2019年度 数理論理学 復習問題(16)

問題 1 シグニチャを $L = \langle \{\mathtt{a}^0, \mathtt{f}^1, \mathtt{g}^2\}, \emptyset \rangle$ とし，$\mathbb{N}$ 上の定数および関数として $\mathtt{a}^{\mathcal{M}} = 0$，$\mathtt{f}^{\mathcal{M}}(x) = x \times 2$，$\mathtt{g}^{\mathcal{M}}(x, y) = x^y$ を考える．L-構造 $\mathcal{M} = \langle \mathbb{N}, \mathtt{a}^{\mathcal{M}}, \mathtt{f}^{\mathcal{M}}, \mathtt{g}^{\mathcal{M}} \rangle$ と，$v(\mathtt{x}) = 2$，$v(\mathtt{y}) = 3$ なる付値 v を考えるとき，以下の値を答えよ．

(1) $[\![\mathtt{a}]\!]_v$　(2) $[\![\mathtt{f}(\mathtt{x})]\!]_v$　(3) $[\![\mathtt{g}(\mathtt{f}(\mathtt{x}), \mathtt{y})]\!]_v$

問題 2 シグニチャを $L = \langle \emptyset, \{<^2\} \rangle$ とし，$M = \{a, b, c\}$, $<^{\mathcal{M}} = \{\langle a, b \rangle, \langle c, b \rangle\}$ とおくとき，L-構造 $\mathcal{M} = \langle M, <^{\mathcal{M}} \rangle$ を考える．このとき，以下の値を答えよ．

(1) $[\![\mathtt{y} < \mathtt{x}]\!]_{v[a/\mathtt{x}][b/\mathtt{y}]}$　(2) $[\![\mathtt{y} < \mathtt{x}]\!]_{v[b/\mathtt{x}][a/\mathtt{y}]}$　(3) $[\![\mathtt{x} \approx \mathtt{y}]\!]_{v[a/\mathtt{x}][a/\mathtt{y}]}$

問題 3 シグニチャを $L = \langle \{\mathtt{f}^2\}, \{\mathtt{P}^1\} \rangle$，$L$-構造 $\mathcal{M}$ を $\mathcal{M} = \langle \{-1, 0, 1\}, \mathtt{f}^{\mathcal{M}}, \mathtt{P}^{\mathcal{M}} \rangle$，$\mathtt{f}^{\mathcal{M}}(x, y) = x \times y$，$\mathtt{P}^{\mathcal{M}} = \{1\}$ と定める．v を任意の付値とするとき，以下の値を求めよ．

(1) $[\![\mathtt{P}(\mathtt{x})]\!]_{v[1/\mathtt{x}]}$　(2) $[\![\mathtt{f}(\mathtt{x}, \mathtt{y})]\!]_{v[-1/\mathtt{x}][0/\mathtt{y}]}$　(3) $[\![\mathtt{P}(\mathtt{f}(\mathtt{x}, \mathtt{y}))]\!]_{v[1/\mathtt{x}][-1/\mathtt{y}]}$

問題 4 シグニチャを $L = \langle \{\mathtt{f}^1\}, \{\mathtt{P}^2\} \rangle$ とするとき，L-構造 $\mathcal{M}$ を $\mathcal{M} = \langle \mathbb{N}, \mathtt{f}^{\mathcal{M}}, \mathtt{P}^{\mathcal{M}} \rangle$，$\mathtt{f}^{\mathcal{M}}(x) = x + 1$，$\mathtt{P}^{\mathcal{M}} = \{\langle x, y \rangle \mid 2x + y \leq 7\}$ と定める．また，付値 v を $v(\mathtt{x}) = 2$，$v(\mathtt{y}) = 3$ により定める．このとき，以下の値を求めよ．

(1) $[\![\mathtt{f}(\mathtt{y})]\!]_v$　(2) $[\![\mathtt{f}(\mathtt{y})]\!]_{v[5/\mathtt{y}]}$　(3) $[\![\mathtt{P}(\mathtt{x}, \mathtt{y})]\!]_v$　(4) $[\![\mathtt{P}(\mathtt{x}, \mathtt{f}(\mathtt{y}))]\!]_v$

問題 5 $\mathcal{P}(X)$ は X の部分集合全体の集合を表わすものとする．シグニチャを $L = \langle \{\mathtt{f}^2\}, \{\mathtt{P}^2\} \rangle$ とするとき，L-構造 $\mathcal{M}$ を $\mathcal{M} = \langle \mathcal{P}(\{0, 1, 2\}), \mathtt{f}^{\mathcal{M}}, \mathtt{P}^{\mathcal{M}} \rangle$，$\mathtt{f}^{\mathcal{M}}(x, y) = x \cap y$，$\mathtt{P}^{\mathcal{M}} = \{\langle x, y \rangle \mid x \subseteq y\}$ と定める．このとき，以下の値を求めよ．

