On Composable Properties of Term Rewriting Systems

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Abstract. A property of term rewriting system (TRS, for short) is said to be composable if it is preserved under unions. We present composable properties of TRSs on the base of modularity results for direct sums of TRSs. We propose a decomposition by a naive sort attachment, and show that modular properties for direct sums of TRSs are τ-composable for a naive sort attachment τ. Here, a decomposition of a TRS $\mathcal{R}$ is a pair $\langle \mathcal{R}_1, \mathcal{R}_2 \rangle$ of (not necessary disjoint) subsets of $\mathcal{R}$ such that $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$; and for a naive sort attachment τ a property ϕ of TRSs is said to be τ-composable if for any TRS $\mathcal{R}$ such that τ is consistent with $\mathcal{R}$, $\phi(\mathcal{R}_1) \wedge \phi(\mathcal{R}_2)$ implies $\phi(\mathcal{R})$, where $\langle \mathcal{R}_1, \mathcal{R}_2 \rangle$ is the decomposition of $\mathcal{R}$ by τ.

1 Introduction

A decomposition of a term rewriting system (TRS, for short) $\mathcal{R}$ is a pair $\langle \mathcal{R}_1, \mathcal{R}_2 \rangle$ of (not necessary disjoint) subsets of $\mathcal{R}$ such that $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$. In this paper, we propose a *decomposition by a naive sort attachment*, and present *τ-composable properties* for a naive sort attachment τ. Here, for a naive sort attachment τ a property ϕ of TRSs is said to be τ-composable if for any TRS $\mathcal{R}$ such that τ is consistent with $\mathcal{R}$, $\phi(\mathcal{R}_1) \wedge \phi(\mathcal{R}_2)$ implies $\phi(\mathcal{R})$, where $\langle \mathcal{R}_1, \mathcal{R}_2 \rangle$ is the decomposition of $\mathcal{R}$ by τ.

A property of TRSs is said to be *composable* if it is preserved under unions. Composable properties, in particular, preserved under non-disjoint unions have been studied by several authors; see [10] [11].

The key idea of our composability result is a top-down labelling—a particular mapping from terms to terms, which is applied top-down. Using this labelling, we show that properties that are modular for direct sums are τ-composable for a naive sort attachment τ. Here, a property of TRSs is said be *modular (for a particular kind of disjoint unions)* if it is preserved under (for such kind of) disjoint unions; and a disjoint union of TRSs is called a *direct sum* when the sets of function symbols that appear in those TRSs are disjoint. Modularity for direct sums were studied in e.g. [3] [6] [7] [8] [12] [13] [14] [15]. Thus, on the base of these modularity results, we uniformly obtain composability results for decompositions by a naive sort attachment.

The τ-composability of a property for a naive sort attachment τ can be also obtained by the persistency of the property. Here, a property ϕ of TRSs

is said to be *persistent* if for any TRS $\mathcal{R}$ and a sort attachment τ consistent with $\mathcal{R}$ it holds that $\phi(\mathcal{R}^\tau)$ if and only if $\phi(\mathcal{R})$, where $\mathcal{R}^\tau$ is the many-sorted TRS induced from $\mathcal{R}$ and τ. It was shown in [17] that for component-closed properties their persistency implies their modularity for direct sums of TRSs. For several properties, their persistency has been proved by modifying the proofs of their modularity for direct sums of TRSs; see [1] [17]. Whether the modularity for direct sums of TRSs implies the persistency (uniformly for component-closed properties) is still a conjecture, and once the conjecture is solved affirmatively the result in this paper will be subsumed. Note, however, that for component-closed properties their persistency coincides with their modularity for direct sums of many-sorted TRSs [16].

The rest of this paper is organized as follows. To introduce a notion of a decomposition by a naive sort attachment, we need a notion of sort attachment and of sort elimination. For this, we review many-sorted term rewriting in section 2. In section 3, we develop a top-down labelling technique and show that how reductions are preserved under the top-down labelling. In section 4, we give various composability results together with some examples.

2 Preliminaries

Our language is given by a set $\mathcal{S}$ of sorts (denoted by $X, Y, Z, \ldots$), a set $\mathcal{V}$ of variables (denoted by $x, y, z, \ldots$), and a set $\mathcal{F}$ of function symbols (denoted by $f, g, h, \ldots$). Each variable is given with its sort; we assume that there are countably infinite variables of sort X for each sort $X \in \mathcal{S}$. Similarly, each function symbol is given with the sorts of its arguments and the sort of its output. We write $f : X_1 \times \cdots \times X_n \to Y$ if f takes n arguments of sorts $X_1, \ldots, X_n$ respectively to a value of sort Y.

With such language, one can build up terms (of sort X) in a usual way: (1) a variable of sort X is a term of sort X; (2) if $f : X_1 \times \cdots \times X_n \to X$ is a function symbol and $t_1, \ldots, t_n$ are terms of sort $X_1, \ldots, X_n$ respectively, then $f(t_1, \ldots, t_n)$ is a term of sort X. Let $\mathcal{T}$ (and $\mathcal{T}^X$) denote the set of terms (of sort X, respectively). We also write "$t : X$" to indicate that $t \in \mathcal{T}^X$. Syntactical equality is denoted by $\equiv$. $\mathcal{V}(t)$ is the set of variables that appear in a term t.

For each sort X, let $\Box^X$ be a special constant—called a *hole*—of sort X. A *context* is a term possibly containing holes. The set of contexts is denoted by $\mathcal{C}$. We write $C : X_1 \times \cdots \times X_n \to X$ when $C \in \mathcal{C}$ has the sort X (as a term) and has n holes $\Box^{X_1}, \ldots, \Box^{X_n}$ from left to right in it. If $C : X_1 \times \cdots \times X_n \to Y$ and $t_1 : X_1, \ldots, t_n : X_n$ then $C[t_1, \ldots, t_n]$ is the term obtained from C by replacing holes with $t_1, \ldots, t_n$ from left to right. A context C is written as $C[\,]$ when C contains precisely one hole. A term t is said to be a *subterm* of s ($t \trianglelefteq s$, in symbol) if $s \equiv C[t]$ for some context $C[\,]$.

A *substitution* σ is a mapping from $\mathcal{V}$ to $\mathcal{T}$ such that x and $\sigma(x)$ have the same sort. A substitution is extended to a homomorphism from $\mathcal{T}$ to $\mathcal{T}$ in an obvious way. For a substitution σ and a term t, we customarily write $t\sigma$ instead of $\sigma(t)$.

A *(many-sorted) rewrite rule* is a pair $\langle l, r\rangle$ of terms such that (1) l and r have the same sort, (2) $l \notin \mathcal{V}$, (3) $\mathcal{V}(r) \subseteq \mathcal{V}(l)$. We conventionally write $l \to r$ instead of $\langle l, r\rangle$. A rewrite rule $l \to r$ is *collapsing* if $r \in \mathcal{V}$; it is *duplicating* if r contains more occurrences of some variable than l does. A *many-sorted term rewriting system* (*STRS*, for short) is a set of rewrite rules. A STRS is said to be *non-collapsing* (*non-duplicating*) if it contains no collapsing (resp. duplicating) rules. For $\mathcal{T}' \subseteq \mathcal{T}$, a STRS $\{l \to r \in \mathcal{R} \mid l, r \in \mathcal{T}'\}$ is denoted by $\mathcal{R}|_{\mathcal{T}'}$.

