Confluent Term Rewriting Systems with Membership Conditions

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Abstract

We propose a new type of conditional term rewriting system: the membership-conditional term rewriting system, in which, each rewriting rule can have membership conditions which restrict the substitution values for the variables occurring in the rule. For example, the rule $f(x,x,y) \triangleright g(x,y)$ *if* $x \in T'$ yields the reduction $f(M,M,N) \rightarrow g(M,N)$ only when M is in the term set T'. Thus, by using membership-conditional rewriting, we can easily provide a strategy for term reduction. We study the confluence of membership-conditional term rewriting systems that are *nonterminating* and *nonlinear*. It is shown that a restricted *nonlinear* term rewriting system in which membership conditions satisfy the closure and termination properties is confluent if the system is nonoverlapping.

1. Introduction

Many term rewriting systems and their modifications are considered in logic, automated theorem proving, and programming language [2,3,4,6,8,9]. A fundamental property of term rewriting systems is the confluence property. A few sufficient criteria for the confluence are well known [2,3,4,5,8,9]. However, if a term rewriting system is nonterminating and nonlinear, we know few criteria for the confluence of the system [7,10].

In this paper, we study the confluence of membership-conditional term rewriting systems that are nonterminating and nonlinear. In a membership-conditional term rewriting system, the rewriting rule can have membership conditions.

We explain this concept with an example. We first consider a classical term rewriting system R that is nonterminating and nonlinear:

$$R \quad \begin{cases} f(x,x) \triangleright 0 \\ f(g(x),x) \triangleright 1 \\ 2 \triangleright g(2) \end{cases}$$

The diagram in Figure 1 illustrates that R is not confluent:

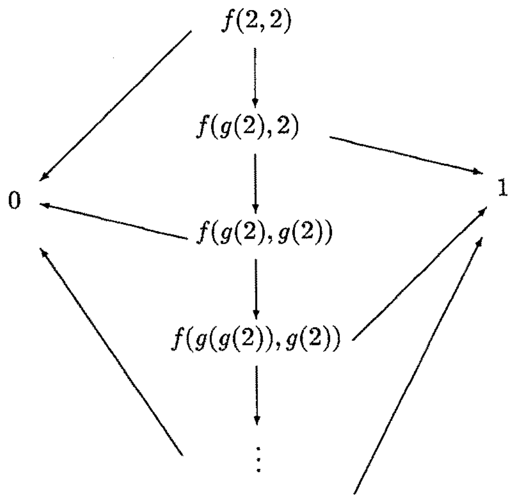

Figure 1

Now, let T' be the set of terms containing no constant symbol 2. By adding the membership condition $x \in T'$ to the first and second rules in R, we obtain the membership-conditional term rewriting system R':

$$R' \quad \begin{cases} f(x,x) \triangleright 0 \ \textit{if}\ x \in T' \\ f(g(x),x) \triangleright 1 \ \textit{if}\ x \in T' \\ 2 \triangleright g(2) \end{cases}$$

The membership condition $x \in T'$ restricts the substitution values for variable x; for example, the first rule $f(x,x) \triangleright 0$ *if* $x \in T'$ defines the reduction $f(M,M) \to 0$ only when $M \in T'$. Then, we can prove that R' is confluent (see Example 5.2 in Section 5), though it is nonterminating and nonlinear. Thus, by adding appropriate membership conditions, nonlinear systems can easily have the confluence property.

Our idea of membership-conditional rewriting was inspired by Church's δ-rule in λ-calculus [1,7]:

$$\delta_C \quad \begin{cases} \delta MM \triangleright \mathbf{T} \ \ \textit{if } M \textit{ is a closed normal form} \\ \delta MN \triangleright \mathbf{F} \ \ \textit{if } M,N \textit{ are closed normal forms and } M \not\equiv N. \end{cases}$$

It is well known that λ-calculus with δ_C is confluent [1,7]. However, if λ-calculus has Staples's δ-rule

$$\delta_S \quad \left\{ \delta MM \triangleright \epsilon \right.$$

instead of δ_C, then it is not confluent [1,7]. Thus, the membership conditions in δ_C (i.e., M,N *must be in the set of closed normal forms*) play an important role for the confluence of λ-calculus with nonlinear rules.

