of the rule $f(q_{s_1}, \ldots, q_{s_n}) \to q_1$ in the reduction $t \to^*_{\mathcal{A}_i} q$. If $m = 0$ then $t \ \to^*_{\mathcal{A}_{i-1}} q$. Thus it follows from induction hypothesis on i that $t \to^*_{\mathcal{I}} s \to^*_{\mathcal{A}_{NF}} q$ for some $s \in \mathcal{T}(\mathcal{F})$. Let $m > 0$. Suppose that

$$t \equiv t[f(t_1, \ldots, t_n)]_p \to^*_{\mathcal{A}_{i-1}} t[f(q_{s_1}, \ldots, q_{s_n})]_p \to_{\mathcal{A}_i} t[q_1]_p \to^*_{\mathcal{A}_i} q.$$

For every $1 \leq j \leq n$, we obtain $u_j \in \mathcal{T}(\mathcal{F})$ with $t_j \to^*_{\mathcal{I}} u_j \to^*_{\mathcal{A}_{NF}} q_{s_j}$ by applying induction hypothesis to $t_j \to^*_{\mathcal{A}_{i-1}} q_{s_j}$. According to Lemma 5.2.3 (ii), $f(s_1, \ldots, s_n) \leq f(u_1, \ldots, u_n)$ and $u_1, \ldots, u_n$ are normal forms. Because $f(l_1, \cdots, l_n)_\Omega \leq f(s_1, \ldots, s_n)$ by the condition (1), we have the following reduction sequence:

$$f(t_1, \ldots, t_n) \to^*_{\mathcal{I}} f(u_1, \ldots, u_n) \equiv f(l_1, \ldots, l_n)\sigma \to_{\mathcal{I}} r\sigma.$$

Let θ be a substitution which is satisfied in the condition (2) of construction. Then we have $r\sigma \to^*_{\mathcal{A}_{NF}} r\theta$ and hence $r\sigma \to^*_{\mathcal{A}_{i-1}} q_1$. Applying induction hypothesis on m to $t[r\sigma]_p \to^*_{\mathcal{A}_{i-1}} t[q_1]_p \to^*_{\mathcal{A}_i} q$, we obtain $s \in \mathcal{T}(\mathcal{F})$ such that $t[r\sigma]_p \to^*_{\mathcal{I}} s \to^*_{\mathcal{A}_{NF}} q$. Thus we have $t \to^*_{\mathcal{I}} s \to^*_{\mathcal{A}_{NF}} q$ since $t \to_{\mathcal{I}} t[r\sigma]_p$. $\square$

Lemma 5.2.5 $L(\mathcal{A}_k) \subseteq \{ t \in \mathcal{T}(\mathcal{F}) \mid t$ is weakly innermost normalizing $\}$.

Proof. From Lemmas 5.2.3 and 5.2.4. □

Lemma 5.2.6 Let $t \in \mathcal{T}(\mathcal{F})$ be a normal form. Then there exists exactly one q in Q such that $t \to^*_{\mathcal{A}_k} q$. Furthermore, q is the state q_s in Q^f such that $s \leq t$ and $u \leq s$ for any $q_u \in Q^f$ with $u \leq t$.

Proof. By Lemma 5.2.4, $t \to^*_{\mathcal{A}_k} q$ iff $t \to^*_{\mathcal{A}_{NF}} q$. Thus, from Lemma 5.2.3 the lemma follows. □

Lemma 5.2.7 $L(\mathcal{A}_k) \supseteq \{ t \in \mathcal{T}(\mathcal{F}) \mid t$ is weakly innermost normalizing $\}$.

Proof. Assume that $t \to^*_{\mathcal{I}} s$ for some normal form s. We show that $t \in L(\mathcal{A}_k)$ by induction on the length m of this reduction. Let $m = 0$. Then t is a normal form and hence $t \in L(\mathcal{A}_k)$ from Lemma 5.2.6. Let $m > 0$. We assume that

$$t \equiv t[f(l_1, \ldots, l_n)\sigma]_p \to_{\mathcal{I}} t[r\sigma]_p \to^*_{\mathcal{I}} s$$

with $f(l_1, \ldots, l_n) \to r \in \mathcal{R}$. By induction hypothesis, $t[r\sigma]_p$ is accepted by $\mathcal{A}_k$, i.e., $t[r\sigma]_p \to^*_{\mathcal{A}_k} q$ for some $q \in Q^f$. Because $x\sigma$ is a normal form for every $x \in r$, Lemma 5.2.6 yields $\theta : \mathcal{V} \to Q$ such that

$$t[r\sigma]_p \to^*_{\mathcal{A}_k} t[r\theta]_p \to^*_{\mathcal{A}_k} t[q_1]_p \to^*_{\mathcal{A}_k} q$$

where $q_1 \in Q$. For any $1 \leq j \leq n$, by Lemma 5.2.6 we have exactly one $q_{s_j} \in Q$ with $l_j\sigma \to^*_{\mathcal{A}_k} q_{s_j}$ because $l_j\sigma$ is a normal form. Note that if $l_j \equiv x$ and $x \in r$ then $x\theta = q_{s_j}$. For any $1 \leq j \leq n$, $q_{l_{j\Omega}} \in Q^f$ since $l_{j\Omega} \in \mathcal{S}^*_{\mathcal{R}}$ and $l_{j\Omega}$ does not contain redexes. According to Lemma 5.2.6 $f(l_1, \ldots, l_n)_\Omega \leq f(s_1, \ldots, s_n)$. Therefore $q_{s_1} \in Q^f, \ldots, q_{s_n} \in Q^f$, $f(l_1, \ldots, l_n) \to r \in \mathcal{R}$ and $q_1 \in Q$ satisfy the conditions (1) and (2) of construction. By the construction of $\mathcal{A}_k$, Δ_k has the rule $f(q_{s_1}, \ldots, q_{s_n}) \to q_1$. Thus, since

$$t \equiv t[f(l_1, \ldots, l_n)\sigma]_p \to^*_{\mathcal{A}_k} t[f(q_{s_1}, \ldots, q_{s_n})]_p \to_{\mathcal{A}_k} t[q_1]_p \to^*_{\mathcal{A}_k} q \in Q^f,$$

t is accepted by $\mathcal{A}_k$. □

Thus we obtain the following result.

Lemma 5.2.8 Let $\mathcal{R}$ be a left-linear growing TRS. The set of ground terms being weakly innermost normalizing is recognized by a tree automaton. □

Theorem 5.2.9 Termination is decidable for almost orthogonal growing TRSs.

Proof. Let $\mathcal{R}$ be an almost orthogonal growing TRS. According to Lemmas 5.2.2 and 5.2.1, $\mathcal{R}$ is strongly normalizing iff every ground term is weakly innermost normalizing. From Lemmas 2.4.5, 2.4.6 and 5.2.8, it is decidable whether every ground term is weakly innermost normalizing. □

Chapter 6

Normalizability of the E-Strategy

In this chapter, we study the E-strategy which is adopted by the OBJ algebraic specification languages such that OBJ2 [9], OBJ3 [11] and CafeOBJ [24]. In Section 6.1, we define the E-strategy and discuss how to give local strategies to function symbols so that every normal form with respect to the E-strategy is a normal form. In Section 6.2, we introduce two properties. One is the carefulness of local strategies and the other is the index-transitivity of orthogonal TRSs. We show that if $\mathcal{R}$ is an index-transitive orthogonal TRS in which careful local strategies are given to function symbols then the E-strategy is normalizing for $\mathcal{R}$. In general, these properties are undecidable. Section 6.3 gives a decidable sufficient condition for carefulness. In Section 6.4, we consider a necessary and sufficient condition for index-transitivity which is useful to prove index-transitivity of orthogonal TRSs.

6.1 The Evaluation Strategy

The *evaluation strategy* (*E-strategy*) chooses a redex according to *local strategies* which are given to function symbols. A local strategy is a list of natural numbers telling the order to try reductions. If a function symbol f has arity n then the local strategy of f consists natural numbers ranging from 0 to n. A positive integer i in the local strategy of f means that the E-strategy reduces ith argument of f. Zero means rewriting at the top. We now give the definition of the E-strategy. Our definition demonstrates the process of search for a redex.

The set of all lists consisting of natural numbers is denoted by $\mathcal{L}$. The empty list is denoted by nil. $\mathcal{L}_n$ denotes the set of all lists consisting natural numbers ranging from 0 to n. Hence $\mathcal{L} = \bigcup_{n \geq 0} \mathcal{L}_n$. Let $\mathcal{F}$ be a signature. Then the signature $\mathcal{F}_{\mathcal{L}}$ is defined by $\mathcal{F}_{\mathcal{L}} = \{ f_l \mid f \in \mathcal{F}_n \text{ and } l \in \mathcal{L}_n \text{ for some } n \}$. The set $\mathcal{V}_{\mathcal{L}}$ of variables is defined by $\mathcal{V}_{\mathcal{L}} = \{ x_{nil} \mid x \in \mathcal{V} \}$. The set $\mathcal{T}(\mathcal{F}_{\mathcal{L}}, \mathcal{V}_{\mathcal{L}})$ is abbreviated to $\mathcal{T}_{\mathcal{L}}$.