Given a STRS $\mathcal{R}$, a term s reduces to a term t ($s \to_{\mathcal{R}} t$, in symbol) when $s \equiv C[l\sigma]$ and $t \equiv C[r\sigma]$ for some $C[\,] \in \mathcal{C}$, $l \to r \in \mathcal{R}$ and substitution σ. We call $s \to_{\mathcal{R}} t$ a *rewrite step* (or a *reduction*). The *redex* of this rewrite step is $l\sigma$. The term t is called a *reduct* of the term s. One can readily check that s and t have the same sort whenever $s \to_{\mathcal{R}} t$. The transitive reflexive closure and the transitive reflexive symmetric closure of $\to_{\mathcal{R}}$ are denoted by $\overset{*}{\to}_{\mathcal{R}}$ and $\overset{*}{\leftrightarrow}_{\mathcal{R}}$, respectively. Henceforth, the subscript $_{\mathcal{R}}$ will be omitted when $\mathcal{R}$ is obvious from the context.

When $\mathcal{S} = \{*\}$, a STRS is called a TRS. Given an arbitrary STRS $\mathcal{R}$, by identifying each sort with $*$, we obviously obtain a TRS $\Theta(\mathcal{R})$—called the underlying TRS of $\mathcal{R}$.

Let $\mathcal{F}$ and $\mathcal{V}$ be sets of function symbols and variables, respectively, on a trivial set $\{*\}$ of sorts. Terms built from this language are called *unsorted terms*. Let $\mathcal{S}$ be another set of sorts. A *sort attachment* τ *on* $\mathcal{S}$ is a mapping from $\mathcal{F} \cup \mathcal{V}$ to the set $\mathcal{S}^*$ of finite sequences of elements from $\mathcal{S}$ such that $\tau(x) \in \mathcal{S}$ for any $x \in \mathcal{V}$ and $\tau(f) \in \mathcal{S}^{n+1}$ for any n-ary function symbol $f \in \mathcal{F}$. We write $\tau(f) = X_1 \times \cdots \times X_n \to Y$ instead of $\tau(f) = X_1, \ldots, X_n, Y$. Given a TRS $\mathcal{R}$, a sort attachment τ is said to be *consistent with* $\mathcal{R}$ if for any $l \to r \in \mathcal{R}$ there exists $Y \in \mathcal{S}$ such that $\vdash_\tau l : Y$ and $\vdash_\tau r : Y$ are provable in the following inference system:

$$\frac{\tau(x) = X}{\vdash_\tau x : X} \tag{1}$$

$$\frac{\tau(f) = X_1 \times \cdots \times X_n \to Y \quad \vdash_\tau t_1 : X_1 \quad \cdots \quad \vdash_\tau t_n : X_n}{\vdash_\tau f(t_1, \ldots, t_n) : Y.} \tag{2}$$

From a given TRS $\mathcal{R}$ and a sort attachment τ consistent with $\mathcal{R}$, by regarding each function symbol f to be of sort $\tau(f)$, and each variable x to be of sort $\tau(x)$, we get a STRS $\mathcal{R}^\tau$—called a STRS induced from $\mathcal{R}$ and τ. Note that $\mathcal{R}^\tau$ acts on $\mathcal{T}^\tau = \{t \in \mathcal{T} \mid \vdash_\tau t : X \text{ for some } X \in \mathcal{S}\}$, i.e. $s, t \in \mathcal{T}^\tau$ whenever $s \to_{\mathcal{R}^\tau} t$. Elements of $\mathcal{T}^\tau$ are called *well-sorted terms*. Clearly, $\mathcal{T}^\tau \subseteq \mathcal{T}$. For a fixed τ, we put $\mathcal{F}^Y = \{f \in \mathcal{F} \mid \tau(f) = X_1 \times \cdots \times X_n \to Y \text{ for some } X_1, \ldots, X_n \in \mathcal{S}\}$ and $\mathcal{T}^Y = \{t \in \mathcal{T} \mid \vdash_\tau t : Y\}$ for each $Y \in \mathcal{S}$. We also write $f : X_1 \times \cdots \times X_n \to Y$ instead of $\tau(f) = X_1 \times \cdots \times X_n \to Y$ when τ is obvious from the context.

3 Decomposition by a naive sort attachment

Using the notions presented in the previous section, we now define a decomposition by a naive sort attachment.

Definition 1. 1. A sort attachment τ on $\mathcal{S}$ is said to be *naive* when $\mathcal{S} = \{0,1,2\}$ and for any $f \in \mathcal{F}$ we have either
   - $\tau(f) = 0 \times \cdots \times 0 \to 0$,
   - $\tau(f) = X_1 \times \cdots \times X_n \to 1$ with $X_1, \ldots, X_n \in \{0,1\}$ or
   - $\tau(f) = Y_1 \times \cdots \times Y_n \to 2$ with $Y_1, \ldots, Y_n \in \{0,2\}$.
2. Let $\mathcal{R}$ be a TRS and τ a naive sort attachment consistent with $\mathcal{R}$. Suppose that $\mathcal{R}_X = \Theta(\mathcal{R}^\tau \restriction (\mathcal{T}^0 \cup \mathcal{T}^X))$ for each $X \in \{1,2\}$. Then a pair $\langle \mathcal{R}_1, \mathcal{R}_2 \rangle$ of TRSs is said to be a *decomposition of* $\mathcal{R}$ *by* τ.
3. A pair $\langle \mathcal{R}_1, \mathcal{R}_2 \rangle$ of TRSs is said to be a *decomposition of* $\mathcal{R}$ *by a naive sort attachment* if there exists a naive sort attachment τ consistent with $\mathcal{R}$ such that $\langle \mathcal{R}_1, \mathcal{R}_2 \rangle$ is a decomposition of $\mathcal{R}$ by τ.

Our theorems (Theorem 12$\sim$17) are consequences of the preservation of reductions under a top-down labelling. So, our first aim is to develop a top-down labelling technique. To this end, we fix a TRS $\mathcal{R}$ on the terms built from a set $\mathcal{F}$ of function symbols and a set $\mathcal{V}$ of variables (on a trivial set $\{*\}$ of sorts). Moreover, we put $\mathcal{S} = \{0,1,2\}$ and assume that τ is a naive sort attachment on $\mathcal{S}$ consistent with $\mathcal{R}$. It should be clear from the definition that well-sorted terms of sort 0 contain neither function symbols from $\mathcal{F}^1$ nor those from $\mathcal{F}^2$, and that well-sorted terms of sort 1 contain no function symbols from $\mathcal{F}^2$ and vice versa.

In the sequel, we need the following notation. For $C \in \mathcal{C}$, we write $C : X_1 \times \cdots \times X_n \to Y$ if $\vdash_\tau C[\Box^{X_1}, \ldots, \Box^{X_n}] : Y$ is derivable by rules (1), (2) with an additional rule:

$$\frac{X \in \mathcal{S}}{\vdash_\tau \Box^X : X.} \tag{3}$$

Definition 2. 1. The *top sort* of a term $t \in \mathcal{T}$ is defined by

$$\mathrm{top}(t) = \begin{cases} \tau(t) & \text{if } t \in \mathcal{V}, \\ Y & \text{if } t \equiv f(t_1, \ldots, t_n) \text{ with } \tau(f) = X_1 \times \cdots \times X_n \to Y. \end{cases}$$

2. Let $t \equiv C[t_1, \ldots, t_n] \in \mathcal{T}$ $(n \geq 0)$ be a term with $C \not\equiv \Box$. We write $t \equiv C[\![t_1, \ldots, t_n]\!]$ if (1) $C : X_1 \times \cdots \times X_n \to Y$, and (2) $\mathrm{top}(t_i) \neq X_i$ for $i = 1, \ldots, n$. If this is the case, terms $t_1, \ldots, t_n$ are called the *principal* subterms of t. Clearly, a term t is uniquely written as $C[\![t_1, \ldots, t_n]\!]$ for some $C \in \mathcal{C}$ and terms $t_1, \ldots, t_n$.
3. The *rank* of a term $t \in \mathcal{T}$ is defined by

$$\mathrm{rank}(t) = \begin{cases} 1 & \text{if } t \in \mathcal{T}^\tau, \\ 1 + max\{\mathrm{rank}(t_i) \mid 1 \leq i \leq n\} & \text{if } t \equiv C[\![t_1, \ldots, t_n]\!] \text{ with } n \geq 1. \end{cases}$$

Example 1. Let $\mathcal{F} = \{f, g, a, b, c, F\}$ and

$$\tau \begin{cases} f : 0 \times 1 \to 1 \\ g : 0 \to 1 \\ a, b, c : 0 \\ F : 0 \times 0 \to 2. \end{cases}$$

Then $f(b, g(a)) \in \mathcal{T}^1$, $\mathrm{top}(f(a,b)) = 1$ and $\mathrm{top}(F(c, f(a,b))) = 2$. Also, $F(c, f(a,b)) \equiv C[\![f(a,b)]\!]$ is a term of rank 3 where $C \equiv F(c, \Box) : 0 \to 2$.