We will extend the idea of membership-conditional rewriting offered in Church's δ-rule to nonlinear term rewriting systems. Section 2 and Section 3 introduce preliminary concepts of reduction systems and of term rewriting systems respectively. In the next section, we present the concept of membership-conditional term rewriting systems. In Section 5, we discuss the sufficient criteria for the confluence of membership-conditional term rewriting systems that are nonterminating and nonlinear. We show that a restricted nonlinear system in which the membership conditions satisfy the closure and termination properties is confluent if the system is nonoverlapping.

2. Reduction Systems

We explain notions of reduction systems and give definitions for the following sections. Since these reduction systems have only an abstract structure, they are called abstract reduction systems [3,7] .

A reduction system is a structure $R = \langle A, \rightarrow \rangle$ consisting of some object set A and some binary relation $\rightarrow$ on A (i.e., $\rightarrow \subseteq A \times A$), called a reduction relation. A reduction (starting with x_0) in R is a finite or infinite sequence $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots$. The identity of elements of A (or syntactical equality) is denoted by $\equiv$. $\overset{*}{\rightarrow}$ is the transitive reflexive closure of $\rightarrow$ and $=$ is the equivalence relation generated by $\rightarrow$ (i.e., the transitive reflexive symmetric closure of $\rightarrow$). If $x \in A$ is minimal with respect to $\rightarrow$, i.e., $\neg \exists y \in A[x \rightarrow y]$, then we say that x is a normal form, or $\rightarrow$ normal form; let NF be the set of normal forms. If $x \overset{*}{\rightarrow} y$ and $y \in NF$ then we say x has a normal form y and y is a normal form of x. $x\downarrow$ indicates a normal form of x.

Definition. $R = \langle A, \rightarrow \rangle$ is terminating (or $\rightarrow$ is terminating), iff every reduction in R terminates, i.e., there is no infinite sequence $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots$.

Definition. $R = \langle A, \rightarrow \rangle$ is confluent (or $\rightarrow$ is confluent), iff
$\forall x, y, z \in A[x \overset{*}{\rightarrow} y \wedge x \overset{*}{\rightarrow} z \Rightarrow \exists w \in A, y \overset{*}{\rightarrow} w \wedge z \overset{*}{\rightarrow} w]$.

We express this property with the diagram in Figure 2. In this sort of diagram, dashed arrows denote (existential) reductions depending on the (universal) reductions shown by full arrows.

$$\begin{tikzcd} x \arrow[rr, "*"] \arrow[dd, "*"'] & & z \arrow[dd, dashed, "*"] \\ & & \\ y \arrow[rr, dashed, "*"] & & w \end{tikzcd}$$

Figure 2

Definition. $R = \langle A, \rightarrow \rangle$ is locally confluent (or $\rightarrow$ is locally confluent), iff $\forall x, y, z \in A[x \rightarrow y \wedge x \rightarrow z \Rightarrow \exists w \in A, y \xrightarrow{*} w \wedge z \xrightarrow{*} w]$.

The following propositions are well known [1,3,7] .

Proposition 2.1. Let R is confluent, then,

(1) $\forall x, y \in A[x = y \Rightarrow \exists w \in A, x \xrightarrow{*} w \wedge y \xrightarrow{*} w]$,

(2) $\forall x, y \in NF[x = y \Rightarrow x \equiv y]$,

(3) $\forall x \in A \forall y \in NF[x = y \Rightarrow x \xrightarrow{*} y]$.

Proposition 2.2. Let R be terminating and locally confluent, then R is confluent.

3. Term Rewriting Systems

Term rewriting systems are reduction systems having a term set as an object set A. Assuming that the reader is familiar with the basic concepts concerning term rewriting systems, we briefly summarize the important notions below [3,4].

Let F be an enumerable set of function symbols denoted by $f, g, h, \cdots$, and let V be an enumerable set of variable symbols denoted by $x, y, z, \cdots$ where $F \cap V = \phi$. By $T(F, V)$, we denote the set of terms constructed from F and V. If V is empty, $T(F, V)$, denoted as $T(F)$, is the set of ground terms. A term set is sometimes denoted by T.

A substitution θ is a mapping from a term set $T(F, V)$ to $T(F, V)$ such that for term M, $\theta(M)$ is completely determined by its values on the variable symbols occurring in M. Following common usage, we write this as $M\theta$ instead of $\theta(M)$.