Definition 6.1.1 Let $\mathcal{F}$ be a signature. An *E-strategy mapping* φ *of* $\mathcal{F}$ is a mapping from $\mathcal{F} \cup \mathcal{V}$ to $\mathcal{L}$ such that $\varphi(f) \in \mathcal{L}_n$ if $f \in \mathcal{F}_n$ and $\varphi(x) = nil$ if $x \in \mathcal{V}$. The E-strategy mapping φ of $\mathcal{F}$ is extended to a mapping from $\mathcal{T}$ to $\mathcal{T}_{\mathcal{L}}$ as follows:

$$\varphi(t) \equiv \begin{cases} x_{nil} & \text{if } t \equiv x, \\ f_{\varphi(f)}(\varphi(t_1), \ldots, \varphi(t_n)) & \text{if } t \equiv f(t_1, \ldots, t_n). \end{cases}$$

The mapping e form $\mathcal{T}_\mathcal{L}$ to $\mathcal{T}$ which erase all lists is defined as follows:

$$\mathsf{e}(t) \equiv \begin{cases} x & \text{if } t \equiv x_{nil}, \\ f(\mathsf{e}(t_1), \ldots, \mathsf{e}(t_n)) & \text{if } t \equiv f_l(t_1, \ldots, t_n). \end{cases}$$

Example 6.1.2 Let $\mathcal{F} = \{f, g, h, a\}$ where $\mathcal{F}_2 = \{f, g\}$, $\mathcal{F}_1 = \{h\}$ and $\mathcal{F}_0 = \{a\}$. Let φ be a mapping from $\mathcal{F} \cup \mathcal{V}$ to $\mathcal{L}$ which is defined by $\varphi(f) = (1, 0, 2, 0)$, $\varphi(g) = (2, 1)$, $\varphi(h) = (1, 1)$, $\varphi(a) = (0)$ and $\varphi(x) = nil$ for any $x \in \mathcal{V}$. Then φ is an E-strategy mapping of $\mathcal{F}$. We have $\varphi(f(h(x), a)) = f_{(1,0,2,0)}(h_{(1,1)}(x_{nil}), a_{(0)})$ and $\varphi(g(x, g(y, z))) = g_{(2,1)}(x_{nil}, g_{(2,1)}(y_{nil}, z_{nil}))$.

Definition 6.1.3 Let $(\mathcal{F}, \mathcal{R})$ be a TRS. Let φ be an E-strategy mapping of $\mathcal{F}$. The reduction relation $\Rightarrow_\varphi$ on $\mathcal{T}_\mathcal{L} \times \mathcal{N}_+^*$ is defined as follows: $\langle t, p\rangle \Rightarrow_\varphi \langle s, q\rangle$ iff $p \in \mathcal{P}os(t)$ and one of the following conditions is satisfied.

(i) $t(p) = e_{nil}$ for some $e \in \mathcal{F} \cup \mathcal{V}$, $t \equiv s$ and $p = q.i$,

(ii) $t(p) = f_{(0,\ldots)}$, $p = q$ and $t|_p \equiv l'\theta$, $\mathsf{e}(l') \equiv l$, $s \equiv t[\varphi(r)\theta]_p$ for some $\theta : \mathcal{V}_\mathcal{L} \to \mathcal{T}_\mathcal{L}$ and $l \to r \in \mathcal{R}$,

(iii) $t \equiv t[f_{(0,\ldots)}(t_1, \ldots, t_n)]_p$, $\mathsf{e}(t|_p)$ is not a redex, $s \equiv t[f_{(\ldots)}(t_1, \ldots, t_n)]_p$ and $p = q$,

(iv) $t \equiv t[f_{(i,\ldots)}(t_1, \ldots, t_n)]_p$ with $i \neq 0$, $s \equiv t[f_{(\ldots)}(t_1, \ldots, t_n)]_p$ and $q = p.i$.

Note that if $\langle t, p\rangle \Rightarrow_\varphi \langle s, q\rangle$ then $q \in \mathcal{P}os(s)$ and $\mathsf{e}(t) \to^{\equiv} \mathsf{e}(s)$. Let t be a term in $\mathcal{T}$. If we have the $\Rightarrow_\varphi$-reduction sequence $\langle \varphi(t), \varepsilon\rangle \Rightarrow_\varphi \langle t_1, p_1\rangle \Rightarrow_\varphi \langle t_2, p_2\rangle \Rightarrow_\varphi \cdots$ then the *E-strategy* reduction from t is defined as $t \to^{\equiv} \mathsf{e}(t_1) \to^{\equiv} \mathsf{e}(t_2) \to^{\equiv} \cdots$.

Example 6.1.4 Let $\mathcal{F} = \{\, add, s, 0 \,\}$ and

$$\mathcal{R} = \begin{cases} add(x, 0) \to x \\ add(x, s(y)) \to s(add(x, y)). \end{cases}$$

Let φ be an E-strategy mapping of $\mathcal{F}$ which is defined by $\varphi(add) = (2, 0)$, $\varphi(s) = (1)$ and $\varphi(0) = nil$. Then $\langle add_{nil}(s_{(1)}(0_{(0)}), 0_{nil}), 1\rangle \Rightarrow_\varphi \langle add_{nil}(s_{nil}(0_{(0)}), 0_{nil}), 1.1\rangle$ and $\langle add_{(0)}(s_{nil}(0_{nil}), s(0_{(0)})), \varepsilon\rangle \Rightarrow_\varphi \langle s_{(1)}(add_{(2,0)}(s_{nil}(0_{nil}), 0_{(0)})), \varepsilon\rangle$. The E-strategy reduces the term $add(add(s(0), 0), add(s(0), 0))$ as follows (at each step the underlined redex is contracted):

$$\begin{array}{rcl} add(add(s(0), 0), \underline{add(s(0), 0)}) & \to & \underline{add(add(s(0), 0), s(0))} \\ & \to & s(\underline{add(add(s(0), 0), 0)}) \\ & \to & s(\underline{add(s(0), 0)}) \\ & \to & s(s(0)). \end{array}$$

However, even if $\Rightarrow_\varphi$-reduction from $\langle \varphi(t), \varepsilon\rangle$ terminates at $\langle s, p\rangle$, $\mathsf{e}(s)$ is not always a normal form as shown by the following example.

Example 6.1.5 Let $\mathcal{F} = \{ f,\, g,\, a,\, b,\, c \}$ and

$$\mathcal{R} = \begin{cases} f(a) \to c \\ g(b, x) \to b \\ b \to a. \end{cases}$$

We define the E-strategy mapping φ of $\mathcal{F}$ by $\varphi(f) = (0,1)$, $\varphi(g) = (1,0)$, $\varphi(a) = \varphi(c) = nil$ and $\varphi(b) = (0)$. Consider the term $t \equiv g(a,b)$. The $\Rightarrow_\varphi$-reduction from $\langle \varphi(t), \varepsilon \rangle$ terminates at $\langle g_{nil}(a_{nil}, b_{(0)}), \varepsilon \rangle$:

$$\langle g_{(1,0)}(a_{nil}, b_{(0)}), \varepsilon \rangle \Rightarrow_\varphi \langle g_{(0)}(a_{nil}, b_{(0)}), 1 \rangle \Rightarrow_\varphi \langle g_{(0)}(a_{nil}, b_{(0)}), \varepsilon \rangle \Rightarrow_\varphi \langle g_{nil}(a_{nil}, b_{(0)}), \varepsilon \rangle.$$

But $g(a,b)$ is not a normal form. The E-strategy must reduce all arguments of g in order to get the normal form $g(a,a)$ of t. Next consider the term $s \equiv f(b)$. We have the following $\Rightarrow_\varphi$-reduction sequence:

$$\langle f_{(0,1)}(b_{(0)}), \varepsilon \rangle \Rightarrow_\varphi \langle f_{(1)}(b_{(0)}), \varepsilon \rangle \Rightarrow_\varphi \langle f_{nil}(b_{(0)}), 1 \rangle \Rightarrow_\varphi \langle f_{nil}(a_{nil}), 1 \rangle \Rightarrow_\varphi \langle f_{nil}(a_{nil}), \varepsilon \rangle.$$

Although $\langle f_{nil}(a_{nil}), \varepsilon \rangle$ is a normal form w.r.t. $\Rightarrow_\varphi$, $f(a)$ is not a normal form. After the reducing arguments of the function symbol f, it is necessary to try to match with the left-hand sides of rewrite rules at the position occurring f.

To avoid this problem, we will give a restriction on E-strategy mappings.

Definition 6.1.6 Let $(\mathcal{F}, \mathcal{R})$ be a TRS. The set $\mathcal{D}$ of *defined function symbols* is defined as follows:

$$\mathcal{D} = \{ \; l(\varepsilon) \mid l \to r \in \mathcal{R} \; \}.$$

Definition 6.1.7 Let $(\mathcal{F}, \mathcal{R})$ be a TRS. Let φ be an E-strategy mapping of $\mathcal{F}$. We say that φ is an *E-strategy mapping of* $(\mathcal{F}, \mathcal{R})$ if it satisfies following condition: for every $f \in \mathcal{F}$

- **(i)** $\varphi(f)$ contains $1, \ldots, n$ if $f \in \mathcal{F}_n$ with $n \geq 1$,
- **(ii)** the last element of $\varphi(f)$ is 0 if $f \in \mathcal{D}$.

Example 6.1.8 Let $(\mathcal{F}, \mathcal{R})$ be the TRS of Example 6.1.5. Then $\mathcal{D} = \{ f,\, g,\, b \}$. Let φ be the E-strategy mapping of $\mathcal{F}$ such that $\varphi(f) = (0,1,0)$, $\varphi(g) = (1,0,2,0)$, $\varphi(a) = \varphi(c) = nil$ and $\varphi(b) = (0)$. Then φ is the E-strategy mapping of $(\mathcal{F}, \mathcal{R})$

In the rest of this section, we assume that φ is an E-strategy mapping of a TRS $(\mathcal{F}, \mathcal{R})$. Under this assumption, we will show that if the $\Rightarrow_\varphi$-reduction sequence from $\langle \varphi(t), \varepsilon \rangle$ ends in $\langle s, p \rangle$ then $\mathsf{e}(s)$ is a normal form. We denote the concatenation of lists l and l' by $l; l'$.

