

2021 年度 数理論理学 復習問題 (14)

問題 1 以下の述語論理式の証明図を書け.

- (1) $\forall x \forall y P(x, y) \rightarrow \exists z P(z, z)$
- (2) $\forall x P(x, f(x)) \rightarrow \forall x \exists y P(x, y)$
- (3) $\neg \exists x (\neg P(x) \wedge P(x))$
- (4) $\forall x (P(x) \wedge Q(x)) \rightarrow (\exists x P(x) \wedge \exists x Q(x))$
- (5) $\exists x (P(x) \rightarrow Q(x)) \rightarrow (\forall x P(x) \rightarrow \exists x Q(x))$
- (6) $\exists x P(x) \vee \exists x Q(x) \rightarrow \exists x (P(x) \vee Q(x))$
- (7) $\forall x (P(x) \rightarrow \exists y Q(x, y)) \rightarrow P(a) \rightarrow \exists y Q(a, y)$

問題 2 以下の述語論理式の証明図を書け.

- (1) $\forall x \forall y \exists z (x + y \approx z)$
- (2) $\forall x \forall y (P(x) \wedge x \approx y \rightarrow P(y))$
- (3) $\forall x \exists y (x \approx f(y)) \rightarrow \forall x \exists y (x \approx f(f(y)))$

問題 3 以下の述語論理式の証明図を書け.

- (1) $\forall x P(x) \wedge \forall y Q(y) \rightarrow \exists z (P(z) \wedge Q(z))$
- (2) $\forall x P(x) \rightarrow \exists z P(z) \wedge \exists y P(y)$
- (3) $\exists x P(x) \rightarrow \exists y \exists z (P(y) \wedge P(z))$
- (4) $\exists x P(x) \rightarrow \forall x \forall y (P(x) \rightarrow Q(y)) \rightarrow \forall y Q(y)$
- (5) $\exists x (Q \rightarrow P(x)) \rightarrow (Q \rightarrow \exists x P(x))$
- (6) $\exists x P(x, x, x) \rightarrow \exists y \exists z P(y, y, z)$
- (7) $\exists x \forall y P(x, x, y) \rightarrow \exists x \exists y P(x, y, y)$
- (8) $\forall x \exists y (P(x) \rightarrow P(y))$
- (9) $\forall x (P(x) \vee Q(x)) \rightarrow \exists x \neg P(x) \rightarrow \exists x Q(x)$
- (10) $\exists x \neg P(x) \rightarrow \neg \forall x P(x)$
- (11) $\neg \forall x P(x) \rightarrow \exists x \neg P(x)$

2021 年度 数理論理学 復習問題解答 (14)

問題 1

(1)

$$\frac{\frac{\frac{[\forall x \forall y P(x, y)]^1}{\forall y P(z, y)} \forall E}{\frac{P(z, z)}{\exists I}} \exists I}{\forall x \forall y P(x, y) \rightarrow \exists z P(z, z)} \rightarrow I^1$$

(2)

$$\frac{\frac{\frac{[\forall x P(x, f(x))]^1}{P(x, f(x))} \forall E}{\frac{\exists y P(x, y)}{\exists I}} \exists I}{\forall x \exists y P(x, y)} \forall I \rightarrow I^1$$

(3)

$$\frac{\frac{[\exists x (P(x) \wedge \neg P(x))]^1}{\perp} \perp}{\neg \exists x (\neg P(x) \wedge P(x))} \neg I^2$$

$$\frac{\frac{[P(z) \wedge \neg P(z)]^1}{\neg P(z)} \wedge E \quad \frac{[P(z) \wedge \neg P(z)]^1}{P(z)} \wedge E}{\perp} \exists E^1$$

(4)

$$\frac{\frac{\frac{[\forall x (P(x) \wedge Q(x))]^1}{P(x) \wedge Q(x)} \forall E}{\frac{P(x)}{\exists I}} \exists I \quad \frac{\frac{[\forall x (P(x) \wedge Q(x))]^1}{P(x) \wedge Q(x)} \forall E}{\frac{Q(x)}{\exists I}} \exists I}{\frac{\exists x P(x) \wedge \exists x Q(x)}{\wedge I}} \wedge I \rightarrow I^1$$

(5)

$$\frac{\frac{[\exists x (P(x) \rightarrow Q(x))]^3}{\exists x Q(x)} \rightarrow I^2 \quad \frac{[P(x) \rightarrow Q(x)]^1}{Q(x)} \rightarrow E \quad \frac{[\forall x P(x)]^2}{P(x)} \forall E}{\frac{\exists x Q(x)}{\exists I}} \exists I \rightarrow I^3$$

(6)

$$\frac{\frac{[\exists x P(x) \vee \exists x Q(x)]^3}{\exists x (P(x) \vee Q(x))} \vee I \quad \frac{[P(x)]^1}{P(x) \vee Q(x)} \vee I \quad \frac{[Q(x)]^1}{P(x) \vee Q(x)} \vee I}{\frac{\frac{[\exists x P(x)]^2}{\exists x (P(x) \vee Q(x))} \exists I \quad \frac{[\exists x Q(x)]^2}{\exists x (P(x) \vee Q(x))} \exists I}{\exists x (P(x) \vee Q(x))} \exists E^1 \quad \frac{[\exists x P(x) \vee \exists x Q(x)]^3}{\exists x (P(x) \vee Q(x))} \vee E^2} \rightarrow I^3$$