Definition 3. 1. A rewrite step $s \to t$ is said to be *inner* (written as $s \to^i t$) if

$$s \equiv C[\![s_1, \ldots, C'[l\sigma], \ldots, s_n]\!] \to C[s_1, \ldots, C'[r\sigma], \ldots, s_n] \equiv t$$

for some terms $s_1, \ldots, s_n$, a substitution σ, $l \to r \in \mathcal{R}$, and $C' \in \mathcal{C}$; otherwise it is *outer* (written as $s \to^o t$).

2. A rewrite step $s \to^o t$ is said to be *destructive* if $\mathrm{top}(s) \neq \mathrm{top}(t)$.

The following lemma shows that a destructive rewrite step occurs only when the applied rule is collapsing, and that the reduct of a destructive rewrite step results from one of the principal subterms.

Lemma 4. *A rewrite step $s \to^o t$ is destructive if and only if $t \equiv \sigma(x)$ and $s \equiv C[\![s_1, \ldots, \sigma(x), \ldots, s_n]\!]$ for some terms $s_1, \ldots, s_n$, a substitution σ, and $C \in \mathcal{C}$ such that $C[s_1, \ldots, \Box, \ldots, s_n] \equiv C'\sigma$ for some $C'[x] \to x \in \mathcal{R}$.*

Proof. ($\Leftarrow$) Suppose $\mathrm{top}(s) = \mathrm{top}(t)$. Then, since $C' \not\equiv \Box$ by the definition of rewrite rules, $\mathrm{top}(C'[x]) = \mathrm{top}(C'[x]\sigma) = \mathrm{top}(\sigma(x))$. Also, by consistency, $\mathrm{top}(x) = \mathrm{top}(C'[x])$, so $\mathrm{top}(x) = \mathrm{top}(t)$. But then, t can not be principal, since $C[s_1, \ldots, \Box, \ldots, s_n] \equiv C'\sigma : \mathrm{top}(x) \to \mathrm{top}(s)$. ($\Rightarrow$) Suppose $\mathrm{top}(s) \neq \mathrm{top}(t)$. By consistency, the rewrite step $s \to_{\mathcal{R}} t$ is an application of a collapsing rule, and the redex of the rewrite step is s. Let the rule be $C'[x] \to x$, and suppose $s \equiv C'[x]\sigma$ and $t \equiv \sigma(x)$. Since $C[s_1, \ldots, \Box, \ldots, s_n] : \mathrm{top}(x) \to \mathrm{top}(s)$, it suffices to show $\mathrm{top}(x) \neq \mathrm{top}(t)$. But $\mathrm{top}(x) = \mathrm{top}(C'[x])$ by consistency, and $\mathrm{top}(C'[x]) = \mathrm{top}(C'[x]\sigma) = \mathrm{top}(s)$ since $C' \not\equiv \Box$. Hence $\mathrm{top}(x) = \mathrm{top}(s) \neq \mathrm{top}(t)$. □

The next lemma is proved in a straightforward way; it analyzes the structure of a rewrite step. We write $t \equiv C\langle\langle t_1, \ldots, t_n \rangle\rangle$ when either $t \equiv C[\![t_1, \ldots, t_n]\!]$ or $C \equiv \Box$ and $t \equiv t_1$.

Lemma 5. *1. If $s \to^o t$ then*

$$\begin{cases} s \equiv C[\![s_1, \ldots, s_n]\!], \\ t \equiv C'\langle\langle s_{i_1}, \ldots, s_{i_m} \rangle\rangle, \end{cases} \qquad \text{where } i_1, \ldots, i_m \in \{1, \ldots, n\}$$

for some $C, C', s_1, \ldots, s_n$, and either

(a) $n = m = 0$ and $s, t \in \mathcal{T}^\tau$;

(b) $n \neq 0$, $s \to^o t$ is destructive, $C' \equiv \Box$ and $t \equiv s_j$ for some $1 \leq j \leq n$; or

(c) $n \neq 0$, $s \to^o t$ is not destructive and $t \equiv C'[\![s_{i_1}, \ldots, s_{i_m}]\!]$.

2. If $s \to^i t$ then

$$\begin{cases} s \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!], \\ t \equiv C[s_1, \ldots, t_j, \ldots, s_n] \text{ and} \\ s_j \to t_j \end{cases}$$

for some $C, s_1, \ldots, s_n, t_j$, and either

(a) $t \equiv C[\![s_1, \ldots, t_j, \ldots, s_n]\!]$; or

(b) $s_j \to t_j$ *is destructive,* t_j *is a principal subterm of* s_j*,*

$$\begin{cases} t_j \equiv C'[\![u_1, \ldots, u_l]\!], \\ t \equiv C''[\![s_1, \ldots, u_1, \ldots, u_l, \ldots, s_n]\!] \end{cases}$$

and $C'' \equiv C[\Box, \ldots, C', \ldots, \Box]$ *for some* $C', u_1, \ldots, u_l$*.*

The following lemma can be proved without any difficulties.

Lemma 6. *Suppose that* $C[\![s_1, \ldots, s_n]\!] \to^o C'\langle\!\langle s_{i_1}, \ldots, s_{i_m}\rangle\!\rangle$ *and* $C : X_1 \times \cdots \times X_n \to Y$*. Then for any* $t_1, \ldots, t_n$ *such that* $X_i = X_j \wedge s_i \equiv s_j$ *implies* $t_i \equiv t_j$ *(for any* $1 \leq i, j \leq n$*), we have a rewrite step* $C[t_1, \ldots, t_n] \to^o C'[t_{i_1}, \ldots, t_{i_m}]$*.*

Before presenting the actual definition of our top-down labelling, we explain the idea of the top-down labelling. Note that our function symbols are divided into the three categories: $\mathcal{F}^0$, $\mathcal{F}^1$ and $\mathcal{F}^2$. By our labelling, the function symbols from $\mathcal{F}^1$ or $\mathcal{F}^2$ remain unlabelled, and the function symbols from $\mathcal{F}^0$ are labelled with 1 or 2.

Suppose a function symbol from $\mathcal{F}^0$ occurs in a term. It is not hard to observe that every term can be uniquely partitioned by well-sorted parts. If the function symbol occurs in a well-sorted part of top sort 1 or 2, then we label the function symbol with 1 or 2, respectively. The difficult case is when the function symbol belongs to a well-sorted part of top sort 0. In this case, all symbols in a well-sorted part of top sort 0 are uniformly labelled with 1 or 2, according to with which well-sorted part of top sort 1 or 2 the well-sorted part of sort 0 might be connected.