Lemma 6.1.9 Let $t \in \mathcal{T}$. Let $\langle \varphi(t), \varepsilon \rangle \Rightarrow_\varphi^* \langle s, p \rangle$ and $q \in \mathcal{P}os(s)$. If $s(q) = e_l$ for some $e \in \mathcal{F} \cup \mathcal{V}$ then there exists a list l' such that $\varphi(e) = l'; l$.

Proof. Trivial. $\square$

We write $\mathsf{s}(s, q)$ for the list l' in Lemma 6.1.9.

Lemma 6.1.10 Let $t \in \mathcal{T}$. If $\langle \varphi(t), \varepsilon \rangle \Rightarrow_{\varphi}^{*} \langle s, p \rangle$ and $p = p_1.i.p_2$ then the last element of $\mathsf{s}(s, p_1)$ is i.

Proof. We prove the lemma by induction on the length n of $\langle \varphi(t), \varepsilon \rangle \Rightarrow_{\varphi}^{*} \langle s, p \rangle$. The case of $n = 0$ is trivial. We assume that $\langle \varphi(t), \varepsilon \rangle \Rightarrow_{\varphi}^{*} \langle s', q \rangle \Rightarrow_{\varphi} \langle s, p \rangle$. Then $\langle s', q \rangle \Rightarrow_{\varphi} \langle s, p \rangle$ satisfies one of the conditions of Definition 6.1.3.

Case 1. The condition (i), (ii) or (iii) is satisfied. Then $q = p_1.i.p_2.j$ for some j or $p = q$. From induction hypothesis, the last element of $\mathsf{s}(s', p_1)$ is i. Since $\mathsf{s}(s, p_1) = \mathsf{s}(s', p_1)$, the lemma holds.

Case 2. The condition (iv) is satisfied. If $p_2 = \varepsilon$ then the lemma is trivial. If $p_2 = p_2'.j$ for some p_2' and j then $q = p_1.i.p_2'$. Thus it follows from induction hypothesis the the last element of $\mathsf{s}(s', p_1)$ is i. Since $\mathsf{s}(s, p_1) = \mathsf{s}(s', p_1)$, the lemma holds. $\square$

Lemma 6.1.11 Let $t \in \mathcal{T}$. Let $\langle \varphi(t), \varepsilon \rangle \Rightarrow_{\varphi}^{*} \langle s, p \rangle$ and $q \in \mathcal{P}os(s)$.

(1) If $q < p$ then $\mathsf{e}(s|_{q.i})$ is a normal form for every $i \in \mathsf{s}(s, q)$ which is neither zero nor the last element of $\mathsf{s}(s, q)$

(2) Otherwise,

(2-1) $\mathsf{e}(s|_{q.i})$ is a normal form for every $i \in \mathsf{s}(s, q)$ with $i \neq 0$,

(2-2) If the last element of $\mathsf{s}(s, q)$ is zero then $\mathsf{e}(s|_q)$ is not a redex.

Proof. We prove the lemma by induction on the length of $\langle \varphi(t), \varepsilon \rangle \Rightarrow_{\varphi}^{*} \langle s, p \rangle$. If the length is zero then $\mathsf{s}(s, q) = nil$ for every $q \in \mathcal{P}os(s)$. Thus the lemma holds trivially. Suppose $\langle \varphi(t), \varepsilon \rangle \Rightarrow_{\varphi}^{*} \langle s', p' \rangle \Rightarrow_{\varphi} \langle s, p \rangle$. $\langle s', p' \rangle \Rightarrow_{\varphi} \langle s, p \rangle$ satisfies one of the conditions of Definition 6.1.3.

Case 1. The condition (i) is satisfied. Let $p' = p.j$ and $s'(p') = e_{nil}$ where $e \in \mathcal{F} \cup \mathcal{V}$. If $q \neq p$ then it follows from induction hypothesis that (1) and (2) hold. Suppose $q = p$. By Lemma 6.1.10, the last element of $\mathsf{s}(s', q)$ is j. Since $\mathsf{s}(s, q) = \mathsf{s}(s', q)$, (2-2) holds. Let $i \in \mathsf{s}(s, q)$ with $i \neq 0$. If $i \neq j$ then $i \in \mathsf{s}(s', q)$ and i is not the last element of $\mathsf{s}(s', q)$. From induction hypothesis, it is obtained that $\mathsf{e}(s'|_{q.i})$ is a normal form. Since $\mathsf{e}(s|_{q.i}) = \mathsf{e}(s'|_{q.i})$, $\mathsf{e}(s|_{q.i})$ is a normal form. If $i = j$ then $q.i = p'$. Since $s'(p') = e_{nil}$, $\mathsf{s}(s', p') = \varphi(e)$. From induction hypothesis and the assumption that φ is an E-strategy mapping of $(\mathcal{F}, \mathcal{R})$, $\mathsf{e}(s'|_{p'})$ is a normal form. Since $s \equiv s'$, $\mathsf{e}(s|_{p'})$ is a normal form.

Case 2. The condition (ii) is satisfied. If $q < p$ then $\mathsf{s}(s, q) = \mathsf{s}(s', q)$. Let $i \in \mathsf{s}(s, q)$ be a natural number which is neither zero nor the last element of $\mathsf{s}(s, q)$. Using Lemma 6.1.10, we obtain that $q.i \perp p$. Thus $s|_{q.i} \equiv s'|_{q.i}$. Since $\mathsf{e}(s'|_{q.i})$ is normal form by induction hypothesis, $\mathsf{e}(s|_{q.i})$ is a normal form. If $q \perp p$ then from induction hypothesis the lemma follows. Finally we consider the case of $q \geq p$. We assume that $l \to r \in \mathcal{R}$ is applied in this reduction. If $q \in \mathcal{P}os_{\mathcal{F}}(r)$ then $\mathsf{s}(s, q) = nil$ and thus the lemma holds. Otherwise, there exists $q' \in \mathcal{P}os(s')$ such that $p' < q'$ and $s'|_{q'} \equiv s|_q$. By using induction hypothesis, we obtain the lemma.

Case 3. The condition (iii) is satisfied. Then the last element of $\mathsf{s}(s, p)$ is zero and $\mathsf{e}(s'|_p)$ is not a redex. Since $\mathsf{e}(s) \equiv \mathsf{e}(s')$, we can obtain the lemma by using induction hypothesis.

Case 4. The condition (iv) is satisfied. Using induction hypothesis, we can show the lemma. $\square$

Theorem 6.1.12 Let φ be an E-strategy mapping of a TRS $(\mathcal{F}, \mathcal{R})$ and let $t \in \mathcal{T}$. If $\langle \varphi(t), \varepsilon \rangle \Rightarrow_{\varphi}^{*} \langle s, p \rangle$ and $\langle s, p \rangle$ is a normal form w.r.t. $\Rightarrow_{\varphi}$ then $\mathsf{e}(s)$ is a normal form of t.

Proof. Clearly $t \to^{*} \mathsf{e}(s)$. By the definition of $\Rightarrow_{\varphi}$, we have $p = \varepsilon$ and $s(\varepsilon) = e_{nil}$ for some $e \in \mathcal{F} \cup \mathcal{V}$. Since φ is an E-strategy mapping of $(\mathcal{F}, \mathcal{R})$, it follows from Lemma 6.1.11 that $\mathsf{e}(s)$ is a normal form. $\square$

Example 6.1.13 Let $(\mathcal{F}, \mathcal{R})$ be the TRS of Example 6.1.5 and let φ be the E-strategy mapping defined in Example 6.1.8. Then from Theorem 6.1.12, every normal form w.r.t. E-strategy is also a normal form of $(\mathcal{F}, \mathcal{R})$. We have the following $\Rightarrow_{\varphi}$-reduction sequence from $\langle \varphi(g(a, b)), \varepsilon \rangle$:

$$\begin{array}{rcl} \langle g_{(1,0,2,0)}(a_{nil}, b_{(0)}), \varepsilon \rangle & \Rightarrow_{\varphi}^{+} & \langle g_{(0)}(a_{nil}, b_{(0)}), 2 \rangle \\ & \Rightarrow_{\varphi} & \langle g_{(0)}(a_{nil}, a_{nil}), 2 \rangle \\ & \Rightarrow_{\varphi}^{+} & \langle g_{nil}(a_{nil}, a_{nil}), \varepsilon \rangle. \end{array}$$

The $\Rightarrow_{\varphi}$-reduction sequence starting with $\langle \varphi(f(b)), \varepsilon \rangle$ is

$$\begin{array}{rcl} \langle f_{(0,1,0)}(b_{(0)}), \varepsilon \rangle & \Rightarrow_{\varphi}^{+} & \langle f_{(0)}(b_{(0)}), 1 \rangle \\ & \Rightarrow_{\varphi} & \langle f_{(0)}(a_{nil}), 1 \rangle \\ & \Rightarrow_{\varphi} & \langle f_{(0)}(a_{nil}), \varepsilon \rangle \\ & \Rightarrow_{\varphi} & \langle c_{nil}, \varepsilon \rangle. \end{array}$$

6.2 Normalizability

In this section, we investigate the normalizability of the E-strategy. Relying on the theorem of Huet and Lévy [13] (Theorem 2.3.5), henceforth we will be dealing with orthegonal TRSs only.