(1) $[\![\mathtt{f}(\mathtt{x}, \mathtt{y})]\!]_{v[\{0,1\}/\mathtt{x}][\{0,2\}/\mathtt{y}]}$　(2) $[\![\mathtt{P}(\mathtt{x}, \mathtt{y})]\!]_{v[\{0\}/\mathtt{x}][\{0,2\}/\mathtt{y}]}$　(3) $[\![\mathtt{P}(\mathtt{f}(\mathtt{x}, \mathtt{y}), \mathtt{x})]\!]_v$

問題 6 シグニチャを $L = \langle \{\emptyset, \{\mathtt{P}^2\} \rangle$ とするとき，L-構造 $\mathcal{M}_i = \langle |\mathcal{M}_i|, \mathtt{P}^{\mathcal{M}_i} \rangle$ $(1 \leq i \leq 3)$ を以下のように定める．

$\mathcal{M}_1 = \langle \mathbb{N}, \{\langle x, y \rangle \mid x < y\} \rangle$
$\mathcal{M}_2 = \langle \mathbb{N}, \{\langle x, y \rangle \mid x \times 3 = y \times 2\} \rangle$
$\mathcal{M}_3 = \langle \mathbb{N}, \{\langle x, y \rangle \mid x + y = 9\} \rangle$

このとき，$\mathcal{M}_1$，$\mathcal{M}_2$，$\mathcal{M}_3$ のそれぞれについて $[\![\mathtt{P}(\mathtt{x}, \mathtt{x})]\!]_v = \mathrm{T}$ となる付値 v はあるか？また，ある場合には 1 つ与えよ．

2019年度 数理論理学 復習問題解答 (16)

問題 1

(1) $[\![\mathtt{a}]\!]_v = \mathtt{a}^{\mathcal{M}} = 0$.

(2) $[\![\mathtt{f}(\mathtt{x})]\!]_v = [\![\mathtt{x}]\!]_v \times 2 = v(\mathtt{x}) \times 2 = 2 \times 2 = 4$

(3) $[\![\mathtt{g}(\mathtt{f}(\mathtt{x}),\mathtt{y})]\!]_v = ([\![\mathtt{f}(\mathtt{x})]\!]_v)^{[\![\mathtt{y}]\!]_v} = ([\![\mathtt{x}]\!]_v \times 2)^{[\![\mathtt{y}]\!]_v} = 4^3 = 64$.

問題 2

(1) $[\![\mathtt{y} < \mathtt{x}]\!]_{v[a/\mathtt{x}][b/\mathtt{y}]} = \mathrm{T} \Leftrightarrow \langle [\![\mathtt{y}]\!]_{v[a/\mathtt{x}][b/\mathtt{y}]}, [\![\mathtt{x}]\!]_{v[a/\mathtt{x}][b/\mathtt{y}]} \rangle \in <^{\mathcal{M}} \Leftrightarrow \langle b, a\rangle \in <^{\mathcal{M}}$ が成立する．よって，$\langle b, a\rangle \notin <^{\mathcal{M}}$ より，$[\![\mathtt{y} < \mathtt{x}]\!]_{v[a/\mathtt{x}][b/\mathtt{y}]} = \mathrm{F}$.

(2) (1) と同様にして，$[\![\mathtt{y} < \mathtt{x}]\!]_{v[b/\mathtt{x}][a/\mathtt{y}]} = \mathrm{T} \Leftrightarrow \langle a, b\rangle \in <^{\mathcal{M}}$ が成立する．よって，$\langle a, b\rangle \in <^{\mathcal{M}}$ より，$[\![\mathtt{y} < \mathtt{x}]\!]_{v[a/\mathtt{x}][b/\mathtt{y}]} = \mathrm{T}$.

(3) $[\![\mathtt{x} \approx \mathtt{y}]\!]_{v[a/\mathtt{x}][a/\mathtt{y}]} = \mathrm{T} \Leftrightarrow [\![\mathtt{x}]\!]_{v[a/\mathtt{x}][a/\mathtt{y}]} = [\![\mathtt{y}]\!]_{v[a/\mathtt{x}][a/\mathtt{y}]}$ が成立する．よって，$[\![\mathtt{x}]\!]_{v[a/\mathtt{x}][a/\mathtt{y}]} = a = [\![\mathtt{y}]\!]_{v[a/\mathtt{x}][a/\mathtt{y}]}$ より，$[\![\mathtt{x} \approx \mathtt{y}]\!]_{v[a/\mathtt{x}][a/\mathtt{y}]} = \mathrm{T}$.