To explain the last case more precisely, suppose $t \equiv C[\![t_1, \ldots, t_n]\!]$ is a subterm of a term s and C is a well-sorted part in s. Let us call the "connection" from the position of t_i (in s) and the position of t (in s) a X_i–Y connection when $C : X_1 \times \cdots \times X_n \to Y$. So, in each well-sorted parts in a term, there are connections from the positions of its principal subterms to its root position. Now, observe that for any term t with a top sort X, its principal subterm t_i can be lifted (by a destructive rewrite step) only when t_i is placed at the hole of sort X. Thus, only X–X connections possibly eliminated (for $X \in \{0, 1, 2\}$). We have to label a well-sorted part of top sort 0 with 1 or 2 beforehand for the case it is used as a part of a well-sorted part of top sort 1 or 2. Therefore, we decide the label of a well-sorted part C of top sort 0 as follows. Starting from the root position of C, we follow up connections in each well-sorted parts above it; on the way, we skip 0–0, 1–1 and 2–2 connections, and if we eventually encounter a 0–X ($X = 1$ or 2) connection then we decide the label X. Figure 1 is an example of the labelling—the function symbols from $\mathcal{F}^0$ placed in white parts will be labelled with 1, and those placed in black parts will be labelled with 2. There, we consider function symbols $\mathcal{F}^0 = \{d : 0 \times 0 \to 0, e : 0 \to 0, a : 0\}$, $\mathcal{F}^1 = \{f : 0 \times 1 \to 1, g : 1 \times 1 \to 1, h : 0 \to 1, k : 1 \to 1, b : 1\}$, $\mathcal{F}^2 = \{E : 0 \times 2 \to 2, F : 2 \times 2 \to 2, G : 0 \to 2, H : 2 \to 2, A : 2\}$ and a term $f(e(E(a, H(d(x, e(g(k(a), b)))))), h(F(H(H(e(y))), G(e(g(h(e(A)), k(k(a)))))))))$.

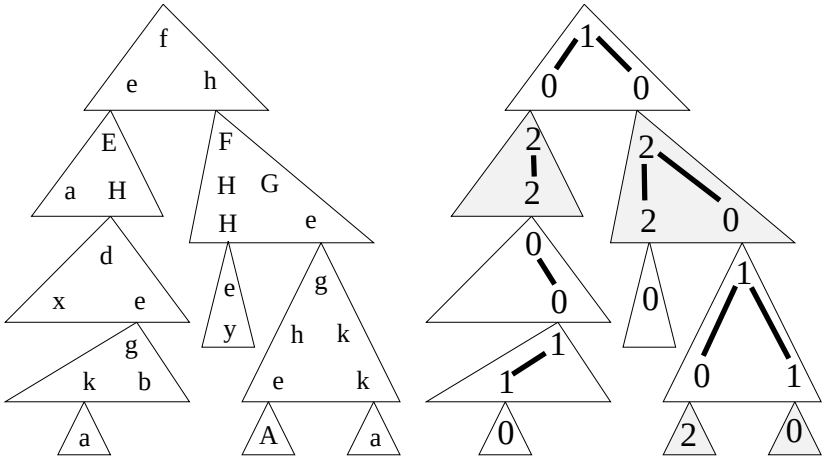

Fig. 1. The labelling of a term

Now, we present these considerations in a formal way. The set of labelled function symbols is introduced like this. For each $f \in \mathcal{F}^0$ and $X \in \{1,2\}$, f_X is a new function symbol—f labelled with X. For $X \in \{1,2\}$, let $\mathrm{lab}_X(\mathcal{F}^0) = \{f_X \mid f \in \mathcal{F}^0\}$. Elements of the set $\bigcup_{X\in\{1,2\}} \mathrm{lab}_X(\mathcal{F}^0)$ are called *labelled symbols*; we assume that these are fresh function symbols, i.e. $\mathrm{lab}_1(\mathcal{F}^0)$, $\mathrm{lab}_2(\mathcal{F}^0)$ and $\mathcal{F}$ are mutually disjoint. And finally, let $\mathrm{lab}(\mathcal{F}) = \bigcup_{X\in\{1,2\}}(\mathrm{lab}_X(\mathcal{F}^0) \cup \mathcal{F}^X)$.

Terms built from $\mathrm{lab}(\mathcal{F})$ and $\mathcal{V}$ are called *labelled terms*; those from $\mathcal{F}$ and $\mathcal{V}$ are *unlabelled*. For a term t and $X \in \{1,2\}$, $\mathrm{lab}_X(t)$ is the term obtained from t by labelling all symbols in $\mathcal{F}^0$ with X. This convention is also adapted for contexts.

Example 2. In Example 1, $\mathrm{lab}_1(\mathcal{F}^0) = \{a_1, b_1, c_1\}$ and $\mathrm{lab}_2(\mathcal{F}^0) = \{a_2, b_2, c_2\}$. We also have $\mathrm{lab}_1(f(g(a), c)) \equiv f(g(a_1), c_1)$ and $\mathrm{lab}_2(F(c, g(b))) \equiv F(c_2, g(b_2))$.

Two kinds of top-down labelling ∇_1 and ∇_2 are defined simultaneously as follows:

Definition 7. Let t be an unlabelled term such that $t \equiv C[\![t_1, \ldots, t_n]\!]$ with $C : X_1 \times \cdots \times X_n \to Y$. Labelled terms $\nabla_X t$ ($X \in \{1,2\}$) are defined by

$$\nabla_X t \equiv \begin{cases} \mathrm{lab}_X(C)[\nabla_X t_1, \ldots, \nabla_X t_n] & \text{if } Y \in \{0, X\}, \\ \mathrm{lab}_Y(C)[\Phi_1 t_1, \ldots, \Phi_n t_n] & \text{otherwise,} \end{cases}$$

where $\Phi_j = \nabla_Y$ if $X_j = 0$, and $\Phi_j = \nabla_X$ otherwise i.e. $X_j = Y$ (thus it makes a Y–Y connection).

Example 3. In Example 1, we have $\nabla_1(a) \equiv a_1$ and $\nabla_2(F(a, f(b, c))) \equiv F(a_2, f(b_1, c_2))$.

Lemma 8. *Suppose that* $C[\![s_1,\ldots,s_n]\!] \to^o s_j$ *is a destructive rewrite step. If* $\nabla_X(C[\![s_1,\ldots,s_n]\!]) \equiv C^*[\Phi_1 s_1,\ldots,\Phi_n s_n]$, *where* $\Phi_1,\ldots,\Phi_n \in \{\nabla_1,\nabla_2\}$, $C^* \equiv \mathrm{lab}_{\mathrm{top}(C)}C$ *if* $\mathrm{top}(C) \in \{1,2\}$, *and* $C^* \equiv \mathrm{lab}_X(C)$ *otherwise, then* $\Phi_j = \nabla_X$.

Proof. Suppose $C : X_1 \times \cdots \times X_n \to Y$. Then it suffices to show $X_j = Y$. By Lemma 4, the destructive step is an application of a collapsing rule. Let the rule be $C'[x] \to x$. But then, $X_j = \mathrm{top}(x) = \mathrm{top}(C'[x]) = \mathrm{top}(C[\![s_1,\ldots,s_n]\!]) = Y$. □

We define the labelled TRS of $\mathcal{R}$ like this:

$$\mathrm{lab}_X(\mathcal{R}) = \{\mathrm{lab}_X(l) \to \mathrm{lab}_X(r) \mid l \to r \in \Theta(\mathcal{R}^\tau \restriction (\mathcal{T}^0 \cup \mathcal{T}^X))\}$$

for $X \in \{1,2\}$, and

$$\mathrm{lab}(\mathcal{R}) = \mathrm{lab}_1(\mathcal{R}) \cup \mathrm{lab}_2(\mathcal{R}).$$

Then, in Lemma 9, it will be shown that the labelling ∇_1 and ∇_2 map every rewrite step of $\mathcal{R}$ to that of $\mathrm{lab}(\mathcal{R})$.