Let φ an E-strategy mapping of a TRS $(\mathcal{F}, \mathcal{R})$. We say that the reduction $\Rightarrow_{\varphi}$ is *normalizing* for $(\mathcal{F}, \mathcal{R})$ if for every $t \in \mathcal{T}$ having a normal form there exists no infinite sequence $\langle \varphi(t), \varepsilon \rangle \Rightarrow_{\varphi} \langle s_1, p_1 \rangle \Rightarrow_{\varphi} \langle s_2, p_2 \rangle \Rightarrow_{\varphi} \cdots$. According to the theorem of Huet and Lévy, the reduction $\Rightarrow_{\varphi}$ is normalizing for an orthogonal TRS if only needed redexes are contracted by $\Rightarrow_{\varphi}$. In general, for a given orthogonal TRS we can not define an E-strategy mapping φ to contract a needed redex for any term not being a norma form. Consider the following orthogonal TRS:

$$\mathcal{R} = \left\{ \begin{array}{l} f(a, x, a) \to a \\ f(b, b, x) \to b \\ c \to c. \end{array} \right.$$

In a term $f(r_1, r_2, r_3)$ where r_1, r_2 and r_3 are redexes, we first contract r_1 since r_1 is needed. A redex that must be contracted in the next step depends on the contractum of r_1. If the contractum of r_1 is a then r_2 may be not needed because r_3 may reduce to a. Similarly, r_3 may be not needed if the contractum of r_1 is b. Thus we can not give the evaluation order for the function symbol f. Because of this problem, we will formulate a property of E-strategy mappings.

We also say that a position p of an Ω-term t is an index with respect to nf if $p \in I_{nf}(t[\Omega]_p)$. The set $I'_{nf}(t)$ is defined by $I'_{nf}(t) = \{ \ p \in \mathcal{P}os(t) \mid p \in I_{nf}(t[\Omega]_p) \ \}$.

Definition 6.2.1 Let $(\mathcal{F}, \mathcal{R})$ be a TRS. An E-strategy mapping φ of $(\mathcal{F}, \mathcal{R})$ is *careful* if it satisfies the following condition: for every $l \in \mathit{Red}$ such that $l(\varepsilon) = f$ and $\varphi(f) = (n_1, \ldots, n_k)$, there exists i such that

(i) $n_i = 0$,

(ii) for any $1 \leq j < i$, n_j is zero or an index w.r.t. nf of l,

(iii) for any $i < j \leq k$, n_j is zero or an Ω-position of l.

Example 6.2.2 Let

$$\mathcal{R} = \begin{cases} f(a, x, a) \to a \\ f(b, x, y) \to b \\ c \to c. \end{cases}$$

Let φ be an E-strategy mapping of $\mathcal{R}$ such that $\varphi(f) = (1, 0, 3, 0, 2, 0)$ and $\varphi(c) = (0)$. Then $\mathit{Red} = \{\, f(a, \Omega, a),\ f(b, \Omega, \Omega),\ c \,\}$. Because $I'_{nf}(f(a, \Omega, a)) = \{\, \varepsilon,\ 1,\ 3 \,\}$, $i = 4$ satisfies the conditions of carefulness for $f(a, \Omega, a)$. Similarly, we can see that $i = 2$ and $i = 1$ satisfy the conditions of carefulness for $f(b, \Omega, \Omega)$ and c, respectively. Thus φ is careful.

However, carefulness is not sufficient for normalizability of $\Rightarrow_\varphi$. Consider the following orthogonal TRS:

$$\mathcal{R} = \begin{cases} f(g(x)) \to a \\ b \to g(b). \end{cases}$$

Let $\varphi(f) = (1, 0)$, $\varphi(g) = (1)$, $\varphi(b) = (0)$ and $\varphi(a) = \mathit{nil}$. Then φ is a careful E-strategy mapping of $\mathcal{R}$. Although the term $t \equiv f(g(b))$ has the normal form a, there exists the infinite sequence

$$\begin{array}{rl} \langle\, f_{(1,0)}(g_{(1)}(b_{(0)})), \varepsilon, \rangle & \Rightarrow_\varphi^+ \ \langle\, f_{(0)}(g_{nil}(b_{(0)})),\ 1.1 \,\rangle \\ & \Rightarrow_\varphi \ \langle\, f_{(0)}(g_{nil}(g_{(1)}(b_{(0)}))),\ 1.1 \,\rangle \\ & \Rightarrow_\varphi \ \langle\, f_{(0)}(g_{nil}(g_{nil}(b_{(0)}))),\ 1.1.1 \,\rangle \\ & \Rightarrow_\varphi \ \langle\, f_{(0)}(g_{nil}(g_{nil}(g_{(1)}(b_{(0)})))),\ 1.1.1 \,\rangle \\ & \Rightarrow_\varphi \ \cdots. \end{array}$$

In the E-strategy, if the redex at a position p was contracted then the search for a redex start at p in the next step. But in general, needed redexes cannot be found locally. For example, the redex b in $g(b)$ is needed but it is not a needed redex in $f(g(b))$. Thus we need the following transitivity property for indices.

Definition 6.2.3 A TRS $\mathcal{R}$ is *index-transitive* if for every term t in $\mathcal{T}$, $p \in I'_{nf}(t)$ and $q \in I'_{nf}(t|_p)$ imply $p.q \in I'_{nf}(t)$.

In the following we prove that if $\mathcal{R}$ is an index-transitive TRS having a careful E-strategy mapping φ then $\Rightarrow_\varphi$ is normalizing for $\mathcal{R}$. Indices w.r.t. nf have the following property.

Lemma 6.2.4 ([18]) Let $t \in \mathcal{T}_\Omega$. If $p \in I'_{nf}(t)$ and $q \leq p$ then $q \in I'_{nf}(t)$. $\square$

Lemma 6.2.5 Let $f(l_1, \ldots, l_n) \to r \in \mathcal{R}$ with $n > 0$. Let $1 \leq i \leq n$. If $l_i \notin \mathcal{V}$ then $i \in I'_{nf}(f(l_1, \ldots, l_n)_\Omega)$.

Proof. From the orthogonality of $\mathcal{R}$. $\Box$

The *non-root reduction* $t \to_{nr} s$ is defined as $t \xrightarrow{p} s$ and $p \neq \varepsilon$. We can easily see that if a term t in $\mathcal{T}$ is not root-stable then there exists a redex s such that $t \to_{nr}^{*} s$.

Lemma 6.2.6 Let t be a term in $\mathcal{T}$ such that $t \to_{nr}^{*} l\theta$ for some $l \to r \in \mathcal{R}$. If $i \in I'_{nf}(l_\Omega)$ and $|\, i \,| = 1$ then $i \in I'_{nf}(t)$.

Proof. We assume that $i \notin I'_{nf}(t)$. Clearly $i \in \mathcal{P}os(t)$. Thus $t[\Omega]_i \to^{*} s$ for some normal form s. Since $t[\Omega]_i \to^{*} l\theta[\Omega]_i$, by the Church-Rosser property of $\mathcal{R}$ we obtain $l\theta[\Omega]_i \to^{*} s$. Because $l\theta[\Omega]_i \geq l_\Omega[\Omega]_i$, $i \notin I'_{nf}(l_\Omega)$. $\Box$

Lemma 6.2.7 Let $\mathcal{R}$ be an index-transitve TRS having a careful E-strategy mapping φ. If $\langle \varphi(t), \varepsilon \rangle \Rightarrow_{\varphi}^{*} \langle\, s, p \,\rangle$ then $p \in I'_{nf}(\mathsf{e}(s))$.

Proof. We prove the lemma by induction on the length of this reduction. The case of zero length is trival. We assume that $\langle \varphi(t), \varepsilon \rangle \Rightarrow_{\varphi}^{*} \langle s', q \rangle \Rightarrow_{\varphi} \langle s, p \rangle$. From induction hypothesis, $q \in I'_{nf}(\mathsf{e}(s'))$. The last step $\langle s', q \rangle \Rightarrow_{\varphi} \langle s, p \rangle$ satisfies one of the conditions of Definition 6.1.3.

Case 1. The condition (i) is satisfied. Let $q = p.i$. From Lemma 6.2.4, $p \in I'_{nf}(\mathsf{e}(s'))$. Since $s \equiv s'$, $p \in I'_{nf}(\mathsf{e}(s))$.

Case 2. The condition (ii) or (iii) is satisfied. Because $p = q$ and $\mathsf{e}(s)[\Omega]_p \equiv \mathsf{e}(s')[\Omega]_p$, $p \in I'_{nf}(\mathsf{e}(s))$.

Case 3. The condition (iv) is satisfied. Then $\mathsf{e}(s) \equiv \mathsf{e}(s')$. Let $s' \equiv s'[f_{(i,\ldots)}(t_1, \ldots, t_n)]_q$ and $p = q.i$. If $\mathsf{e}(f_{(i,\ldots)}(t_1, \ldots, t_n))$ is root-stable then $i \in I'_{nf}(\mathsf{e}(f_{(i,\ldots)}(t_1, \ldots, t_n)))$. Since $q \in I'_{nf}(\mathsf{e}(s'))$ and $\mathcal{R}$ is index-transitive, $p \in I'_{nf}(\mathsf{e}(s'))$. Thus $p \in I'_{nf}(\mathsf{e}(s))$. Otherwise, we assume that $\mathsf{e}(f_{(i,\ldots)}(t_1, \ldots, t_n)) \to_{nr}^{*} f(l_1, \ldots, l_n)\theta$ for some $f(l_1, \ldots, l_n) \to r \in \mathcal{R}$. Let $\varphi(f) = (n_1, \ldots, n_k)$. By the carefulness of φ, there exists i' satisfying **(i), (ii)** and **(iii)** in Definiton 6.2.1 for $f(l_1, \ldots, l_n)_\Omega$. We will show the claim that $i = n_j$ for some j with $j < i'$. Suppse the contrary. Then we have the reduction sequence