問題 3

(1) $[\![\mathtt{P}(\mathtt{x})]\!]_{v[1/\mathtt{x}]} = \mathrm{T} \Leftrightarrow [\![\mathtt{x}]\!]_{v[1/\mathtt{x}]} \in \mathtt{P}^{\mathcal{M}} \Leftrightarrow 1 \in \mathtt{P}^{\mathcal{M}} = \{1\}$．よって，$[\![\mathtt{P}(\mathtt{x})]\!]_{v[1/\mathtt{x}]} = \mathrm{T}$.

(2) $[\![\mathtt{f}(\mathtt{x},\mathtt{y})]\!]_{v[-1/\mathtt{x}][0/\mathtt{y}]} = \mathtt{f}^{\mathcal{M}}([\![\mathtt{x}]\!]_{v[-1/\mathtt{x}][0/\mathtt{y}]}, [\![\mathtt{y}]\!]_{v[-1/\mathtt{x}][0/\mathtt{y}]}) = \mathtt{f}^{\mathcal{M}}(-1, 0) = -1 \times 0 = 0$.

(3) $[\![\mathtt{P}(\mathtt{f}(\mathtt{x},\mathtt{y}))]\!]_{v[1/\mathtt{x}][-1/\mathtt{y}]} = \mathrm{T} \Leftrightarrow [\![\mathtt{f}(\mathtt{x},\mathtt{y})]\!]_{v[1/\mathtt{x}][-1/\mathtt{y}]} \in \mathtt{P}^{\mathcal{M}} \Leftrightarrow \mathtt{f}^{\mathcal{M}}(1,-1) \in \mathtt{P}^{\mathcal{M}} \Leftrightarrow -1 \in \{1\}$．よって，$[\![\mathtt{P}(\mathtt{f}(\mathtt{x},\mathtt{y}))]\!]_{v[1/\mathtt{x}][-1/\mathtt{y}]} = \mathrm{F}$.

問題 4

(1) $[\![\mathtt{f}(\mathtt{y})]\!]_v = \mathtt{f}^{\mathcal{M}}([\![\mathtt{y}]\!]_v) = [\![\mathtt{y}]\!]_v + 1 = v(\mathtt{y}) + 1 = 3 + 1 = 4$.

(2) $[\![\mathtt{f}(\mathtt{y})]\!]_{v[5/\mathtt{y}]} = \mathtt{f}^{\mathcal{M}}([\![\mathtt{y}]\!]_{v[5/\mathtt{y}]}) = [\![\mathtt{y}]\!]_{v[5/\mathtt{y}]} + 1 = v[5/\mathtt{y}](\mathtt{y}) + 1 = 5 + 1 = 6$.

(3) $[\![\mathtt{P}(\mathtt{x},\mathtt{y})]\!]_v = \mathrm{T} \Leftrightarrow \langle [\![\mathtt{x}]\!]_v, [\![\mathtt{y}]\!]_v\rangle \in \mathtt{P}^{\mathcal{M}} \Leftrightarrow \langle 2,3\rangle \in \mathtt{P}^{\mathcal{M}} \Leftrightarrow \langle 2,3\rangle \in \{\langle x,y\rangle \mid 2x+y \leq 7\} \Leftrightarrow 2\times 2+3 \leq 7 \Leftrightarrow 7 \leq 7$．よって，$[\![\mathtt{P}(\mathtt{x},\mathtt{y})]\!]_v = \mathrm{T}$.

(4) (3) と同様に，$[\![\mathtt{P}(\mathtt{x},\mathtt{f}(\mathtt{y})]\!]_v = \mathrm{T} \Leftrightarrow \langle [\![\mathtt{x}]\!]_v, [\![\mathtt{f}(\mathtt{y})]\!]_v\rangle \in \mathtt{P}^{\mathcal{M}} \Leftrightarrow \langle 2, \mathtt{f}^{\mathcal{M}}([\![\mathtt{y}]\!]_v)\rangle \in \mathtt{P}^{\mathcal{M}} \Leftrightarrow \langle 2, \mathtt{f}^{\mathcal{M}}(3)\rangle \in \mathtt{P}^{\mathcal{M}} \Leftrightarrow \langle 2,4\rangle \in \mathtt{P}^{\mathcal{M}} \Leftrightarrow \langle 2,4\rangle \in \{\langle x,y\rangle \mid 2x+y \leq 7\} \Leftrightarrow 2\times 2+4 \leq 7 \Leftrightarrow 8 \leq 7$．よって，$[\![\mathtt{P}(\mathtt{x},\mathtt{y})]\!]_v = \mathrm{F}$.