Consider an extra constant $\square$ called a hole and the set $T(F \cup \{\square\}, V)$. Then $C \in T(F \cup \{\square\}, V)$ is called a context on F. We use the notation $C[\ ,\dots,\ ]$ for the context containing n holes ($n \geq 0$), and if $N_1, \dots, N_n \in T(F, V)$, then $C[N_1, \dots, N_n]$ denotes the result of placing $N_1, \dots, N_n$ in the holes of $C[\ ,\dots,\ ]$ from left to right. In particular, $C[\ ]$ denotes a context containing precisely one hole.

N is called a subterm of $M \equiv C[N]$. Let N be a subterm occurrence of M; then, we write $N \subseteq M$, and if $N \not\equiv M$, then we write $N \subset M$. If N_1 and N_2 are subterm occurrences of M having no common symbol occurrences (i.e., $M \equiv C[N_1, N_2]$), then N_1, N_2 are called disjoint (denoted by $N_1 \perp N_2$).

A rewriting rule on T is a pair $\langle M_l, M_r \rangle$ of terms in T such that $M_l \notin V$ and any variable in M_r also occurs in M_l. The notation $\triangleright$ denotes a set of rewriting rules on T and we write $M_l \triangleright M_r$ for $\langle M_l, M_r \rangle \in \triangleright$. A $\rightarrow$redex, or redex, is a term $M_1\theta$, where $M_l \triangleright M_r$, and in this case $M_r\theta$ is called a $\rightarrow$contractum, of $M_l\theta$. The set $\triangleright$ of rewriting rules on T defines a reduction relation $\rightarrow$ on T as follows:

$$M \rightarrow N \text{ iff } M \equiv C[M_l\theta],\ N \equiv C[M_r\theta], \text{ and } M_l \triangleright M_r \\ \text{for some } M_l, M_r, C[\ ], \text{ and } \theta.$$

When we want to specify the redex occurrence $\Delta \equiv M_l\theta$ of M in this reduction, write $M \xrightarrow{\Delta} N$.

Definition. A term rewriting system R on T is a reduction system $R = \langle T, \rightarrow \rangle$ such that the reduction relation $\rightarrow$ is defined by a set $\triangleright$ of rewriting rules on T. If R has $M_l \triangleright M_r$, then we write $M_l \triangleright M_r \in R$.

If every variable in term M occurs only once, then M is called linear. We say that R is left-linear (or linear) iff for any $M_l \triangleright M_r \in R$, M_l is linear. R is called nonlinear if R is not left-linear.

Let $M \triangleright N$ and $P \triangleright Q$ be two rules in R. We assume that we have renamed the variables appropriately, so that M and P share no variables. Assume $S \notin V$ is a subterm occurrence in M, i.e., $M \equiv C[S]$, such that S and P are unifiable, i.e., $S\theta \equiv P\theta$, with a minimal unifier θ [3,8] . Since $M\theta \equiv C[S]\theta \equiv C\theta[P\theta]$, two reductions starting with $M\theta$, i.e., $M\theta \rightarrow C\theta[Q\theta] \equiv C[Q]\theta$ and $M\theta \rightarrow N\theta$, can be obtained by using $P \triangleright Q$ and $M \triangleright N$. Then we say that $P \triangleright Q$ and $M \triangleright N$ are overlapping, and that the pair $\langle C[Q]\theta, N\theta \rangle$ of terms is critical in R [3,4] . We may choose $M \triangleright N$ and $P \triangleright Q$ to be the same rule, but in this case we shall not consider the case $S \equiv M$, which gives the trivial pair $\langle N, N \rangle$. If R has no critical pair, then we say that R is nonoverlapping [3,4,8,10] .

The following sufficient conditions for the confluence of R are well known [3,4,8,9] .

Proposition 3.1. Let R be terminating, and let P and Q have the same normal form for any critical pair $\langle P, Q \rangle$ in R. Then R is confluent.

Proposition 3.2. Let R be left-linear and nonoverlapping. Then R is confluent.

For more discussions concerning the confluence of term rewriting systems having overlapping or nonlinear rules, see [3,7,10] .

4. Membership-Conditional Rewriting

In this section, we propose membership-conditional term rewriting systems. A membership-conditional term rewriting system R on T is a term rewriting system on T in which the rewriting rule $M_l \triangleright M_r$ can have the membership conditions $x \in T', y \in T'', \cdots, z \in T'''$. Here, $T', T'', \cdots, T'''$ are any subsets of T.

