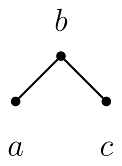
2019年度 数理論理学 復習問題(17)

問題 1 シグニチャを $L = \langle\{\mathtt{f}^1\}, \{\mathtt{P}^1\}\rangle$，$L$-構造 $\mathcal{M}$ を $\mathcal{M} = \langle\{-1, 0, 1\}, \mathtt{f}^{\mathcal{M}}, \mathtt{P}^{\mathcal{M}}\rangle$，$\mathtt{f}^{\mathcal{M}}(x) = -x$，$\mathtt{P}^{\mathcal{M}} = \{0, 1\}$ と定める．v を任意の付値とするとき，以下の値を求めよ．

(1) $[\![\neg\mathtt{P}(\mathtt{x})]\!]_{v[1/\mathtt{x}]}$ (2) $[\![\mathtt{x} \approx \mathtt{f}(\mathtt{x}) \wedge \mathtt{P}(\mathtt{f}(\mathtt{x}))]\!]_{v[0/\mathtt{x}]}$ (3) $[\![\neg\mathtt{P}(\mathtt{f}(\mathtt{x})) \to \mathtt{P}(\mathtt{x})]\!]_{v[1/\mathtt{x}]}$

問題 2 シグニチャを $L = \langle\emptyset, \{\mathtt{P}^2\}\rangle$ とするとき，L-構造 $\mathcal{M}$ を $\mathcal{M} = \langle\mathbb{Z}, \mathtt{P}^{\mathcal{M}}\rangle$，$\mathtt{P}^{\mathcal{M}} = \{\langle x, -x\rangle \mid x \in \mathbb{Z}\}$ と定める．このとき，v を任意の付値とするとき，$[\![\mathtt{P}(\mathtt{x}, \mathtt{y}) \to \mathtt{P}(\mathtt{y}, \mathtt{x})]\!]_v = \mathrm{T}$ となることを示せ．

問題 3–6 では，次のハッセ図で与えられる半順序集合による L-構造を考える．



問題 3 v を任意の付値として，以下の真理値を計算せよ．

(1) $[\![\forall\mathtt{x}\,\exists\mathtt{y}\,(\mathtt{x} \leq \mathtt{y})]\!]_v$
(2) $[\![\exists\mathtt{x}\,\forall\mathtt{y}\,(\mathtt{x} \leq \mathtt{y})]\!]_v$

問題 4 v を任意の付値として，以下の真理値を計算せよ．

(1) $[\![\forall\mathtt{y}\,\exists\mathtt{x}\,(\mathtt{x} \leq \mathtt{y})]\!]_v$
(2) $[\![\exists\mathtt{y}\,\forall\mathtt{x}\,(\mathtt{x} \leq \mathtt{y})]\!]_v$

問題 5 v を任意の付値として，以下の真理値を計算せよ．

(1) $[\![\forall\mathtt{x}\,\forall\mathtt{y}\,((\mathtt{x} \approx \mathtt{y}) \vee (\mathtt{x} < \mathtt{y}))]\!]_v$
(2) $[\![\exists\mathtt{x}\,\exists\mathtt{y}\,(\neg(\mathtt{x} \approx \mathtt{y}) \wedge (\mathtt{y} \leq \mathtt{x}))]\!]_v$

問題 6 v を任意の付値として，以下の真理値を計算せよ．

(1) $[\![\forall\mathtt{x}\,\neg\forall\mathtt{y}\,(\mathtt{x} \leq \mathtt{y})]\!]_v$
(2) $[\![\forall\mathtt{x}\,(\forall\mathtt{y}\,(\mathtt{y} \leq \mathtt{x}) \to \exists\mathtt{y}\,(\mathtt{y} < \mathtt{x}))]\!]_v$

2019 年度 数理論理学 復習問題解答 (17)