$$\begin{array}{lll} \langle \varphi(t), \varepsilon \rangle & \Rightarrow_{\varphi}^{*} & \langle s'[f_{(0,n_{i'+1},\ldots,n_k)}(s_1, \ldots s_n)]_q, q \rangle \\ & \Rightarrow_{\varphi} & \langle s'[f_{(n_{i'+1},\ldots,n_k)}(s_1, \ldots s_n)]_q, q \rangle = \langle s'_1, q_1 \rangle \\ & \Rightarrow_{\varphi} & \langle s'_2, q_2 \rangle \\ & \vdots & \\ & \Rightarrow_{\varphi} & \langle\, s'_m, q_m \,\rangle = \langle\, s', q \,\rangle. \end{array}$$

such that $f(\mathsf{e}(s_1), \ldots \mathsf{e}(s_n))$ is not a redex and for any $1 \leq j \leq m-1$, $q_j \geq q$ and if $q_j = q$ then $\langle\, s'_j, q_j \,\rangle \Rightarrow_{\varphi} \langle\, s'_{j+1}, q_{j+1} \,\rangle$ does not satisfy (ii) of Definition 6.1.3. If $l_j \notin \mathcal{V}$ where $1 \leq j \leq n$ then from Lemma 6.2.5, j is an index w.r.t. nf of $f(l_1, \ldots, l_n)_\Omega$. By the carefulness of φ, $j \in \mathsf{s}(s'_1, q)$. According to Lemma 6.1.11 $\mathsf{e}(s_j)$ is a normal form. Thus $f(\mathsf{e}(s_1), \ldots \mathsf{e}(s_n))$ is an instance of $f(l_1, \ldots, l_n)$ because $f(\mathsf{e}(s_1), \ldots, \mathsf{e}(s_n)) \to_{nr}^{*} l\theta$. This is a contradiction and hence we are done. By the claim and **(ii)** in Definition 6.2.1, $i \in I'_{nf}(f(l_1, \ldots, l_n)_\Omega)$. From Lemma 6.2.6 it follows that i is an index w.r.t. nf of $f(\mathsf{e}(t_1), \ldots \mathsf{e}(t_n))$. Because $q \in I'_{nf}(\mathsf{e}(s'))$ and $\mathcal{R}$ is index-transitive, $p \in I'_{nf}(\mathsf{e}(s'))$. Therefore $p \in I'_{nf}(\mathsf{e}(s))$. $\Box$

The length of a list l is denoted by $|\,l\,|$. We define $\|\,t\,\|$ for $t \in \mathcal{T}_{\mathcal{L}}$ as follows:

$$\|\,t\,\| = \begin{cases} 0 & \text{if } t \in \mathcal{V}_{\mathcal{L}}, \\ |\,l\,| + \|t_1\| + \cdots + \|t_n\| & \text{if } t \equiv f_l(t_1, \ldots, t_n). \end{cases}$$

If $t \in \mathcal{T}$ has a normal form then $\mathsf{d}(t)$ denotes the maximum length of needed reduction sequences from t to the normal form.

Theorem 6.2.8 Let $\mathcal{R}$ be an index-transitive orthogonal TRS having a careful E-strategy mapping φ. Then $\Rightarrow_\varphi$ is normalizing for $\mathcal{R}$.

Proof. We define $\|\langle s, p\rangle\|$ as $(\mathsf{d}(\mathsf{e}(s)), \|\,s\,\|, |\,p\,|)$ if $\mathsf{e}(s)$ has a normal form. Let t be a term in $\mathcal{T}$ which has a normal form. Now we show that if $\langle \varphi(t), \varepsilon\rangle \equiv \langle s_0, p_0\rangle \Rightarrow_\varphi \langle s_1, p_1\rangle \Rightarrow_\varphi \cdots$ then $\|\langle s_n, p_n\rangle\| >_{lex} \|\langle s_{n+1}, p_{n+1}\rangle\|$ for any $n \geq 0$, where $>_{lex}$ is the lexicographic order on $\mathcal{N}^3$. The step $\langle s_n, p_n\rangle \Rightarrow_\varphi \langle s_{n+1}, p_{n+1}\rangle$ satisfies one of the conditions of Definition 6.1.3. If (i) is satisfied then $\mathsf{b}(\mathsf{e}(s_n)) = \mathsf{b}(\mathsf{e}(s_{n+1}))$, $\|s_n\| = \|s_{n+1}\|$ and $|p_n| > |p_{n+1}|$. If (ii) is satisfied then $\mathsf{b}(\mathsf{e}(s_n)) > \mathsf{b}(\mathsf{e}(s_{n+1}))$ by Lemma 6.2.7. If (iii) or (iv) is satisfied then $\mathsf{b}(\mathsf{e}(s_n)) = \mathsf{b}(\mathsf{e}(s_{n+1}))$ and $\|s_n\| > \|s_{n+1}\|$. Therefore we obtain $\|\langle s_n, p_n\rangle\| >_{lex} \|\langle s_{n+1}, p_{n+1}\rangle\|$. Since $>_{lex}$ is well-founded, i.e., there exists no infinite sequence $a_0 >_{lex} a_1 >_{lex} a_2 >_{lex} \cdots$ of elements of $\mathcal{N}^3$, the reduction $\Rightarrow_\varphi$ is normalizing.
$\square$

6.3 A Sufficient Condition for Carefulness

In general, it is undecidable whether there exists a careful E-strategy mapping φ for a given TRS $\mathcal{R}$. Because it is undecidable whether an position in a term is an index w.r.t. nf. In this section, we introduce a sufficient condition for existence of a careful E-strategy mapping. We explain how to define a careful E-strategy mapping if a TRS satisfies this condition.

Lemma 6.2.5 expresses a sufficient condition for indices w.r.t. nf of an Ω-term l in *Red*, i.e., non-Ω-positions of l are indices w.r.t. nf. This formalizes the following property of E-strategy mappings.

Definition 6.3.1 Let $(\mathcal{F}, \mathcal{R})$ be a TRS. An E-strategy mapping φ of $(\mathcal{F}, \mathcal{R})$ is *semi-careful* if it satisfies the following condition: for every $l \in Red$ such that $l(\varepsilon) = f$ and $\varphi(f) = (n_1, \ldots, n_k)$, there exists i such that

- **(i)** $n_i = 0$,
- **(ii)** for any $1 \leq j < i$, n_j is zero or non-Ω-position of l,
- **(iii)** for any $i < j \leq k$, n_j is zero or an Ω-position of l.

Lemma 6.3.2 Let φ be a semi-careful E-strategy mapping of $(\mathcal{F}, \mathcal{R})$. Then φ is careful.

Proof. By using Lemma 6.2.5. $\square$

In the following we present the class of TRSs having a semi-careful E-strategy mapping.

Definition 6.3.3

1. If t is an Ω-term then the set $N(t)$ is defined as follows:

$$N(t) = \begin{cases} \{\, i \mid t_i \not\equiv \Omega,\ 1 \le i \le n \,\} & \text{if } t \equiv f(t_1, \ldots, t_n) \text{ and } n > 0, \\ \phi & \text{otherwise.} \end{cases}$$

2. If $f \in \mathcal{F}$ then the set N_f is defined by

$$N_f = \{\ N(l) \mid l \in \mathit{Red} \ \text{ and } \ l(\varepsilon) = f\ \}.$$

Example 6.3.4 Let $\mathcal{F} = \{\, f, g, a, b, c \,\}$ and

$$\mathcal{R} = \begin{cases} f(x, a, b) \to x \\ f(x, b, y) \to y \\ g(x) \to a \\ c \to g(c). \end{cases}$$

Then $N_f = \{\{2,3\}, \{2\}\}$, $N_g = N_c = \{\phi\}$ and $N_a = N_b = \phi$.

Definition 6.3.5 A TRS $(\mathcal{F}, \mathcal{R})$ is N-*total* if for all $f \in \mathcal{D}$, $S_1 \subseteq S_2$ or $S_1 \supseteq S_2$ whenever $S_1,\ S_2 \in N_f$.

We define the E-strategy mapping φ_t for an N-total TRS $(\mathcal{F}, \mathcal{R})$ as follows. Let $f \in \mathcal{F}_n$. If $f \notin \mathcal{D}$ and $n = 0$ then we define $\varphi_t(f)$ as nil. If $f \notin \mathcal{D}$ and $n > 0$ then we define $\varphi_t(f)$ as $(1, \ldots, n)$. If $f \in \mathcal{D}$ and $n = 0$ then we define $\varphi_t(f)$ as (0). We finally assume that $f \in \mathcal{D}$ and $n > 0$. Let $N_f = \{S_1, \ldots, S_m\}$. Then there exists the sequence $S_{i_1} \subset S_{i_2} \subset \cdots \subset S_{i_m}$ of elements of N_f. Let $l_1, \ldots, l_{m+1}$ be the following lists:

$$\begin{cases} l_1 = S_{i_1}, & \\ l_j = (n_1, \ldots, n_k) & \text{if } 1 < j \le m,\ S_{i_j} \backslash S_{i_{j-1}} = \{n_1, \ldots, n_k\} \text{ and} \\ & \quad n_i < n_{i+1} \text{ for any } 1 \le i \le k-1, \\ l_{m+1} = (n_1, \ldots, n_k) & \text{if } \{1, \ldots, n\} \backslash S_{i_m} = \{n_1, \ldots, n_k\} \text{ and} \\ & \quad n_i < n_{i+1} \text{ for any } 1 \le i \le k-1. \end{cases}$$

We define $\varphi_t(f)$ by $\varphi_t(f) = l_1; (0); l_2; \cdots; l_m; (0); l_{m+1}; (0)$.

Example 6.3.6 Consider the TRS $(\mathcal{F}, \mathcal{R})$ of Example 6.3.4. Then $(\mathcal{F}, \mathcal{R})$ is N-total. We have $\varphi_t(f) = (2, 0, 3, 0, 1, 0)$, $\varphi_t(g) = (0, 1, 0)$, $\varphi_t(c) = (0)$ and $\varphi_t(a) = \varphi_t(b) = \mathit{nil}$.