Example 4. Let

$$\mathcal{R}\begin{cases} f(x,y) \to x & (r1)\\ f(x,y) \to f(x,g(y)) & (r2)\\ g(x) \to h(x) & (r3)\\ F(g(x),x) \to F(x,g(x)) & (r4)\\ F(h(x),x) \to F(x,h(x)) & (r5), \end{cases} \qquad \tau\begin{cases} f : 1\times 0 \to 1\\ g : 0 \to 0\\ h : 0 \to 0\\ F : 0\times 0 \to 2. \end{cases}$$

Then τ is consistent with $\mathcal{R}$, and we have

$$\mathrm{lab}(\mathcal{R})\begin{cases} f(x,y) \to x\\ f(x,y) \to f(x,g_1(x))\\ g_1(x) \to h_1(x)\\ g_2(x) \to h_2(x)\\ F(g_2(x),x) \to F(x,g_2(x))\\ F(h_2(x),x) \to F(x,h_2(x)). \end{cases}$$

Lemma 9. *For any* $X \in \{1,2\}$, *we have* $s \to_{\mathcal{R}} t$ *if and only if* $\nabla_X s \to_{\mathrm{lab}(\mathcal{R})} \nabla_X t$.

Proof. ($\Leftarrow$) It suffices to erase all labels. ($\Rightarrow$) We suppose $s \to_{\mathcal{R}} t$ and show that for any $X \in \{1,2\}$, $\nabla_X s \to_{\mathrm{lab}(\mathcal{R})} \nabla_X t$. Our proof proceeds by induction on the rank of s. Base step is obvious. Suppose that $\mathrm{rank}(s) > 1$. We distinguish all possible cases according to Lemma 5:

1. $s \to^o t$. Then $s \equiv C[\![s_1,\ldots,s_n]\!]$ and $t \equiv C'\langle\!\langle s_{i_1},\ldots,s_{i_m}\rangle\!\rangle$ $(i_1,\ldots,i_m \in \{1,\ldots,n\})$ for some $C, C', s_1,\ldots,s_n$. But then, on the base of Lemma 6 and Lemma 8, one can easily show that $\nabla_X s \to_{\mathrm{lab}(\mathcal{R})} \nabla_X t$.
2. $s \to^i t$. Then $s \equiv C[\![s_1,\ldots,s_j,\ldots,s_n]\!]$, $t \equiv C[s_1,\ldots,t_j,\ldots,s_n]$ and $s_j \to t_j$ for some $C, s_1,\ldots,s_n,t_j$. If $t \equiv C[\![s_1,\ldots,t_j,\ldots,s_n]\!]$, then $\nabla_X s \to_{\mathrm{lab}(\mathcal{R})} \nabla_X t$ immediately follows from induction hypothesis. Otherwise, suppose $s_j \to t_j$ is destructive, t_j is a principal subterm of s_j, $t_j \equiv C'[\![u_1,\ldots,u_l]\!]$, $t \equiv C''[\![s_1,\ldots,u_1,\ldots,u_l,\ldots,s_n]\!]$ and $C'' \equiv C[\Box,\ldots,C',\ldots,\Box]$ for some $C', u_1,\ldots,u_l$.

(a) $\mathrm{top}(C) \in \{0, X\}$.

$$\begin{aligned}\nabla_X s &\equiv \mathrm{lab}_X(C)[\nabla_X s_1, \ldots, \nabla_X s_n]\\ \nabla_X t &\equiv \mathrm{lab}_X(C'')[\nabla_X s_1, \ldots, \nabla_X u_1, \ldots, \nabla_X u_l, \ldots, \nabla_X s_n]\\ &\equiv \mathrm{lab}_X(C)[\nabla_X s_1, \ldots, \mathrm{lab}_X(C')[\nabla_X u_1, \ldots, \nabla_X u_l], \ldots, \nabla_X s_n]\\ &\equiv \mathrm{lab}_X(C)[\nabla_X s_1, \ldots, \nabla_X t_j, \ldots, \nabla_X s_n].\end{aligned}$$

Hence, $\nabla_X s \to_{\mathrm{lab}(\mathcal{R})} \nabla_X t$ by the induction hypothesis.

(b) $\mathrm{top}(C) = Z \notin \{0, X\}$. Let $C : X_1 \times \cdots \times X_n \to Z$. Then,

$$\begin{aligned}\nabla_X s &\equiv \mathrm{lab}_Z(C)[\Phi_1 s_1, \ldots, \Phi_n s_n],\\ \nabla_X t &\equiv \mathrm{lab}_Z(C'')[\Phi_1 s_1, \ldots, \Phi'_1 u_1, \ldots, \Phi'_l u_l, \ldots, \Phi_n s_n]\\ &\equiv \mathrm{lab}_Z(C)[\Phi_1 s_1, \ldots, \mathrm{lab}_Z(C')[\Phi'_1 u_1, \ldots, \Phi'_l u_l], \ldots, \Phi_n s_n].\end{aligned}$$

i. $\mathrm{top}(C') = 0$. Then $C' : 0 \times \cdots \times 0 \to 0$, and hence $\Phi'_i = \Phi_j$ for all $i = 1, \ldots, l$. Also, $X_j = \mathrm{top}(C') = 0$, so $\Phi_j = \nabla_Z$. Therefore,

$$\begin{aligned}\nabla_X t &\equiv \mathrm{lab}_Z(C)[\Phi_1 s_1, \ldots, \mathrm{lab}_Z(C')[\nabla_Z u_1, \ldots, \nabla_Z u_l], \ldots, \Phi_n s_n]\\ &\equiv \mathrm{lab}_Z(C)[\Phi_1 s_1, \ldots, \Phi_j(C'[u_1, \ldots, u_l]), \ldots, \Phi_n s_n]\\ &\equiv \mathrm{lab}_Z(C)[\Phi_1 s_1, \ldots, \Phi_j t_j, \ldots, \Phi_n s_n].\end{aligned}$$

ii. $\mathrm{top}(C') = Z$. Then $C' : Y_1 \times \cdots \times Y_l \to Z$, $C'' : X_1 \times \cdots \times X_{j-1} \times Y_1 \times \cdots \times Y_l \times X_{j+1} \times \cdots \times X_n \to Z$, $X_j = Z$, and $\Phi_j = \nabla_X$. Therefore,

$$\begin{aligned}\nabla_X t &\equiv \mathrm{lab}_Z(C)[\Phi_1 s_1, \ldots, \mathrm{lab}_Z(C')[\Phi'_1 u_1, \ldots, \Phi'_l u_l], \ldots, \Phi_n s_n]\\ &\equiv \mathrm{lab}_Z(C)[\Phi_1 s_1, \ldots, \nabla_X(C'[u_1, \ldots, u_l]), \ldots, \Phi_n s_n]\\ &\equiv \mathrm{lab}_Z(C)[\Phi_1 s_1, \ldots, \Phi_j t_j, \ldots, \Phi_n s_n].\end{aligned}$$

Hence in both cases, $\nabla_X s \to_{\mathrm{lab}(\mathcal{R})} \nabla_X t$ follows from the induction hypothesis.

□

4 Composable properties

For a property ϕ of TRSs and a TRS $\mathcal{R}$, we write $\phi(\mathcal{R})$ when ϕ holds for $\mathcal{R}$.