問題 1 (1) $[\![\neg\mathtt{P}(\mathtt{x})]\!]_{v[1/\mathtt{x}]} = \mathsf{not}([\![\mathtt{P}(\mathtt{x})]\!]_{v[1/\mathtt{x}]}) = \mathsf{not}(\mathrm{T}) = \mathrm{F}$.
(2) $[\![\mathtt{x} \approx \mathtt{f}(\mathtt{x})]\!]_{v[0/\mathtt{x}]} = \mathrm{T} \Leftrightarrow [\![\mathtt{x}]\!]_{v[0/\mathtt{x}]} = [\![\mathtt{f}(\mathtt{x})]\!]_{v[0/\mathtt{x}]} \Leftrightarrow 0 = -0$ より，$[\![\mathtt{x} \approx \mathtt{f}(\mathtt{x})]\!]_{v[0/\mathtt{x}]} = \mathrm{T}$ が成立．また，$[\![\mathtt{f}(\mathtt{x})]\!]_{v[0/\mathtt{x}]} = \mathtt{f}^{\mathcal{M}}([\![\mathtt{x}]\!]_{v[0/\mathtt{x}]}) = \mathtt{f}^{\mathcal{M}}(0) = 0$．よって，$[\![\mathtt{P}(\mathtt{f}(\mathtt{x}))]\!]_{v[0/\mathtt{x}]} = \mathrm{T} \Leftrightarrow [\![\mathtt{f}(\mathtt{x})]\!]_{v[0/\mathtt{x}]} \in \mathtt{P}^{\mathcal{M}} \Leftrightarrow 0 \in \mathtt{P}^{\mathcal{M}}$ となるので，$[\![\mathtt{P}(\mathtt{f}(\mathtt{x}))]\!]_{v[0/\mathtt{x}]} = \mathrm{T}$．したがって，$[\![\mathtt{x} \approx \mathtt{f}(\mathtt{x}) \land \mathtt{P}(\mathtt{f}(\mathtt{x}))]\!]_{v[0/\mathtt{x}]} = \mathsf{and}([\![\mathtt{x} \approx \mathtt{f}(\mathtt{x})]\!]_{v[0/\mathtt{x}]}, [\![\mathtt{P}(\mathtt{f}(\mathtt{x}))]\!]_{v[0/\mathtt{x}]}) = \mathsf{and}(\mathrm{T},\mathrm{T}) = \mathrm{T}$.
(3) $[\![\mathtt{P}(\mathtt{f}(\mathtt{x}))]\!]_{v[1/\mathtt{x}]} = \mathrm{T} \Leftrightarrow -1 \in \mathtt{P}^{\mathcal{M}}$ より，$[\![\mathtt{P}(\mathtt{f}(\mathtt{x}))]\!]_{v[1/\mathtt{x}]} = \mathrm{F}$．よって，$[\![\neg\mathtt{P}(\mathtt{f}(\mathtt{x})) \to \mathtt{P}(\mathtt{x})]\!]_{v[1/\mathtt{x}]} = \mathsf{imp}(\mathsf{not}([\![\mathtt{P}(\mathtt{f}(\mathtt{x}))]\!]_{v[1/\mathtt{x}]}, [\![\mathtt{P}(\mathtt{x})]\!]_{v[1/\mathtt{x}]})) = \mathsf{imp}(\mathsf{not}(\mathrm{F}), \mathrm{T}) = \mathrm{T}$.

問題 2 $\mathtt{P}^{\mathcal{M}} = \{\langle x, -x\rangle \mid x \in \mathbb{Z}\} = \{\langle x, y\rangle \in \mathbb{Z}^2 \mid x = -y\}$ に注意すると，$[\![\mathtt{P}(\mathtt{x},\mathtt{y})]\!]_v = \mathrm{T} \Leftrightarrow \langle v(\mathtt{x}), v(\mathtt{y})\rangle \in \mathtt{P}^{\mathcal{M}} \Leftrightarrow v(\mathtt{x}) = -v(\mathtt{y})$．また，$[\![\mathtt{P}(\mathtt{y},\mathtt{x})]\!]_v = \mathrm{T} \Leftrightarrow v(\mathtt{y}) = -v(\mathtt{x})$．任意の整数 x, y について，$x = -y$ ならば $y = -x$ となるから，$[\![\mathtt{P}(\mathtt{x},\mathtt{y})]\!]_v = \mathrm{T}$ であるとき $[\![\mathtt{P}(\mathtt{y},\mathtt{x})]\!]_v = \mathrm{T}$ が成立する．したがって，$[\![\mathtt{P}(\mathtt{x},\mathtt{y}) \to \mathtt{P}(\mathtt{y},\mathtt{x})]\!]_v = \mathrm{T}$ となる．

問題 3

(1) $[\![\forall \mathrm{x}\, \exists \mathrm{y}\, (\mathrm{x} \le \mathrm{y})]\!]_v = \mathrm{T}$

$[\![\forall \mathrm{x}\, \exists \mathrm{y}\, A]\!]$	x	$[\![\exists \mathrm{y}\, A]\!]$	y	$[\![A]\!]$ $[\![\mathrm{x} \le \mathrm{y}]\!]$
	a		a	T
			b	T
		T	c	F
	b		a	F
			b	T
		T	c	F
	c		a	F
			b	T
T		T	c	T

(2) $[\![\exists \mathrm{x}\, \forall \mathrm{y}\, (\mathrm{x} \le \mathrm{y})]\!]_v = \mathrm{F}$

$[\![\exists \mathrm{x}\, \forall \mathrm{y}\, A]\!]$	x	$[\![\forall \mathrm{y}\, A]\!]$	y	$[\![A]\!]$ $[\![\mathrm{x} \le \mathrm{y}]\!]$
	a		a	T
			b	T
		F	c	F
	b		a	F
			b	T
		F	c	F
	c		a	F
			b	T
F		F	c	T

問題 4

(1) $[\![\forall \mathrm{y}\, \exists \mathrm{x}\, (\mathrm{x} \le \mathrm{y})]\!]_v = \mathrm{T}$