From the definition of φ_t we can easily prove the following lemmas.

Lemma 6.3.7 Let $(\mathcal{F}, \mathcal{R})$ be an N-total TRS. Then φ_t is a semi-careful E-strategy mapping of $(\mathcal{F}, \mathcal{R})$. □

Theorem 6.3.8 Let $(\mathcal{F}, \mathcal{R})$ be an N-total index-transitive orthogonal TRS. Then $\Rightarrow_{\varphi_t}$ is normalizing for $(\mathcal{F}, \mathcal{R})$.

Proof. From theorem 6.2.8 and Lemmas 6.3.2 and 6.3.7. □

It is not difficult to see that if a TRS $(\mathcal{F}, \mathcal{R})$ has a semi-careful E-strategy mapping then $\mathcal{R}$ is N-total. Thus we have the following theorem.

Theorem 6.3.9 Let $(\mathcal{F}, \mathcal{R})$ be a TRS. Then $(\mathcal{F}, \mathcal{R})$ is N-total iff $(\mathcal{F}, \mathcal{R})$ has a semi-careful E-strategy mapping. □

6.4 A Necessary and Sufficient Condition for Index-Transitivity

In this section we give a necessary and sufficient condition for index-transitivity, which is useful to prove index-transitivity of orthogonal TRSs. For this purpose, we need several lemmas that express properties of indices w.r.t. nf.

Lemma 6.4.1 Let $t \in \mathcal{T}$ and $p \in \mathcal{P}os(t)$. Let $A : t \to^* s$ and $q \in p\backslash A$. If $q \in I'_{nf}(s)$ then $p \in I'_{nf}(t)$.

Proof. We prove the lemma by induction on the length n of A. The case $n = 0$ is trivial. Let $A : t \xrightarrow{p_1} t' \to^* s$. Suppose that $p' \in p\backslash A[1]$ and $q \in p'\backslash A[1, n]$. From induction hypothesis it follows that $p' \in I'_{nf}(t')$. If $p \leq p_1$ then $p = p'$ and $t[\Omega]_p \equiv t'[\Omega]_p$. Thus $p \in I'_{nf}(t)$. If $p \not\leq p_1$ then there exists $s' \in \mathcal{T}_\Omega$ such that $t[\Omega]_p \xrightarrow{p_1} s'$ and $s' \leq t'[\Omega]_{p'}$. Since $p' \in I'_{nf}(t')$, $nf(t'[\Omega]_p) = false$. From the monotonicity of nf it follows that $nf(s') = false$. By the Church-Rosser property of $\mathcal{R}$ we obtain $nf(t[\Omega]_p) = false$. Therefore $p \in I'_{nf}(t)$. □

Definition 6.4.2 Let p be a redex position of t, i.e., $t \equiv t[l\theta]_p$ for some $l \to r \in \mathcal{R}$. The set $C(t, p)$ is defined by

$$C(t,p) = \{\ p.q \mid q \in \mathcal{P}os_{\mathcal{F}}(l)\ \}.$$

Lemma 6.4.3 Let t_0 be a term in $\mathcal{T}$ which has a normal form and let $p \in I'_{nf}(t_0)$. Let $A : t_0 \xrightarrow{q_0} t_1 \xrightarrow{q_1} \cdots \xrightarrow{q_{n-1}} t_n$ be a reduction sequence such that for every $0 \leq i \leq n-1$

$$C(t_i, q_i) \cap \{\ p' \in \mathcal{P}os(t_i) \mid p' \in p\backslash A[i]\ \} = \phi.$$

Then t_n has a descendant of p by A, i.e., $p\backslash A \neq \phi$.

Proof. We assume that t_n does not have a descendant of p. For each $0 \leq i < n$, let s_i be a Ω-term obtained from t_i by replacing all subterms at descendants of p by $A[i]$ with Ω. From the assumption for A and the left-linearity of $\mathcal{R}$ we can obtain the reduction sequence $s_0 \to^{\equiv} s_1 \to^{\equiv} \cdots \to^{\equiv} s_n$. Since $s_0 \equiv t_0[\Omega]_p$ and $s_n \equiv t_n$, we have $t_0[\Omega]_p \to^* t_n$. By the Church-Rosser property of $\mathcal{R}$, t_n has a normal form and hence $nf(t_0[\Omega]_p) = true$. However, this contradicts the assumption that $p \in I'_{nf}(t_0)$. □

Lemma 6.4.4 Let $t \in \mathcal{T}$. Let p be a redex position in t and let $q \in C(t, p)$. If $p \in I'_{nf}(t)$ then $q \in I'_{nf}(t)$.

Proof. If $p \in I'_{nf}(t)$ then p is needed. We now suppose that $q \notin I'_{nf}(t)$. Then there exists a reduction sequence $A : t[\Omega]_q \equiv t_0 \xrightarrow{p_0} t_1 \xrightarrow{p_1} \cdots \xrightarrow{p_{n-1}} t_n \in \mathrm{NF}_{\mathcal{R}}$. From the orthogonality of $\mathcal{R}$, $p_i \notin p\backslash A[i]$ for any $0 \leq i \leq n-1$. For each $0 \leq i \leq n$, let $\tilde{t}_i$ be a term obtained from t_i by replacing all Ω's with $t|_q$. Then we get the reduction sequence $A' : \tilde{t}_0 \equiv t \xrightarrow{p_0} \tilde{t}_1 \xrightarrow{p_1} \cdots \xrightarrow{p_{n-1}} \tilde{t}_n \equiv t_n$ such that $p_i \notin p\backslash A'[i]$ for any $0 \leq i \leq n-1$. However this contradicts the fact that a redex position p is needed. □

Lemma 6.4.5 Let t be a term in $\mathcal{T}$ which has normal form and let $p \in I'_{nf}(t)$. Let $A : t \xrightarrow{q} s$ and $p \notin C(t, q)$. Then there exists p' in $p\backslash A$ such that $p' \in I'_{nf}(s)$.

Proof. There exists a reduction sequence $A' : t \xrightarrow{q} s \equiv s_0 \xrightarrow{q_0}_{\mathcal{N}} s_1 \xrightarrow{q_1}_{\mathcal{N}} \cdots \xrightarrow{q_{n-1}}_{\mathcal{N}} s_n \in \mathrm{NF}_{\mathcal{R}}$. Let $S_i = C(s_i, q_i) \cap \{\, q' \in \mathcal{P}os(s_i) \mid q' \in p\backslash A'[i+1] \,\}$ for each $0 \leq i \leq n-1$. We consider the following two cases.

Case 1. $S_i = \phi$ for every $0 \leq i \leq n-1$. According to Lemma 6.4.3, s_n has a descendant p_1 of p by A'. Let $p_2 \in p\backslash A$ with $p_1 \in p_2\backslash A'[1, n+1]$. Since s_n is a normal form, $p_1 \in I'_{nf}(s_n)$. By Lemma 6.4.1 $p_2 \in I'_{nf}(s)$.

Case 2. $S_i \neq \phi$ for some i. Let $p_1 \in S_i$ and let $p_2 \in p\backslash A$ with $p_1 \in p_2\backslash A'[1, i+1]$. By Lemma 6.4.4 $p_1 \in I'_{nf}(s_i)$. Thus it follows from Lemma 6.4.1 that $p_1 \in I'_{nf}(s)$. □

Lemma 6.4.6 Let $l \to r \in \mathcal{R}$ and $\theta : \mathcal{V} \to \mathcal{T}$. Let $p \in \mathcal{P}os_{\mathcal{F}}(l)$ with $p \neq \varepsilon$. Then $l\theta|_p$ has a normal form iff $x\theta$ has a normal form for every $x \in l|_p$

Proof. From the orthogonality of $\mathcal{R}$. □

Lemma 6.4.7 Let t be a term in $\mathcal{T}$ which has a normal form. If $p_1.p_2 \in I'_{nf}(t)$ then $p_2 \in I'_{nf}(t|_{p_1})$.

Proof. We assume that $p_2 \notin I'_{nf}(t|_{p_1})$. Then $t|_{p_1}[\Omega]_{p_2} \to^* s$ for some normal form s. We can obtain that $t|_{p_1} \to^* s$ because s does not contain Ω's. Since $t \to^* t[s]_{p_1}$ and $t[\Omega]_{p_1.p_2} \to^* t[s]_{p_1}$, from the Church-Rosser property of $\mathcal{R}$, $t[\Omega]_{p_1.p_2}$ has a normal form, i.e., $nf(t[\Omega]_{p_1.p_2}) = true$. Hence $p_1.p_2 \notin I'_{nf}(t)$. □

Lemma 6.4.8 ([18]) Let $t \in \mathcal{T}_\Omega$. If $p \in I'_{nf}(t)$ and $t \leq s$ then $p \in I'_{nf}(s)$. □

The following lemma gives a sufficient condition for index-transitivity.

Lemma 6.4.9 Let $\mathcal{R}$ be an orthogonal TRS such that for any $l \in \mathit{Red}$ if $l|_p \equiv \Omega$ and $|\,p\,| \geq 2$ then $p \in I'_{nf}(l)$. Let t be a term in $\mathcal{T}$ which has a normal form. If $p_1 \in I'_{nf}(t)$ and $p_2 \in I'_{nf}(t|_{p_1})$ then

(1) $t|_{p_1}$ has a normal form,

(2) $p_1.p_2 \in I'_{nf}(t)$.

Proof. We prove the lemma by induction on $\mathsf{d}(t)$. The case $\mathsf{d}(t) = 0$ is trivial because t is a normal form. Let $A : t \xrightarrow{p_3}_{\mathcal{N}} s$.

Case 1. $p_1, p_1.p_2 \notin C(t, p_3)$. We can easily show (1) and (2) by using induction hypothesis and Lemmas 6.4.1 and 6.4.5.