(7)

$$\begin{array}{c}
\frac{[\forall x (P(x) \rightarrow \exists y Q(x, y))]^2}{P(a) \rightarrow \exists y Q(a, y)} \forall E \quad \frac{[P(a)]^1}{\exists y Q(a, y)} \rightarrow E \\
\frac{\quad}{P(a) \rightarrow \exists y Q(a, y)} \rightarrow I^1 \\
\frac{\quad}{\forall x (P(x) \rightarrow \exists y Q(x, y)) \rightarrow P(a) \rightarrow \exists y Q(a, y)} \rightarrow I^2
\end{array}$$

問題 2

(1)

$$\begin{array}{c}
\overline{\forall x (x \approx x)} \text{ REFL} \\
\frac{\quad}{x + y \approx x + y} \forall E \\
\frac{\quad}{\exists z (x + y \approx z)} \exists I \\
\frac{\quad}{\forall y \exists z (x + y \approx z)} \forall I \\
\frac{\quad}{\forall x \forall y \exists z (x + y \approx z)} \forall I
\end{array}$$

(2)

$$\begin{array}{c}
\frac{[P(x) \wedge x \approx y]^1}{x \approx y} \wedge E \quad \frac{[P(x) \wedge x \approx y]^1}{P(x)} \wedge E \\
\frac{\quad}{P(y)} \text{ SUBST} \\
\frac{\quad}{P(x) \wedge x \approx y \rightarrow P(y)} \rightarrow I^1 \\
\frac{\quad}{\forall y (P(x) \wedge x \approx y \rightarrow P(y))} \forall I \\
\frac{\quad}{\forall x \forall y (P(x) \wedge x \approx y \rightarrow P(y))} \forall I
\end{array}$$

(3)

$$\begin{array}{c}
\frac{[y \approx f(z)]^1 \quad [x \approx f(y)]^2}{x \approx f(f(z))} \text{ SUBST} \\
\frac{[\forall x \exists y (x \approx f(y))]^3}{\exists z (y \approx f(z))} \forall E \quad \frac{\quad}{\exists y (x \approx f(f(y)))} \exists I \\
\frac{\quad}{\exists y (x \approx f(f(y)))} \exists E^1 \\
\frac{\quad}{\exists y (x \approx f(f(y)))} \exists E^2 \\
\frac{\quad}{\forall x \exists y (x \approx f(f(y)))} \forall E \\
\frac{\quad}{\forall x \exists y (x \approx f(y)) \rightarrow \forall x \exists y (x \approx f(f(y)))} \rightarrow I^3
\end{array}$$

問題 3

(1)

$$\begin{array}{c}
\frac{[\forall x P(x) \wedge \forall y Q(y)]^1}{\forall x P(x)} \wedge E \quad \frac{[\forall x P(x) \wedge \forall y Q(y)]^1}{\forall y Q(y)} \wedge E \\
\frac{\quad}{P(z)} \forall E \quad \frac{\quad}{Q(z)} \forall E \\
\frac{\quad}{P(z) \wedge Q(z)} \wedge I \\
\frac{\quad}{\exists z (P(z) \wedge Q(z))} \exists I \\
\frac{\quad}{\forall x P(x) \wedge \forall y Q(y) \rightarrow \exists z (P(z) \wedge Q(z))} \rightarrow I^1
\end{array}$$

(2)

$$\begin{array}{c}
\frac{[\forall x P(x)]^1}{P(z)} \forall E \quad \frac{[\forall x P(x)]^1}{P(y)} \forall E \\
\frac{\quad}{\exists z P(z)} \exists I \quad \frac{\quad}{\exists y P(y)} \exists I \\
\frac{\quad}{\exists z P(z) \wedge \exists y P(y)} \wedge I \\
\frac{\quad}{\forall x P(x) \rightarrow \exists z P(z) \wedge \exists y P(y)} \rightarrow I^1
\end{array}$$

(3)

$$\begin{array}{c}
\frac{[P(y)]^1 \quad [P(y)]^1}{P(y) \wedge P(y)} \wedge I \\
\frac{P(y) \wedge P(y)}{\exists z (P(y) \wedge P(z))} \exists I \\
\frac{[\exists x P(x)]^2 \quad \exists y \exists z (P(y) \wedge P(z))}{\exists y \exists z (P(y) \wedge P(z))} \exists I \\
\frac{\exists y \exists z (P(y) \wedge P(z))}{\exists x P(x) \rightarrow \exists y \exists z (P(y) \wedge P(z))} \rightarrow I^2
\end{array}$$