The membership-conditional rewriting rule is denoted by

$$M_l \triangleright M_r \ \mathit{if}\ x \in T', y \in T'' \cdots, z \in T'''.$$

The conditions $x \in T', y \in T'' \cdots, z \in T'''$ restrict the substitution's values on the variables $x, y, \cdots, z$ occurring in the rule $M_l \triangleright M_r$. Thus, the rlue $M_l \triangleright M_r \ \mathit{if}\ x \in T', y \in T'' \cdots, z \in T'''$ defines the reduction $M \to N$ only when $M \equiv C[M_l\theta]$, $N \equiv C[M_r\theta]$ for some $C[\,]$ and some θ such that $x\theta \in T', y\theta \in T'', \cdots, z\theta \in T'''$.

Example 4.1. Let $F = \{+, d, s, 0\}$ and $F' = \{+, s, 0\}$. Consider the membership-conditional term rewriting system R on $T(F, V)$ which computes the addition and the double function $d(n) = n + n$ on the set $\mathbf{N}$ of natural numbers represented by $0, s(0), s(s(0)), \ldots$:

$$R \quad \begin{cases} x + 0 \triangleright x \\ x + s(y) \triangleright s(x + y) \\ d(x) \triangleright x + x \ \mathit{if}\ x \in T(F') \end{cases}$$

Then we have the following reduction:

$$d(d(0)) \to d(0 + 0) \to (0 + 0) + (0 + 0) \xrightarrow{*} 0.$$

Note that $d(d(0))$ cannot directly contract into $d(0) + d(0)$ with the third rule in R since $d(0) \notin T(F')$. □

Example 4.2. Let $F = \{-, s, 0\}$. Consider the membership-conditional term rewriting system R on $T(F, V)$ computing the subtraction on the set $\mathbf{N}$:

$$R \quad \begin{cases} x - 0 \triangleright x \ \mathit{if}\ x \in NF \\ s(x) - s(y) \triangleright x - y \ \mathit{if}\ x, y \in NF \\ x - x \triangleright 0 \ \mathit{if}\ x \in NF \end{cases}$$

Then, R contracts only the innermost redex occurrences in a term since the membership conditions prohibit to contract the other redex occurrences. Thus, by using the membership conditions we can explicitly provide the innermost reduction strategy for term rewriting systems. □

Example 4.3. Let $F = \{f\}$. The following membership-conditional term rewriting system R on $T(F, V)$ appears to be paradoxical:

$$R \quad \left\{ \; f(x) \rhd x \;\; if \; x \in \{N | f(N) \in NF\} \right.$$

However, R is not paradoxical against our expectation. It can be easily proven that $\{N|f(N) \in NF\}$ is empty. Thus R is equal to the term rewriting system with no rule. □

Remark. A conditional rule $M_l \rhd M_r$ *if* $P(x)$, where $P(x)$ is some predicate of the variable x, can be translated into a membership-conditional rule $M_l \rhd M_r$ *if* $x \in T$ where $T = \{N| \; P(N)\}$. Conversely, taking $P(x) \equiv x \in T$, we can also translate a membership-conditional rule $M_l \rhd M_r$ *if* $x \in T$ into a conditional rule $M_l \rhd M_r$ *if* $P(x)$. Thus conditional rules of the form

$$M_l \rhd M_r \;\; if \; P'(x) \wedge P''(y) \wedge \cdots \wedge P'''(z)$$

are essentially equal to membership-conditional rules of the form

$$M_l \rhd M_r \;\; if \; x \in T', y \in T'' \cdots, z \in T'''.$$

Hence a membership-conditional term rewriting system can be regarded as a conditional term rewriting system in which every condition $P(x, y, \cdots, z)$ can be translated into a condition $P'(x) \wedge P''(y) \wedge \cdots \wedge P'''(z)$ with separated variables.

5. Confluence of Membership Rewriting

It is well known that if a term rewriting system is terminating, the confluence can be easily proven by the critical pair lemma [3,4,8]. However, if a term rewriting system is nonterminating, it is difficult to prove the confluence of the system. In particular, a system that is nonterminating and nonlinear gives few results to prove the confluence [5,10].

In this section, we study the confluence of membership-conditional term rewriting systems without assuming the termination or the linearity. Our key idea to prove the confluence comes from the observation that with appropriate membership conditions, nonlinear systems behave like left-linear systems.