Case 2. $p_1 \notin C(t, p_3)$ and $p_1.p_2 \in C(t, p_3)$. From Lemma 6.4.4 it follows that $p_1.p_2 \in I'_{nf}(t)$. Using induction hypothesis and Lemma 6.4.5, we can easily show (1).

Case 3. $p_1 \in C(t, p_3)$ and $p_1.p_2 \notin C(t, p_3)$. If $p_1 = p_3$ then we can show (1) and (2) by using induction hypothesis and Lemmas 6.4.1 and 6.4.5. We next consider the case of $p_1 > p_3$. Let $p_1 = p_3.q_1$ where $q_1 \neq \varepsilon$. Suppose that $t \equiv t[l\theta]_{p_3}$ and $s \equiv t[r\theta]_{p_3}$ for $l \to r \in \mathcal{R}$. By Lemma 6.2.4, $p_3 \in I'_{nf}(t)$ and therefore $p_3 \in I'_{nf}(s)$. It follows form induction hypothesis that $r\theta$ has a normal form. Hence $l\theta$ has a normal form. We first show the claim that if $p \in \mathcal{P}os_{\mathcal{V}}(l)$ and $|\,p\,| \geq 2$ then there exists q in $p_3.p\backslash A$ such that $q \in I'_{nf}(s)$.

Poof of the claim. Let p be a position in $\mathcal{P}os_{\mathcal{V}}(l)$ such that $|\,p\,| \geq 2$. Then $p \in I'_{nf}(l_\Omega)$ by the assumption of the lemma. By using Lemma 6.4.8 we obtain that $p \in I'_{nf}(l\theta)$. Let

$A' : l\theta \overset{\varepsilon}{\to} r\theta$. According to Lemma 6.4.5 there exists p' in $p\backslash A'$ such that $p' \in I'_{nf}(r\theta)$. It is clear that $p_3.p' \in p_3.p\backslash A$. Since $p_3 \in I'_{nf}(s)$ and $p' \in I'_{nf}(s|_{p_3})$, we get $p_3.p' \in I'_{nf}(s)$ by induction hypothesis.

We next prove that $t|_{p_1} \equiv l\theta|_{q_1}$ has a normal form. According to Lemma 6.4.6 it is sufficient to show that $x\theta$ has a normal form for any $x \in l|_{q_1}$. Let $x \in l|_{q_1}$. Then $x \equiv l|_p$ for some p with $|\,p\,| \geq 2$. By the claim, there exists q in $p_3.p\backslash A$ such that $q \in I'_{nf}(s)$. From induction hypothesis, $s|_q$ has a normal form. Since $x\theta \equiv s|_q$, $x\theta$ has a normal form. Finally we prove that $p_1.p_2 \in I'_{nf}(t)$. We have $p_2 = q_2.q_3$ for some $q_2 \in \mathcal{P}os_{\mathcal{V}}(l|_{q_1})$ with $|q_2| \geq 1$. Since $q_2.q_3 \in I'_{nf}(t|_{p_1})$ and $t|_{p_1}$ has a normal form, it is obtained that $q_3 \in I'_{nf}(t|_{p_1.q_2})$ by Lemma 6.4.7. Because $q_1.q_2 \in \mathcal{P}os_{\mathcal{V}}(l)$ and $|q_1.q_2| \geq 2$, according to the claim there exists q in $p_3.q_1.q_2\backslash A$ such that $q \in I'_{nf}(s)$. Because $t|_{p_1.q_2} \equiv t|_{p_3.q_1.q_2} \equiv s|_q$, we get $q_3 \in I'_{nf}(s|_q)$. Thus by using induction hypothesis, it is obtained that $q.q_3 \in I'_{nf}(s)$. Since $q.q_3$ is a descendant of $p_1.p_2$ by A, Lemma 6.4.1 yields $p_1.p_2 \in I'_{nf}(t)$.

Case 4. $p_1 \in C(t,p_3)$ and $p_1.p_2 \in C(t,p_3)$. From Lemma 6.4.4 it follows that $p_1.p_2 \in I'_{nf}(t)$. Similar to Case 3, we can show that $t|_{p_1}$ has a normal form. □

We show that the condition in Lemma 6.4.9 is also necessary for index-transitivity.

Theorem 6.4.10 Let $\mathcal{R}$ be an orthogonal TRS. The following are equivalent:

(1) $\mathcal{R}$ is index-transitive,

(2) for any $t \in \mathcal{T}_\Omega$ if $p \in I'_{nf}(t)$ and $q \in I'_{nf}(t|_p)$ then $p.q \in I'_{nf}(t)$,

(3) for any $l \in \mathit{Red}$ if $l|_p \equiv \Omega$ and $|\,p\,| \geq 2$ then $p \in I'_{nf}(l)$.

Proof.

(1)$\Rightarrow$(2) We assume that $p \in I'_{nf}(t)$, $q \in I'_{nf}(t|_p)$ and $p.q \notin I'_{nf}(t)$. Then there exists $s \in \mathcal{T}_\Omega$ such that $t[\Omega]_{p.q} \leq s$, $nf(s) = \mathit{true}$ and $s|_{p.q} \equiv \Omega$. From the monotonicity of nf we get $s' \in \mathcal{T}$ such that $t \leq s'$ and $nf(s'[\Omega]_{p.q}) = \mathit{true}$, i.e., $p.q \notin I'_{nf}(s')$. Using Lemma 6.4.8, we can show that $p \in I'_{nf}(s')$ and $q \in I'_{nf}(s'|_p)$. But this contradicts the assumption that $\mathcal{R}$ is index-transitive.

(2)$\Rightarrow$(3) Let $l \in \mathit{Red}$ and let p be an Ω-position with $|\,p\,| \geq 2$. Then there exists $q \in \mathcal{P}os(l)$ such that $q < p$ and $|\,q\,| = 1$. By the orthogonality of $\mathcal{R}$, $q \in I'_{nf}(l)$ and $p/q \in I'_{nf}(l|_q)$. Thus it follows from the assumption (2) that $p \in I'_{nf}(l)$.

(3)$\Rightarrow$(1) Let $t \in \mathcal{T}$. Let $p \in I'_{nf}(t)$ and $q \in I'_{nf}(t|_p)$. If t has a normal form then $p.q \in I'_{nf}(t)$ by Lemma 6.4.9. Otherwise $nf(t[\Omega]_{p.q}) = \mathit{false}$ by the monotonicity of nf. Thus $p.q \in I'_{nf}(t)$. □

Example 6.4.11 Consider the TRS $\mathcal{R}$ of Example 6.3.4. Every variable in the left-hand side of a rewrite rule occurs at depth one in the left-hand side. From Theorem 6.4.10, $\mathcal{R}$ is index-transitive. Thus $\Rightarrow_{\varphi_t}$ is normalizing for $\mathcal{R}$.

Chapter 7

Conclusion

In this thesis we investigated normalizing strategies for term rewriting systems.

In Chapter 3 we presented the class of NVNF-sequential systems. NVNF-sequentiality is defined by using the same approximate reduction as NV-sequentiality [28]. However, it gives consideration to the reachability only to normal forms without Ω's whereas NV-sequentiality gives consideration to the reachability to terms without Ω's. We showed that the class of NVNF-sequential systems properly includes NV-sequential systems. We proved the decidability of indices with respect to NVNF-sequentiality for left-linear term rewriting systems. This result implies that we can compute at least one of the needed redexes in a term no being a normal form if a term rewriting system is orthogonal NVNF-sequential. Thus, by the theorem of Huet and Lévy [13] every orthogonal NVNF-sequential system has a decidable normalizing call-by-need strategy. Our main purpose was to give a simplified proof of the the decidability of indices with respect to NVNF-sequentiality. The complexity of the decision algorithm for indices w.r.t. NVNF-sequentiality remains open although indices w.r.t. NV-sequentiality are decidable in polynomial time.

In Chapter 4, we investigated the normalizability of Huet and Lévy's strategy [13] (index reduction) and Oyamaguchi's strategy [28] (NV-index reduction) for left-linear overlapping term rewriting systems. We first introduced the notion of stable balanced joinability. A term rewriting system is called stable balanced joinable if every critical pair is joinable with balanced stable reduction. Stable balanced joinable property implies the balanced weakly Church-Rosser property of index reduction. Thus, by Toyama's theorem [30] concerning reduction strategies, index reduction is normalizing for every stable balanced joinable strongly sequential system. The class of stable balanced joinable systems includes all root balanced joinable systems which were defined by Toyama in [30]. We next introduced the notion of NV-stable balanced joinability. It was shown that NV-index reduction is normalizing for every NV-stable balanced joinable NV-sequential system. Stable and NV-stable balanced joinability are semi-decidable properties of left-linear term rewriting systems, however these properties are undecidable. It remains to give decidable sufficient conditions for stable and NV-stable balanced joinability. In Chapter 4, we do not deal with more general sequential systems (NVNF-, shallow [3] or growing [14] sequential systems). Because the index reduction does not have the balanced weakly Church-Rosser property for more general sequential systems. However, we conjecture that our result can be generalized to these sequentiality.

In Chapter 5, we research into decidable properties of growing term rewriting systems.

We first extended Jacquemard's result [14] to left-linear growing systems that may contain non-right-linear rules. It has been shown that the set of reachable terms to some recognizable set is recognized by a tree automaton if a term rewriting system is left-linear growing. This implies the decidability of reachability for left-linear growing term rewriting systems . Moreover, this gives us better approximations of term rewriting systems which extend the class of orthogonal systems having a decidable normalizing call-by-need strategy. By our recognizability result, we can show that the reachability and the joinability of a term rewriting system are decidable if its inverse system is left-linear growing. We believe that this result is useful for the construction of normalizing strategies for non-left-linear term rewriting systems. We next proved that termination for almost orthogonal growing term rewriting systems is decidable.