(5)

$$\begin{array}{c}
\frac{[Q \rightarrow P(x)]^1 \quad [Q]^2}{P(x)} \rightarrow E \\
\frac{[\exists x (Q \rightarrow P(x))]^3 \quad \frac{P(x)}{\exists x P(x)} \exists I}{\exists x P(x)} \exists E^1 \\
\frac{\exists x P(x)}{Q \rightarrow \exists x P(x)} \rightarrow I^2 \\
\frac{Q \rightarrow \exists x P(x)}{\exists x (Q \rightarrow P(x)) \rightarrow (Q \rightarrow \exists x P(x))} \rightarrow I^3
\end{array}$$

(7)

(4)

$$\begin{array}{c}
\frac{[\forall x \forall y (P(x) \rightarrow Q(y))]^2}{\forall y (P(z) \rightarrow Q(y))} \forall E \\
\frac{\forall y (P(z) \rightarrow Q(y))}{P(z) \rightarrow Q(y)} \forall E \quad \frac{[P(z)]^1}{P(z) \rightarrow Q(y)} \rightarrow E \\
\frac{[P(z)]^1}{P(z) \rightarrow Q(y)} \rightarrow E \\
\frac{[Q(y)]}{\forall y Q(y)} \forall I \\
\frac{[\exists x P(x)]^3 \quad \forall y Q(y)}{\forall y Q(y)} \exists E^1 \\
\frac{\forall y Q(y)}{\forall x \forall y (P(x) \rightarrow Q(y)) \rightarrow \forall y Q(y)} \rightarrow I^2 \\
\frac{\forall x \forall y (P(x) \rightarrow Q(y)) \rightarrow \forall y Q(y)}{\exists x P(x) \rightarrow \forall x \forall y (P(x) \rightarrow Q(y)) \rightarrow \forall y Q(y)} \rightarrow I^3
\end{array}$$

(6)

$$\begin{array}{c}
\frac{[P(y, y, y)]^1}{\exists z P(y, y, z)} \exists I \\
\frac{[\exists x P(x, x, x)]^2 \quad \exists y \exists z P(y, y, z)}{\exists y \exists z P(y, y, z)} \exists I \\
\frac{\exists y \exists z P(y, y, z)}{\exists x P(x, x, x) \rightarrow \exists y \exists z P(y, y, z)} \rightarrow I^2
\end{array}$$

(8)

(9)

$$\begin{array}{c}
\frac{[\forall y P(z, z, y)]^1}{P(z, z, z)} \forall E \\
\frac{P(z, z, z)}{\exists y P(z, y, y)} \exists I \\
\frac{[\exists x \forall y P(x, x, y)]^2 \quad \exists x \exists y P(x, y, y)}{\exists x \exists y P(x, y, y)} \exists I \\
\frac{\exists x \exists y P(x, y, y)}{\exists x \forall y P(x, x, y) \rightarrow \exists x \exists y P(x, y, y)} \rightarrow I^2
\end{array}$$

$$\begin{array}{c}
\frac{[P(x)]^1}{P(x) \rightarrow P(x)} \rightarrow I^1 \\
\frac{P(x) \rightarrow P(x)}{\exists y (P(x) \rightarrow P(y))} \exists I \\
\frac{\exists y (P(x) \rightarrow P(y))}{\forall x \exists y (P(x) \rightarrow P(y))} \forall I
\end{array}$$

(10)

$$\begin{array}{c}
\frac{[\forall x (P(x) \vee Q(x))]^4}{P(z) \vee Q(z)} \forall E \\
\frac{[\exists x \neg P(x)]^3 \quad P(z) \vee Q(z)}{\exists x Q(x)} \exists E^2 \\
\frac{\exists x Q(x)}{\exists x \neg P(x) \rightarrow \exists x Q(x)} \rightarrow I^3 \\
\frac{\exists x \neg P(x) \rightarrow \exists x Q(x)}{\forall x (P(x) \vee Q(x)) \rightarrow \exists x \neg P(x) \rightarrow \exists x Q(x)} \rightarrow I^4
\end{array}$$

(11)

$$\begin{array}{c}
\frac{[\neg P(y)]^2 \quad \frac{[\forall x P(x)]^1}{P(y)} \forall E}{\neg P(y)} \neg E \\
\frac{[\exists x \neg P(x)]^3 \quad \frac{\perp}{\neg \forall x P(x)} \neg I^1}{\neg \forall x P(x)} \exists E^2 \\
\frac{\neg \forall x P(x)}{\exists x \neg P(x) \rightarrow \neg \forall x P(x)} \rightarrow I^3
\end{array}$$

$$\begin{array}{c}
\frac{[\neg \exists x \neg P(x)]^2 \quad \frac{[\neg P(x)]^1}{\exists x \neg P(x)} \exists I}{\neg \exists x \neg P(x)} \neg E \\
\frac{\neg \exists x \neg P(x)}{P(x)} RAA^1 \\
\frac{P(x)}{\forall x P(x)} \forall I \\
\frac{[\neg \forall x P(x)]^3 \quad \forall x P(x)}{\perp} \neg E \\
\frac{\perp}{\exists x \neg P(x)} RAA^2 \\
\frac{\exists x \neg P(x)}{\neg \forall x P(x) \rightarrow \exists x \neg P(x)} \rightarrow I^3
\end{array}$$