$[\![\forall \mathrm{y}\, \exists \mathrm{x}\, A]\!]$	y	$[\![\exists \mathrm{x}\, A]\!]$	x	$[\![A]\!]$ $[\![\mathrm{x} \le \mathrm{y}]\!]$
	a		a	T
			b	F
		T	c	F
	b		a	T
			b	T
		T	c	T
	c		a	F
			b	F
T		T	c	T

(2) $[\![\exists \mathrm{y}\, \forall \mathrm{x}\, (\mathrm{x} \le \mathrm{y})]\!]_v = \mathrm{T}$

$[\![\exists \mathrm{y}\, \forall \mathrm{x}\, A]\!]$	y	$[\![\forall \mathrm{x}\, A]\!]$	x	$[\![A]\!]$ $[\![\mathrm{x} \le \mathrm{y}]\!]$
	a		a	T
			b	F
		F	c	F
	b		a	T
			b	T
		T	c	T
	c		a	F
			b	F
T		F	c	T

問題 5

(1) $[\![\forall \mathtt{x}\, \forall \mathtt{y}\, ((\mathtt{x} \approx \mathtt{y}) \vee (\mathtt{x} < \mathtt{y}))]\!]_v = \mathrm{F}$

$[\![\forall \mathtt{x}\, \forall \mathtt{y}\, A]\!]$	x	$[\![\forall \mathtt{y}\, A]\!]$	y	$[\![A]\!]$ $[\![P \vee Q]\!]$	$[\![P]\!]$ $[\![\mathtt{x} \approx \mathtt{y}]\!]$	$[\![Q]\!]$ $[\![\mathtt{x} < \mathtt{y}]\!]$
	a		a	T	T	F
			b	T	F	T
		F	c	F	F	F
	b		a	F	F	F
			b	T	T	F
		F	c	F	F	F
	c		a	F	F	F
			b	T	F	T
F		F	c	T	T	F

(2) $[\![\exists \mathtt{x}\, \exists \mathtt{y}\, (\neg(\mathtt{x} \approx \mathtt{y}) \wedge (\mathtt{y} \leq \mathtt{x}))]\!]_v = \mathrm{T}$

$[\![\exists \mathtt{x}\, \exists \mathtt{y}\, A]\!]$	x	$[\![\exists \mathtt{y}\, A]\!]$	y	$[\![A]\!]$ $[\![\neg P \wedge Q]\!]$	$[\![\neg P]\!]$	$[\![P]\!]$ $[\![\mathtt{x} \approx \mathtt{y}]\!]$	$[\![Q]\!]$ $[\![\mathtt{y} \leq \mathtt{x}]\!]$
	a		a	F	F	T	T
			b	F	T	F	F
		F	c	F	T	F	F
	b		a	T	T	F	T
			b	F	F	T	T
		T	c	T	T	F	T
	c		a	F	T	F	F
			b	F	T	F	F
T		F	c	F	F	T	T

問題 6

(1) $[\![\forall \mathtt{x}\, \neg \forall \mathtt{y}\, (\mathtt{x} \leq \mathtt{y})]\!]_v = \mathrm{T}$

$[\![\forall \mathtt{x}\, \neg \forall \mathtt{y}\, (\mathtt{x} \leq \mathtt{y})]\!]$	x	$[\![\neg \forall \mathtt{y}\, (\mathtt{x} \leq \mathtt{y})]\!]$	$[\![\forall \mathtt{y}\, (\mathtt{x} \leq \mathtt{y})]\!]$	y	$[\![\mathtt{x} \leq \mathtt{y}]\!]$
	a			a	T
				b	T
		T	F	c	F
	b			a	F
				b	T
		T	F	c	F
	c			a	F
				b	T
T		T	F	c	T

(2) $[\![\forall \mathtt{x}\,(\forall \mathtt{y}\,(\mathtt{y} \leq \mathtt{x}) \rightarrow \exists \mathtt{y}\,(\mathtt{y} < \mathtt{x}))]\!]_v = \mathrm{T}$

			$[\![A]\!]$			$[\![B]\!]$		
$[\![\forall \mathtt{x}\,(A \rightarrow B)]\!]$	$\mathtt{x}$	$[\![A \rightarrow B]\!]$	$[\![\forall \mathtt{y}\,(\mathtt{y} \leq \mathtt{x})]\!]$	$\mathtt{y}$	$[\![\mathtt{y} \leq \mathtt{x}]\!]$	$[\![\exists \mathtt{y}\,(\mathtt{y} < \mathtt{x})]\!]$	$\mathtt{y}$	$[\![\mathtt{y} < \mathtt{x}]\!]$
	a			a	T		a	F
				b	F		b	F
		T	F	c	F	F	c	F
	b			a	T		a	T
				b	T		b	F
		T	T	c	T	T	c	T
	c			a	F		a	F
				b	F		b	F
T		T	F	c	T	F	c	F