Definition. A restricted nonlinear rule is a membership-conditional rewriting rule in which the nonlinear variables on the left side of the rule must have membership conditions. For the other variables, membership conditions are optional. We say that R is restricted nonlinear iff every rule in R is restricted nonlinear.

For example, the restricted nonlinear rule $f(x, x, y) \rhd g(x, y, y)$ *if* $x \in T'$ has nonlinear variable x on the left side $f(x, x, y)$. Hence, variable x must have the membership condition $x \in T'$. However, variable y on the left side is linear, thus, membership condition for y is not necessary.

A classical left-linear term-rewriting system is obviously a restricted nonlinear system, because the left-linear system has only linear variables on the left side of the rewriting rules. Thus, the restricted nonlinear system is a natural extension of the classical left-linear system. Indeed, the sufficient criteria for the confluence of restricted nonlinear systems are very similar to that of the classical left-linear systems.

Overlapping between two conditional rewriting rules can be defined in the same way as for two classical rewriting rules except that the substitution must satisfy the membership conditions in the rules. Then, Proposition 3.2 for the confluence of the classical left-linear systems can be extended to the following theorem.

Theorem 5.1. Let a membership-conditional term rewriting system R be nonoverlapping and restricted nonlinear. If every term set T' in the membership conditions is a set of normal forms, i.e., $T' \subseteq NF$, then R is confluent.

Proof. Since nonlinear variables on the left side of the rewriting rules must have normal forms as the substitution's values, the nonlinear variables can be ignored when we treat a sufficient criterion for the confluence. Thus, the confluence of R can be easily proven in the same way as for the classical left-linear and nonoverlapping systems: See the proof in [3,9] of Proposition 3.2. □

Example 5.1. Consider the membership-conditional term rewriting system R:

$$R \quad \begin{cases} f(x,x) \rhd 0 \ \textit{if } x \in NF \\ f(g(x),x) \rhd 1 \ \textit{if } x \in NF \\ 2 \rhd g(2) \end{cases}$$

Note that R is nonterminating and nonlinear. Clearly, R satisfies the conditions in Theorem 5.1. Thus, R is confluent. □

In Theorem 5.1, every set T' in the membership conditions must be a set of normal forms. We are now going to relax this restriction on the membership conditions.

Let T' be a subset of the term set T. We say that T' is closed iff $\forall M \in T' \ \forall N \in T[M \to N \Rightarrow N \in T']$. We say that T' is terminating iff every $M \in T'$ has no infinite reduction $M \to\to\to \cdots$.

Let a membership-conditional term rewriting system $R = \langle T, \to \rangle$ be nonoverlapping and restricted nonlinear, and let every term set T' in the membership conditions be closed and terminating. From now on we will prove the confluence of R. Note that if T' is a set of normal forms, then clearly T' is closed and terminating. Thus, the system in Theorem 5.1 is a particular case of this system.

Lemma 5.1. R is locally confluent.

Proof. It is obvious since R is nonoverlapping and every term set T' in the membership conditions is closed. □

Let $T', \cdots, T'''$ be all the term sets occurring in membership conditions of R. Let $S = \{N \mid N \subseteq M \ \text{ for some } M \in T' \cup \cdots \cup T'''\}$. Note that if $M \in S$ and $N \subseteq M$ then N is also in S.

Lemma 5.2. S is closed and terminating.

Proof. It is trivial from the definitions of closed and terminating. □

By using the above term set S, We define a new reduction relation $\underset{S}{\to}$ on T as follows:

$M \underset{S}{\to} N$ iff $M \equiv C[P]$, $N \equiv C[Q]$, and $P \to Q$
for some $C[\ ]$ and some $P, Q \in S$.

Then P is called $\underset{S}{\to}$ redex. Note that $\underset{S}{\to} \subseteq \to$.

Lemma 5.3. $\underset{S}{\to}$ is terminating.

Proof. By the closure and termination properties of S, it can be easily proven. □

Lemma 5.4. $\underset{S}{\to}$ is confluent.