問題 5

(1) $[\![\mathtt{f}(\mathtt{x},\mathtt{y})]\!]_{v[\{0,1\}/\mathtt{x}][\{0,2\}/\mathtt{y}]} = \mathtt{f}^{\mathcal{M}}([\![\mathtt{x}]\!]_{v[\{0,1\}/\mathtt{x}][\{0,2\}/\mathtt{y}]}, [\![\mathtt{y}]\!]_{v[\{0,1\}/\mathtt{x}][\{0,2\}/\mathtt{y}]}) = \mathtt{f}^{\mathcal{M}}(\{0,1\},\{0,2\}) = \{0,1\} \cap \{0,2\} = \{0\}$.

(2) $[\![\mathtt{P}(\mathtt{x},\mathtt{y})]\!]_{v[\{0\}/\mathtt{x}][\{0,2\}/\mathtt{y}]} = \mathrm{T} \Leftrightarrow \langle [\![\mathtt{x}]\!]_{v[\{0\}/\mathtt{x}][\{0,2\}/\mathtt{y}]}, [\![\mathtt{y}]\!]_{v[\{0\}/\mathtt{x}][\{0,2\}/\mathtt{y}]}\rangle \in \mathtt{P}^{\mathcal{M}} \Leftrightarrow \langle \{0\}, \{0,2\}\rangle \in \mathtt{P}^{\mathcal{M}} \Leftrightarrow \{0\} \subseteq \{0,2\}$．よって，$[\![\mathtt{P}(\mathtt{x},\mathtt{y})]\!]_{v[\{0\}/\mathtt{x}][\{0,2\}/\mathtt{y}]} = \mathrm{T}$.

(3) $[\![\mathtt{P}(\mathtt{f}(\mathtt{x},\mathtt{y}),\mathtt{x})]\!]_v \Leftrightarrow \langle [\![\mathtt{f}(\mathtt{x},\mathtt{y})]\!]_v, [\![\mathtt{x}]\!]_v\rangle \in \mathtt{P}^{\mathcal{M}} \Leftrightarrow \langle v(\mathtt{x}) \cap v(\mathtt{y}), v(\mathtt{x})\rangle \in \mathtt{P}^{\mathcal{M}} \Leftrightarrow v(\mathtt{x}) \cap v(\mathtt{y}) \subseteq v(\mathtt{x})$．ここで，$v(\mathtt{x}) \cap v(\mathtt{y}) \subseteq v(\mathtt{x})$ は，集合の性質よりどのような付値 v についても成立するから，$[\![\mathtt{P}(\mathtt{f}(\mathtt{x},\mathtt{y}),\mathtt{x})]\!]_v = \mathrm{T}$.

問題 6

(1) $\mathcal{M}_1$ について．$[\![\mathtt{P}(\mathtt{x},\mathtt{x})]\!]_v = \mathrm{T} \Leftrightarrow \langle [\![\mathtt{x}]\!]_v, [\![\mathtt{x}]\!]_v\rangle \in \{\langle x,y\rangle \mid x < y\} \Leftrightarrow v(\mathtt{x}) < v(\mathtt{x})$.
任意の $n \in \mathbb{N}$ について $n \not< n$ であるから，$[\![\mathtt{P}(\mathtt{x},\mathtt{x})]\!]_v = \mathrm{T}$ となるような付値は存在しない．

(2) $\mathcal{M}_2$ について．$[\![\mathtt{P}(\mathtt{x},\mathtt{x})]\!]_v = \mathrm{T} \Leftrightarrow \langle [\![\mathtt{x}]\!]_v, [\![\mathtt{x}]\!]_v \rangle \in \{\langle x, y\rangle \mid x \times 3 = y \times 2\}\rangle \Leftrightarrow v(\mathtt{x}) \times 3 = v(\mathtt{x}) \times 2$．よって，$v(\mathtt{x}) = 0$ なる付値 v をとれば成立する．
(3) $\mathcal{M}_3$ について．$[\![\mathtt{P}(\mathtt{x},\mathtt{x})]\!]_v = \mathrm{T} \Leftrightarrow \langle [\![\mathtt{x}]\!]_v, [\![\mathtt{x}]\!]_v \rangle \in \{\langle x, y\rangle \mid x + y = 9\}\rangle \Leftrightarrow v(\mathtt{x}) + v(\mathtt{x}) = 9$．$n + n = 9$ となるような自然数 n は存在しないから，$[\![\mathtt{P}(\mathtt{x},\mathtt{x})]\!]_v = \mathrm{T}$ となるような付値は存在しない．