Definition 10. Let ϕ be a property of TRSs. Then, for a given naive sort attachment τ, the property ϕ is said to be *τ-composable* if for any TRS $\mathcal{R}$ such that τ is consistent with $\mathcal{R}$, $\phi(\mathcal{R}_1) \wedge \phi(\mathcal{R}_2)$ implies $\phi(\mathcal{R})$, where $\langle \mathcal{R}_1, \mathcal{R}_2 \rangle$ is the decomposition of $\mathcal{R}$ by τ.

Proposition 11. *Let ϕ be a property of TRSs, $\mathcal{R}$ a TRS, τ a naive sort attachment consistent with $\mathcal{R}$ and $\langle \mathcal{R}_1, \mathcal{R}_2 \rangle$ the decomposition of $\mathcal{R}$ by τ. If*

1. *ϕ is preserved under the union of $\mathrm{lab}_1(\mathcal{R})$ and $\mathrm{lab}_2(\mathcal{R})$, and*
2. *$\phi(\mathrm{lab}(\mathcal{R}))$ implies $\phi(\mathcal{R})$,*

then $\phi(\mathcal{R}_1) \wedge \phi(\mathcal{R}_2)$ implies $\phi(\mathcal{R})$.

Proof. Suppose that $\phi(\mathcal{R}_1) \wedge \phi(\mathcal{R}_2)$. Then, since $\mathrm{lab}_X(\mathcal{R})$ is a renaming of $\mathcal{R}_X$ ($X \in \{1,2\}$), we have $\phi(\mathrm{lab}_1(\mathcal{R})) \wedge \phi(\mathrm{lab}_2(\mathcal{R}))$. This implies $\phi(\mathrm{lab}_1(\mathcal{R}) \cup \mathrm{lab}_2(\mathcal{R}))$ by the first assumption; hence $\phi(\mathrm{lab}(\mathcal{R}))$ by definition. Therefore, $\phi(\mathcal{R})$ by the second assumption. □

We are now going to give various composable properties on the base of modularity results for direct sums of TRSs. For this, let us review some basic properties of TRSs.

A term t is said to be *terminating* if there is no infinite sequence $t \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} t_3 \to_{\mathcal{R}} \cdots$ of terms starting from t. $\mathcal{R}$ is *terminating* if every term is terminating. Terms t_1 and t_2 are said to be *joinable* if there exists some term s such that $t_1 \stackrel{*}{\to}_{\mathcal{R}} s \stackrel{*}{\leftarrow}_{\mathcal{R}} t_2$. A term t is said to be *confluent* (*locally confluent*) if terms t_1 and t_2 are joinable whenever $t_1 \stackrel{*}{\leftarrow}_{\mathcal{R}} t \stackrel{*}{\to}_{\mathcal{R}} t_2$ (whenever $t_1 \leftarrow_{\mathcal{R}} t \to_{\mathcal{R}} t_2$, respectively). $\mathcal{R}$ is confluent (locally confluent) if every term is confluent (locally confluent, respectively). A term s is *normal form* if there is no term t such that $s \to_{\mathcal{R}} t$. $\mathcal{R}$ has *unique normal forms* (or is *UN*) if $s \stackrel{*}{\leftrightarrow}_{\mathcal{R}} t$ implies $s \equiv t$ for any normal forms s and t; $\mathcal{R}$ has *unique normal forms with respect to reduction* (or is $UN^{\rightarrow}$) if $t_1 \stackrel{*}{\leftarrow}_{\mathcal{R}} s \stackrel{*}{\to}_{\mathcal{R}} t_2$ implies $t_1 \equiv t_2$ for any term s and normal forms t_1 and t_2; $\mathcal{R}$ has the *normal form property* (or is *NF*) if $s \stackrel{*}{\leftrightarrow}_{\mathcal{R}} t$ implies $s \stackrel{*}{\to}_{\mathcal{R}} t$ for any term s and a normal form t. For the relation between these properties, we refer the reader to [7].

We first show sufficient conditions for the composability of termination.

Theorem 12. *Let $\mathcal{R}$ be a TRS and $\langle \mathcal{R}_1, \mathcal{R}_2 \rangle$ a decomposition of $\mathcal{R}$ by a naive sort attachment. Suppose that either one of the following conditions are satisfied:*

1. *$\mathcal{R}$ is non-collapsing;*
2. *$\mathcal{R}$ is non-duplicating;*
3. *one of $\mathcal{R}_1$ or $\mathcal{R}_2$ is non-collapsing and non-duplicating.*

Then, $\mathcal{R}$ is terminating whenever $\mathcal{R}_1$ and $\mathcal{R}_2$ are terminating.

Proof. Let τ be a naive sort attachment consistent with $\mathcal{R}$ such that $\mathcal{R}_X = \Theta(\mathcal{R}^\tau \restriction (\mathcal{T}^0 \cup \mathcal{T}^X))$ for each $X \in \{1,2\}$. It suffices to show that conditions 1, 2 of Proposition 11 are satisfied. Since $\mathrm{lab}_X(\mathcal{R})$ is a renaming of $\mathcal{R}_X$, $\mathrm{lab}_1(\mathcal{R})$ and $\mathrm{lab}_2(\mathcal{R})$ are terminating TRSs. By definition, the sets of function symbols that appear in $\mathrm{lab}_1(\mathcal{R})$ and $\mathrm{lab}_2(\mathcal{R})$ are disjoint. Also, it is clear that if $\mathcal{R}$ is non-collapsing (non-duplicating) then $\mathrm{lab}(\mathcal{R}) = \mathrm{lab}_1(\mathcal{R}) \cup \mathrm{lab}_2(\mathcal{R})$ is also non-collapsing (non-duplicating, respectively); and if $\mathcal{R}_X$ is non-collapsing and non-duplicating then so is $\mathrm{lab}(\mathcal{R}_X)$. Therefore, by the modularity of the termination for direct sums of TRSs [8][12], $\mathrm{lab}(\mathcal{R})$ is terminating whenever $\mathrm{lab}_1(\mathcal{R})$ and $\mathrm{lab}_2(\mathcal{R})$ are terminating. To show the condition 2, suppose that $\mathcal{R}$ is not terminating, i.e. there exists an infinite sequence $t \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} t_3 \to_{\mathcal{R}} \cdots$. Then, by Lemma 9, $\nabla_1 t \to_{\mathrm{lab}(\mathcal{R})} \nabla_1 t_1 \to_{\mathrm{lab}(\mathcal{R})} \nabla_1 t_2 \to_{\mathrm{lab}(\mathcal{R})} \nabla_1 t_3 \to_{\mathrm{lab}(\mathcal{R})} \cdots$. Thus, $\mathrm{lab}(\mathcal{R})$ is not terminating. □

Theorem 12 with condition 1 or 2 captures an application of the persistency of termination; see [17].

To the contrary, confluence is a τ-composable property for a naive sort attachment τ.

Theorem 13. *For any naive sort attachment τ, confluence is τ-composable.*

Proof. We again show that conditions 1, 2 of Proposition 11 are satisfied. For the condition 1, let us mention only the modularity of the confluence for direct sums of TRSs [14]. To show the condition 2, suppose that $\mathrm{lab}(\mathcal{R})$ is confluent and $t_1 \overset{*}{\leftarrow}_{\mathcal{R}} t \overset{*}{\rightarrow}_{\mathcal{R}} t_2$. Then, by Lemma 9, $\nabla_1 t_1 \overset{*}{\leftarrow}_{\mathrm{lab}(\mathcal{R})} \nabla_1 t \overset{*}{\rightarrow}_{\mathrm{lab}(\mathcal{R})} \nabla_1 t_2$. By the assumption, $\nabla_1 t_1 \overset{*}{\rightarrow}_{\mathrm{lab}(\mathcal{R})} s \overset{*}{\leftarrow}_{\mathrm{lab}(\mathcal{R})} \nabla_1 t_2$. Erasing labels, we clearly have $t_1 \overset{*}{\rightarrow}_{\mathcal{R}} s' \overset{*}{\leftarrow}_{\mathcal{R}} t_2$ for some s'. Hence, $\mathcal{R}$ is confluent. Thus, confluence of $\mathrm{lab}(\mathcal{R})$ implies that of $\mathcal{R}$. □

Theorem 13 captures an application of the persistency of confluence; see [1].