Proof. By the locally confluence of R and the closure property of S we obtain the locally confluence of $\underset{S}{\to}$. By Lemma 5.3, $\underset{S}{\to}$ is terminating. Thus, from Proposition 2.2 it follows that $\underset{S}{\to}$ is confluent. □

For every term set T' in the membership conditions, we consider the normalized term set $T'_{nf} = \{M\downarrow \mid M \in T'\}$ where $M\downarrow$ denotes the normal form of M. Note that from the closure and termination properties of T', T'_{nf} is definable and $T'_{nf} \subseteq T'$. Then the normalized membership-conditional system R_{nf} is defined by replacing each rewriting rule
$M_l \triangleright M_r$ *if* $x \in T', \cdots, z \in T''$ in R with
$M_l \triangleright M_r$ *if* $x \in T'_{nf}, \cdots, z \in T''_{nf}$. Note that from Theorem 5.1, the confluence of R_{nf} follows. $\underset{nf}{\longrightarrow}$ denotes the reduction relation of R_{nf}. It is trivial that $\underset{nf}{\longrightarrow} \subseteq \to$.

Lemma 5.5. We have the diagram in Figure 3.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
M \arrow[r, "*", "S"'] \arrow[d, "*"', "S"] & P \arrow[d, dashed, "S"', "*"] \\
N \arrow[r, dashed, "*", "nf"'] & Q
\end{tikzcd}$$

Figure 3

Proof. Take $Q \equiv M\downarrow_S$. Here, $M\downarrow_S$ indicates the $\underset{S}{\rightarrow}$ normal form of M. By the confluence of $\underset{S}{\rightarrow}$ we obtain $N \overset{*}{\underset{S}{\rightarrow}} Q$ and $P \overset{*}{\underset{S}{\rightarrow}} Q$. Since $\underset{S}{\rightarrow}$ is terminating, we can reduce N into Q by rewriting only innermost $\underset{S}{\rightarrow}$ redex occurrences (i.e., innermost $\underset{S}{\rightarrow}$ reduction strategy). Now, consider two $\rightarrow$ redex occurrences Δ and Δ' in a term such that $\Delta' \subset \Delta$. From the definition of $\underset{S}{\rightarrow}$, if Δ is a $\underset{S}{\rightarrow}$ redex occurrence then Δ' is so. Hence, every innermost $\underset{S}{\rightarrow}$ redex occurrence is an innermost $\rightarrow$ redex occurrence; we obtain a reduction $N \overset{*}{\rightarrow} Q$ by rewriting only innermost $\rightarrow$ redex occurrences. By tracing the innermost reduction by R_{nf}, $N \overset{*}{\underset{nf}{\rightarrow}} Q$ follows. □

Lemma 5.6. We have the diagram in Figure 4.

$$\begin{tikzcd}[column sep=large, row sep=huge]
M \arrow[rr, "\Delta'"] \arrow[d, "\Delta"', "S"] & & P \arrow[d, dashed, "S"', "*"] \\
N \arrow[r, dashed, "*", "S"'] & Q' \arrow[r, dashed] & Q
\end{tikzcd}$$

Figure 4

Proof. If $M \underset{S}{\rightarrow} P$, then it is immediate from the confluence of $\underset{S}{\rightarrow}$. Hence, suppose not $M \underset{S}{\rightarrow} P$. Then, $\Delta' \not\subseteq \Delta$ since Δ' is not a $\underset{S}{\rightarrow}$ redex.

Case 1. $\Delta \perp \Delta'$. It is trivial.

Case 2. $\Delta \subset \Delta'$. Since R is nonoverlapping and S is closed, we can apply the rewriting rule used in $M \to P$ to $Q' \to Q$ in the same way after adjusting the nonlinear parts with $N \xrightarrow[S]{*} Q'$. Then $P \xrightarrow[S]{*} Q$ follows. □

Lemma 5.7. We have the diagram in Figure 5.

$$
\begin{tikzcd}
M \arrow[r] \arrow[d, "S", "*"'] & P \arrow[d, dashed, "S"', "*"] \\
N \arrow[r, dashed, "*", "nf"'] & Q
\end{tikzcd}
$$