In Chapter 6, we applied the results on call-by-need strategy to the E-strategy which is adopted by the OBJ algebraic specification languages [9, 11, 24]. The E-strategy chooses a redex according to local strategies which are given to each function symbol. We consider how to give local strategies for a given orthogonal term rewriting system to contract a needed redex only. For this purpose, we introduced the notions of index-transitivity and carefulness. We showed that for every index-transitive orthogonal term rewriting system, if careful local strategies are given to each function symbol then the E-strategy contract needed redexes only, thus E-strategy is normalizing. In general, index-transitivity and the existence of careful local strategies are undecidable. We first gave a decidable sufficient condition for the existence of careful local strategies, which is called N-total, and explained how to give careful local strategies for N-total term rewriting systems. We next gave a necessary and sufficient condition for index-transitivity, which is useful to prove the index-transitivity of a term rewriting systems. I think that our conditions are still strong for the normalizability of the E-strategy. A future work is to weaken our conditions.

Bibliography

[1] S. Antoy and A. Middeldorp, A Sequential Reduction Strategy, *Theoretical Computer Science* **165**(1), pp.75-95, 1996.

[2] F. Baader and T. Nipkow, Term Rewriting and All That, Cambridge University Press, 1998.

[3] H. Comon, Sequentiality, Second Order Monadic Logic and Tree Automata, *Proceedings of the 10th IEEE Symposium on Logic in Computer Science*, pp.508-517, 1995.

[4] H. Comon, M. Dauchet, R. Gilleron, D. Lugies, S. Tison, and M. Tommasi, Tree Automata Techniques and Applications, Preliminary Version, `http://www.grappa.univ-lille3.fr/tata/`, 1997.

[5] M. Dauchet, T. Heuillard, P. Lescanne, and S. Tison, Decidability of the Confluence of Finite Ground Term Rewriting Systems and of Other Related Term Rewriting Systems, *Information and Computation* **88**, pp.187-201, 1990.

[6] M. Dauchet and S. Tison, The Theory of Ground Rewrite Systems is Decidable, *Proceedings of the 5th IEEE Symposium on Logic in Computer Science*, pp.242-248, 1990.

[7] N. Dershowitz and J.-P. Jouannaud, Rewrite Systems, in *Handbook of Theoretical Computer Science*, Vol.B, ed. J. van Leeuwen, North-Holland, pp.243-320, 1990.

[8] I. Durand and A. Middeldorp, Decidable Call by Need Computations in Term Rewriting (Extended Abstract), *Proceedings, 14th International Conference on Automated Deduction*, *Lecture Notes in Artificial Intelligence* **1249**, pp.4-18,1997.

[9] K. Futatsugi, J.A. Goguen, J.-P. Jouannaud, and J. Meseguer, Principles of OBJ2, *Proceedings of 12th ACM Symposium on Principles of Programming Languages*, pp.52-66, 1985.

[10] F. Gécseg and M. Steinby, Tree Automata, Akadémiai Kiadó, Budapest, 1984.

[11] J.A. Goguen, T. Winkler, J. Meseguer, K. Futatsugi, and J.-P. Jouannaud, Introducing OBJ, Technical Reports SRI-CSL-92-03, SRI International, 1992.

[12] B. Gramlich, Abstract Relations between Restricted Termination and Confluence Properties of Rewrite Systems, *Fundamenta Informaticae* **24**, pp.3-23, 1995.

[13] G. Huet and J.-J. Lévy, Computations in Orthogonal Rewriting Systems, I and II, in *Computational Logic, Essays in Honor of Alan Robinson*, eds. J.-L. Lassez and G. Plotkin, MIT Press, pp.396-443, 1991. Previous version: Call by Need Computations in Non-Ambiguous Linear Term Rewriting Systems, Report 359, INRIA, 1979.

[14] F. Jacquemard, Decidable Approximations of Term Rewriting Systems, *Proceedings, 7th International Conference, Rewriting Techniques and Applications*, *Lecture Notes in Computer Science* **1103**, pp.362-376, 1996.

[15] J.-P. Jouannaud and W. Sadfi, Strong Sequentiality of Left-Linear Overlapping Rewrite Systems, *Proceedings, 4th International Workshop on Conditional Term Rewriting Systems*, *Lecture Notes in Computer Science* **968**, pp.235-246, 1995.

[16] J.R. Kennaway, Sequential Evaluation Strategies for Parallel-Or and Related Reduction System, *Annals of Pure and Applied Logic* **43**, pp.31-56, 1989.

[17] J.W. Klop, Term Rewriting Systems, in em Handbook of Logic in Computer Science, Vol.2, eds. S. Abramasky, D. Gabbay, and T. Maibaum, pp.1-116, Oxford University Press, 1992.

[18] J.W. Klop and A. Middeldorp, Sequentiality in Orthogonal Term Rewriting Systems, *Journal of Symbolic Computation* **12**, pp.161-195, 1991.

[19] A. Middeldorp, Call by Need Computations to Root-Stable Form, *Proceedings of the 24th Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, pp.94-105, 1997.

[20] T. Nagaya, M. Matsumoto, K. Ogata, and K. Futatsugi, How to Give Local Strategies to Function Symbols for Equality of Two Implementations of the E-Strategy with and without Evaluated Flags, *Proceedings of Asian Symposium on Computer Mathematics*, pp.71-81, Lanzhou University Press, 1998.

[21] T. Nagaya, M. Sakai, and Y. Toyama, NVNF-Sequentiality of Left-Linear Term Rewriting Systems, *Theory of Rewriting Systems and Its Applications*, *RIMS Kokyuroku* **918**, Kyoto University, pp.109-117, 1995.

[22] T. Nagaya, M. Sakai, and Y. Toyama, Index Reduction of Overlapping Strongly Sequential Systems, *IEICE Transactions on Information and Systems*, **E81-D**(5), pp.419-426,1998.

[23] T. Nagaya and Y. Toyama, Decidability for Left-Linear Growing Term Rewriting Systems, Research Report IS-RR-98-0031F, School of Information Science, JAIST, 1998. Revised version to appear in *Proceedings, 10th International Conference, Rewriting Techniques and Applications*, 1999.

[24] A. Nakagawa, T. Sawada, and K. Futatsugi, CafeOBJ User's Manual -ver.1.4-, `http://caraway.jaist.ac.jp/cafeobj/`, 1998.

[25] M.J. O'Donnell, Computing in Systems Described by Equations, *Lecture Notes in Computer Science* **58**, 1977.

[26] M. Oyamaguchi, The Reachability Problem for Quasi-Ground Term Rewriting Systems, *Journal of Information Processing* **9**, pp.232-236, 1986.

[27] M. Oyamaguchi, The Reachability and Joinability Problems for Right-Ground Term-Rewriting Systems, *Journal of Information Processing* **13**, pp.347-354, 1990.

[28] M. Oyamaguchi, NV-Sequentiality: A Decidable Condition for Call-by-Need Computations in Term Rewriting Systems, *SIAM Journal on Computation* **22**(1), pp.112-135, 1993.

[29] B.K. Rosen, Tree-Manipulating Systems and Church-Rosser Theorems, *Journal of the ACM* **20**(1), pp.160-187, 1973.

[30] Y. Toyama, Strong Sequentiality of Left-Linear Overlapping Term Rewriting Systems, *Proceedings of the 7th IEEE Symposium on Logic in Computer Science*, pp.274-284, 1992.

[31] Y. Toyama, S. Smetsers, M. van Eekelen and R. Plasmeijer, The Functional Strategy and Transitive Term Rewriting Systems, in *Term Graph Rewriting: Theory and Practice*, eds. R. Sleep, R. Plasmeijer and M. van Eekelen, pp.61-75, John Wiley & Sons Ltd, 1993.

Publications

[1] T. Nagaya, M. Sakai, and Y. Toyama, Decidability of Indices on NVNF-Sequential Term Rewriting Systems, *IEICE Technical Report* **COMP94**-63, Vol.94, No.354, pp.87-94, 1994 (*in Japanese*).

[2] T. Nagaya, M. Sakai, and Y. Toyama, NVNF-Sequentiality of Left-Linear Term Rewriting Systems, *Theory of Rewriting Systems and Its Applications*, *RIMS Kokyuroku* **918**, Kyoto University, pp.109-117, 1995.

[3] T. Nagaya, M. Sakai, and Y. Toyama, Index Reduction of Overlapping Strongly Sequential Systems, *IEICE Technical Report* **COMP96**-32, Vol.96, No.250, pp.39-48, 1996.

[4] T. Nagaya, M. Matsumoto, K. Ogata, and K. Futatsugi, The Equivalence of the Reductions with the E-Strategy with and without Marks, Research Report IS-RR-97-0034F, School of Information Science, JAIST, 1997.

[5] T. Nagaya, M. Sakai, and Y. Toyama, Index Reduction of Overlapping Strongly Sequential Systems, *IEICE Transactions on Information and Systems*, **E81-D**(5), pp.419-426,1998.

[6] T. Nagaya, M. Matsumoto, K. Ogata, and K. Futatsugi, How to Give Local Strategies to Function Symbols for Equality of Two Implementations of the E-Strategy with and without Evaluated Flags, *Proceedings of Asian Symposium on Computer Mathematics*, pp.71-81, Lanzhou University Press, 1998.

[7] T. Nagaya and Y. Toyama, Decidability for Left-Linear Growing Term Rewriting Systems, Research Report IS-RR-98-0031F, School of Information Science, JAIST, 1998.

[8] T. Nagaya and Y. Toyama, Decidability for Left-Linear Growing Term Rewriting Systems, *Proceedings, 10th International Conference, Rewriting Techniques and Applications*, 1999, to appear.