A rewrite rule $l \to r$ is said to be *left-linear* if every variable occurs at most once in l. A TRS is left-linear if all rules are left-linear. For left-linear TRSs, we obtain other conditions for the composability of termination.

A TRS $\mathcal{R}$ is said to be *consistent with respect to reduction* (*r-consistent*, for short) if there exists no term t such that $x \overset{*}{\leftarrow}_{\mathrm{lab}(\mathcal{R})} t \overset{*}{\rightarrow}_{\mathrm{lab}(\mathcal{R})} y$ for distinct variables x and y.

Theorem 14. *Let $\mathcal{R}$ be a TRS and $\langle \mathcal{R}_1, \mathcal{R}_2 \rangle$ a decomposition of $\mathcal{R}$ by a naive sort attachment. Suppose that either one of the following conditions are satisfied:*

1. *$\mathcal{R}$ is left-linear and r-consistent;*
2. *one of $\mathcal{R}_1$ or $\mathcal{R}_2$ is left-linear and confluent, and the other is non-collapsing.*

Then, $\mathcal{R}$ is terminating whenever $\mathcal{R}_1$ and $\mathcal{R}_2$ are terminating.

Proof. We again show that conditions 1, 2 of Proposition 11 are satisfied. For the condition 1, we only show that r-consistency of $\mathcal{R}$ implies that of $\mathrm{lab}(\mathcal{R})$; then the condition easily follows from the modularity of the termination for direct sums of TRSs [13] [15]. If $\mathrm{lab}(\mathcal{R})$ is not r-consistent, then there exists a labelled term t and distinct variables x and y such that $x \overset{*}{\leftarrow}_{\mathrm{lab}(\mathcal{R})} t \overset{*}{\rightarrow}_{\mathrm{lab}(\mathcal{R})} y$. Erasing labels, $x \overset{*}{\leftarrow}_{\mathcal{R}} t' \overset{*}{\rightarrow}_{\mathcal{R}} y$ for some t'. Thus, $\mathcal{R}$ is not r-consistent. The condition 2 can be shown similarly to the proof of Theorem 12. □

We next examine τ-composable properties related to normal forms for a naive sort attachment τ. For this, a lemma below, which easily follows from Lemma 9, is useful.

Lemma 15. *For any $X \in \{1, 2\}$, a term s is a normal form if and only if $\nabla_X s$ is a normal form.*

Theorem 16. *For any naive sort attachment τ, UN is τ-composable.*

Proof. We again show that conditions 1, 2 of Proposition 11 are satisfied. For the condition 1, let us mention only the modularity of the UN for direct sums of TRSs [7]. To show the condition 2, suppose that lab($\mathcal{R}$) is UN and $s \overset{*}{\leftrightarrow}_{\mathcal{R}} t$ for normal forms s and t. Then $\nabla_1 s$ and $\nabla_1 t$ are normal forms by Lemma 15 and $\nabla_1 s \overset{*}{\leftrightarrow}_{\mathrm{lab}(\mathcal{R})} \nabla_1 t$ by Lemma 9. Hence $\nabla_1 s \equiv \nabla_1 t$ by the assumption. Erasing labels, we clearly have $s \equiv t$. □

Theorem 17. *Let $\mathcal{R}$ be a left-linear TRS and $\langle \mathcal{R}_1, \mathcal{R}_2 \rangle$ a decomposition of $\mathcal{R}$ by a naive sort attachment. Then, $\mathcal{R}$ is UN$^{\rightarrow}$ (NF) whenever $\mathcal{R}_1$ and $\mathcal{R}_2$ are UN$^{\rightarrow}$ (NF, respectively).*

Proof. We again show that conditions 1, 2 of Proposition 11 are satisfied. For the condition 1, let us mention only the modularity of the UN$^{\rightarrow}$and NF for direct sums of left-linear TRSs [6][9] and the fact that the left-linearity of $\mathcal{R}$ implies that of lab($\mathcal{R}$). The condition 2 for UN$^{\rightarrow}$: Suppose that lab($\mathcal{R}$) is UN$^{\rightarrow}$and $t_1 \overset{*}{\leftarrow}_{\mathcal{R}} s \overset{*}{\rightarrow}_{\mathcal{R}} t_2$. with t_1, t_2 normal forms. Then $\nabla_1 t_1, \nabla t_2$ are normal forms by Lemma 15 and $\nabla_1 t_1 \overset{*}{\leftarrow}_{\mathrm{lab}(\mathcal{R})} \nabla_1 s \overset{*}{\rightarrow}_{\mathrm{lab}(\mathcal{R})} \nabla_1 t_2$ by Lemma 9. Hence $\nabla t_1 \equiv \nabla t_2$ by the assumption. Erasing labels, we have $t_1 \equiv t_2$. The condition 2 for NF: Suppose that lab($\mathcal{R}$) is NF and $s \overset{*}{\leftrightarrow}_{\mathcal{R}} t$ with t normal forms. Then $\nabla_1 t$ is a normal from by Lemma 15 and $\nabla_1 s \overset{*}{\leftrightarrow}_{\mathrm{lab}(\mathcal{R})} \nabla_1 t$ by Lemma 9. Hence $\nabla_1 s \overset{*}{\rightarrow}_{\mathrm{lab}(\mathcal{R})} \nabla_1 t$ by the assumption. Erasing labels, we have $s \overset{*}{\rightarrow}_{\mathcal{R}} t$. □

One can obtain similar τ-composability results for a naive sort attachment τ, on the base of other modularity results for direct sums of TRSs (e.g. [3], [6]), although we will not mention them further.

We end this section with some examples.

Example 5. Let

$$\mathcal{R} \begin{cases} f(x, g(b, a, y)) \to f(x, g(c, y, y)) & (r1) \\ f(x, y) \to y & (r2) \\ a \to b & (r3) \\ F(c, x) \to F(a, x) & (r4), \end{cases} \qquad \tau \begin{cases} f : 0 \times 1 \to 1 \\ g : 0 \times 0 \times 0 \to 1 \\ a, b, c : 0 \\ F : 0 \times 0 \to 2. \end{cases}$$

Note that one can not directly use the recursive path ordering to show the termination of $\mathcal{R}$ because of $(r1)$, nor the lexicographic path ordering because of $(r1), (r3), (r4)$.

It is easy to see that τ is consistent with $\mathcal{R}$. The decomposition of $\mathcal{R}$ by τ is the pair of $\mathcal{R}_1 = \{(r1), (r2), (r3)\}$ and $\mathcal{R}_2 = \{(r3), (r4)\}$.

- Put $a > b > c$. Then $\mathcal{R}_1$ is terminating by the lexicographic path ordering.
- Put $c > a > b$. Then, $\mathcal{R}_2$ is terminating by the lexicographic path ordering.
- $\mathcal{R}_2$ is non-collapsing and non-duplicating.

Hence, by Theorem 12, we conclude that $\mathcal{R}$ is terminating.

Example 6. In Example 4, observe that well known results, e.g. those appearing in [4], [5] or [14], are not directly helpful to infer the confluence of $\mathcal{R}$. The decomposition of $\mathcal{R}$ by τ is the pair of $\mathcal{R}_1 = \{(r1), (r2), (r3)\}$ and $\mathcal{R}_2 = \{(r3), (r4), (r5)\}$.