Figure 5

Proof. We will prove the lemma by induction on the maximal length $d(M)$ of the $\xrightarrow[S]{}$ reductions starting at M. The case $d(M) = 0$ is trivial from the definition of R_{nf}. Assume the lemma for $d(M) < k$. Then, we can show the diagram in Figure 6 for the case $d(M) = k$, proving diagram(1) by Lemma 5.6, diagram(2) by Lemma 5.5, diagram(3) by the induction hypothesis for the lemma since $d(M') < d(M)$. □

$$
\begin{tikzcd}
M \arrow[rr] \arrow[d, "S"] \arrow[drr, phantom, "(1)"] & & P \arrow[d, dashed, "S", "*"'] \\
{} \arrow[r, dashed, "*", "S"'] \arrow[d, "S", "*"'] \arrow[dr, phantom, "(2)"] & M' \arrow[r, dashed] \arrow[d, dashed, "S", "*"'] \arrow[dr, phantom, "(3)"] & {} \arrow[d, dashed, "S", "*"'] \\
N \arrow[r, dashed, "*", "nf"'] & {} \arrow[r, dashed, "*", "nf"'] & Q
\end{tikzcd}
$$

Figure 6

Lemma 5.8. We have the diagram in Figure 7.

$$
\begin{tikzcd}
M \arrow[r, "*"] \arrow[d, "*"', "S"] & P \arrow[d, dashed, "S"', "*"] \\
N \arrow[r, dashed, "*", "nf"'] & Q
\end{tikzcd}
$$

Figure 7

Proof. Using Lemma 5.7, the diagram in Figure 8 can be made. □

$$
\begin{tikzcd}
M \arrow[r] \arrow[d, "*"', "S"] & \cdot \arrow[r] \arrow[d, dashed, "*"', "S"] & \cdot \arrow[r] \arrow[d, dashed, "*"', "S"] & \cdot \arrow[r] \arrow[d, dashed, "*"', "S"] & P \arrow[d, dashed, "*"', "S"] \\
N \arrow[r, dashed, "*", "nf"'] & \cdot \arrow[r, dashed, "*", "nf"'] & \cdot \arrow[r, dashed, "*", "nf"'] & \cdot \arrow[r, dashed, "*", "nf"'] & Q
\end{tikzcd}
$$

Figure 8

Theorem 5.2. Let a membership-conditional term rewriting system R be nonoverlapping and restricted nonlinear. If every term set T' in the membership conditions is closed and terminating, then R is confluent.

Proof. The diagram in Figure 9 can be obtained, proving diagram(1) by Lemma 5.8, diagram(2) by the confluence of R_{nf}. From $\xrightarrow[S]{} \subseteq \rightarrow$ and $\xrightarrow[nf]{} \subseteq \rightarrow$, the confluence of R follows. □

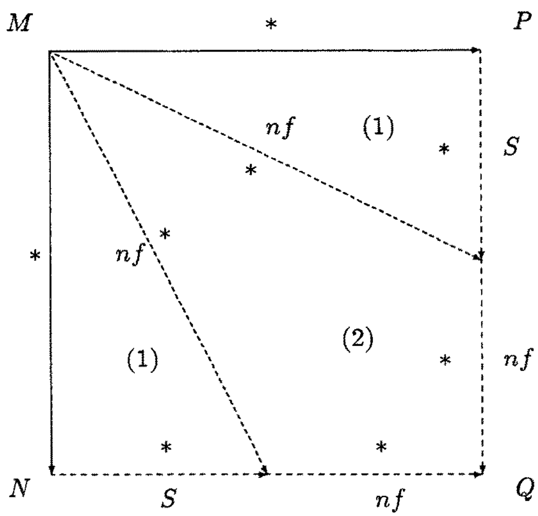

Figure 9

Example 5.2. Let $F' = \{f, g, 0, 1\}$. Consider the membership conditional term rewriting system R:

$$R \quad \begin{cases} f(x,x) \triangleright 0 \ \textit{if } x \in T(F', V) \\ f(g(x), x) \triangleright 1 \ \textit{if } x \in T(F', V) \\ 2 \triangleright g(2) \end{cases}$$

It is clear that R is nonoverlapping and restricted nonlinear. Since $T(F', V)$ is closed and terminating, from Theorem 5.2 it follows that R is confluent. □

6. Conclusion

In this paper, we have proposed a new conditional term rewriting system: the membership-conditional term rewriting system. We have shown the sufficient criteria for the confluence of the system under the restricted nonlinear condition.

Many directions for further research come easily to mind. One direction is application to many-sorted systems. Membership-conditional systems can provide a very useful means of constructing hierarchical many-sorted systems.

Application to functional programs is another very interesting direction. Membership-conditional systems can explicitly provide reduction strategy, such as innermost reduction. Hence, using this property, we can offer effective computation for functional programs.

We believe that further research in these directions will exploit the potential of membership-conditional rewriting techniques.

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