- Since $\mathcal{R}_1$ is orthogonal, it is confluent.
- Since $\mathcal{R}_2$ is terminating and its critical pairs are joinable, it is confluent [5].

Hence, by Theorem 13, we conclude that $\mathcal{R}$ is confluent.

Example 7. Let

$$\mathcal{R}\begin{cases} f(g(x,a,b)) \to x & (r1) \\ g(f(h(c,d)),x,y) \to h(e(x),e(y)) & (r2) \\ e(a) \to c & (r3) \\ e(b) \to d & (r4) \\ F(a,x,x) \to F(x,e(x),e(x)) & (r5), \end{cases} \qquad \tau\begin{cases} f : 1 \to 1 \\ g : 1 \times 0 \times 0 \to 1 \\ h : 0 \times 0 \to 1 \\ a,b,c,d : 0 \\ e : 0 \to 0 \\ F : 0 \times 0 \times 0 \to 2. \end{cases}$$

It is easy to see that τ is consistent with $\mathcal{R}$. The decomposition of $\mathcal{R}$ by τ is the pair of $\mathcal{R}_1 = \{(r1), (r2), (r3), (r4)\}$ and $\mathcal{R}_2 = \{(r3), (r4), (r5)\}$.

- Put $g > h > e > c, d$. Then $\mathcal{R}_1$ is terminating by the lexicographic path ordering.
- If $\mathcal{R}_2$ is not terminating, then it is easy to see that the rewrite rule (r5) should be used infinitely many times. Observe that the only term that has the reduct a is a itself, so these rewrite steps by (r5) must be of the form $F(a,a,a) \to_{\mathcal{R}_2} F(a,e(a),e(a))$. But since $F(a,e(a),e(a))$ is terminating, there is no such infinite reduction sequence. Thus $\mathcal{R}_2$ is terminating.
- $\mathcal{R}_1$ is left-linear and parallel closed; hence confluent [4].
- $\mathcal{R}_2$ is non-collapsing.

Hence, by Theorem 14, we conclude that $\mathcal{R}$ is terminating.

Remark. We now compare our work and composability results in [11].

Firstly, let us recall definitions in [11]. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two TRSs and suppose that for $i \in \{1,2\}$ $\mathcal{F}_i$ is the set of function symbols in $\mathcal{R}_i$. Put $\mathcal{D}_i = \{\mathrm{root}(l) \mid l \to r \in \mathcal{R}_i\}$, $\mathcal{C}_i = \mathcal{F}_i \backslash \mathcal{D}_i$, $\mathcal{B} = \mathcal{F}_1 \cap \mathcal{F}_2$, $\mathcal{Q} = \{l \to r \in \mathcal{R}_1 \cup \mathcal{R}_2 \mid root(l) \in \mathcal{D}_1 \cap \mathcal{D}_2\}$ and $\mathcal{A}_i = \mathcal{F}_i \backslash \mathcal{B}$ $(i \in \{1,2\})$. Then $\mathcal{R}_1$ and $\mathcal{R}_2$ are said to be composable if (1) $\mathcal{C}_1 \cap \mathcal{D}_2 = \mathcal{C}_2 \cap \mathcal{D}_1 = \emptyset$, and (2) $\mathcal{Q} \subseteq \mathcal{R}_1 \cap \mathcal{R}_2$. Also, $\mathcal{R}_i$ is said to be layer-preserving if for any $l \to r \in \mathcal{R}_i$, $\mathrm{root}(l) \in \mathcal{A}_i$ implies $\mathrm{root}(r) \in \mathcal{A}_i$ $(i \in \{1,2\})$.

Now, Theorem 4.3.5 in [11] says that for any two composable layer-preserving confluent TRSs $\mathcal{R}_1$ and $\mathcal{R}_2$, their union is confluent. We claim that this result can not applied to Example 6. In Example 6, $\mathcal{D}_1 = \{f, g\}$, $\mathcal{C}_1 = \{h\}$, $\mathcal{D}_2 = \{F, g\}$ and $\mathcal{C}_2 = \{h\}$; hence $\mathcal{Q} = \{(r3)\}$, $\mathcal{B} = \{g, h\}$, $\mathcal{A}_1 = \{f\}$ and $\mathcal{A}_2 = \{F\}$. Thus, our system is composable. But $\mathcal{R}_1$ is not layer-preserving because of $(r1)$.

Next, Theorem 5.3.8 in [11] says that for any two composable terminating TRSs $\mathcal{R}_1$ and $\mathcal{R}_2$, their union is terminating if either

1. both $\mathcal{R}_1$ and $\mathcal{R}_2$ are layer-preserving;
2. both $\mathcal{R}_1$ and $\mathcal{R}_2$ are non-duplicating; or
3. one of $\mathcal{R}_1$ and $\mathcal{R}_2$ is both layer-preserving and non-duplicating.

We claim that this result can not applied to Example 7. In our example, $\mathcal{D}_1 = \{f, g, e\}$, $\mathcal{C}_1 = \{h, a, b, c, d\}$, $\mathcal{D}_2 = \{F, e\}$ and $\mathcal{C}_2 = \{a, b, c, d\}$; hence $\mathcal{Q} = \{(r3), (r4)\}$, $\mathcal{B} = \{a, b, c, d, e\}$, $\mathcal{A}_1 = \{f, g, h\}$ and $\mathcal{A}_2 = \{F\}$. Thus, our system is composable. But none of the conditions is satisfied, because of $(r1)$ and $(r5)$.

Remark. One can show the termination of $\mathcal{R}$ in Example 5 or 7 by using the semantic labelling technique[18]. However, unlike the case of the semantic labelling, it is decidable whether for a given finite TRS our method can be used; for, the number of naive sort attachments are finite, so we can get all possible decompositions by checking consistency.

Example 8. Let

$$\mathcal{R}\begin{cases} f(a) \to f(b) & (r1) \\ f(b) \to f(b) & (r2) \\ a \to c & (r3) \\ F(x) \to G(F(x), F(a)) & (r4), \end{cases} \quad \tau\begin{cases} f : 0 \to 1 \\ a, b, c : 0 \\ F : 0 \to 2 \\ G : 2 \times 2 \to 2. \end{cases}$$

It is easy to see that τ is consistent with $\mathcal{R}$. The decomposition of $\mathcal{R}$ by τ is the pair of $\mathcal{R}_1 = \{(r1), (r2), (r3)\}$ and $\mathcal{R}_2 = \{(r3), (r4)\}$.

- Since rewrite steps of $\mathcal{R}_1$ can not change the number of occurrences of f in terms, nontrivial $\overset{*}{\leftrightarrow}_{\mathcal{R}_1}$-equivalence classes are of form $\{f^n(a), f^n(b), f^n(c)\}$ for some $n(> 0)$ or $\{a, c\}$. Therefore, every $\overset{*}{\leftrightarrow}_{\mathcal{R}_1}$-equivalence class has a unique normal form in it. Thus, $\mathcal{R}_1$ is UN.
- Since $\mathcal{R}_2$ is orthogonal, it is confluent; hence it is UN.

Therefore, by Theorem 16, we conclude that $\mathcal{R}$ is UN.

5 Concluding remarks

We have proposed a decomposition by a naive sort attachment, and have shown that modular properties for direct sums are τ-composable for a naive sort attachment τ. Thus, we have presented conditions under which properties such as termination, confluence, etc., are τ-composable for a naive sort attachment τ.

Our composition does not impose the disjointness condition. Among the composability results for non-disjoint TRSs, our result is based on a sort attachment, while results in [11] are based on syntactical conditions and those in [10] are based on the constructor discipline.

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