Inductive Theorems for Higher-Order Rewriting*

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Abstract. Based on the simply typed term rewriting framework, inductive reasoning in higher-order rewriting is studied. The notion of higher-order inductive theorems is introduced to reflect higher-order feature of simply typed term rewriting. Then the inductionless induction methods in first-order term rewriting are incorporated to verify higher-order inductive theorems. In order to ensure that higher-order inductive theorems are closed under contexts, the notion of higher-order sufficient completeness is introduced. Finally, the decidability of higher-order sufficient completeness is discussed.

1 Introduction

In a framework based on equational logic such as functional programming and algebraic specification, properties of a particular recursive data structure can be verified by induction. Once an appropriate translation from a specification to a set of equational axioms has been established, inductive properties of the specification can be proved using term rewriting. Automatically proving inductive theorems by term rewriting has been widely investigated since earlier works [5, 8, 17].

Higher-order functions are ubiquitous in functional programming. However, they can not be expressed directly in usual term rewriting, and thus various frameworks of higher-order extension of term rewriting have been proposed in the literature. In most of these frameworks, termination proof is difficult to cope with, and consequently only few attempts have been made to incorporate higher-order functions to inductive theorem proving.

Simply typed term rewriting proposed by Yamada [21] is a simple extension of first-order term rewriting which incorporates higher-order functions. Equational specification using higher-order functions, like functional programs, are naturally expressed in this framework. In contrast to the usual higher-order term rewriting frameworks [6, 9, 15], simply typed term rewriting dispenses with bound variables. In this respect, simply typed term rewriting reflects limited

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higher-order features. On the other hand, simply typed term rewriting framework is succinct and theoretically much easier to deal with.

In this paper, we study inductive theorem proving in higher-order rewriting based on simply typed term rewriting framework. We show that there is an equation that is intuitively inductively valid, but is inductively invalid in terms of first-order inductive reasoning. We propose the notion of higher-order inductive theorem that reflects higher-order feature more reasonably. Then inductionless induction methods in first-order term rewriting are incorporated to simply typed term rewriting to verify higher-order inductive theorems. It also turns out that a higher-order inductive theorem is not always closed under contexts. To ensure that higher-order inductive theorems are closed under contexts, we introduce the notion of higher-order sufficient completeness. The decidability of higher-order sufficient completeness for a subclass of simply typed term rewriting systems is shown by introducing the notion of higher-order quasi-reducibility.

2 Preliminaries

In this section, we first fix our notation on abstract reduction systems, and recall a result about equivalence of abstract reduction systems. Based on this equivalence condition, we review notions and results on inductive theorem proving in first-order term rewriting. We assume the reader to be familiar with abstract reduction systems and (first-order) term rewriting [4, 19].

2.1 Equivalence of Abstract Reduction Systems

An *abstract reduction system* (*ARS*, for short) is a pair $\langle A, \to \rangle$ of a set A and a binary relation $\to$ on A. The reflexive transitive closure and the equivalence closure of $\to$ are denoted by $\to^*$ and $\leftrightarrow^*$, respectively. An element $a \in A$ is *reducible* when $a \to b$ for some $b \in A$; otherwise, it is *normal* or in *normal form*. The sets of reducible and normal elements of an ARS $\mathcal{A}$ are denoted by $\mathrm{RED}(\mathcal{A})$ and $\mathrm{NF}(\mathcal{A})$, respectively. Elements $a \in A$ and $b \in A$ are *joinable* if $a \to^* c \,{}^*\!\leftarrow b$ for some $c \in A$. An ARS $\mathcal{A}$ is *weakly normalizing* ($\mathrm{WN}(\mathcal{A})$) if for any $a \in A$ there exists $b \in \mathrm{NF}(A)$ such that $a \to^* b$, is *terminating* or *strongly normalizing* ($\mathrm{SN}(\mathcal{A})$) if there is no infinite reduction sequence, and is *confluent* ($\mathrm{CR}(\mathcal{A})$) if $a, b \in A$ are joinable whenever $a \,{}^*\!\leftarrow c \to^* b$ for some $c \in A$.

The following proposition provides a sufficient condition for the equivalence of two ARSs, which is the basis of all our results on inductive theorem proving (c.f. [10, 20]).

Proposition 1 (equivalence condition for ARSs). Let $\mathcal{A}_1 = \langle A, \to_1 \rangle$ and $\mathcal{A}_2 = \langle A, \to_2 \rangle$ be ARSs such that $\to_1 \subseteq \to_2$. Suppose that the following conditions are satisfied: (1) $\mathrm{WN}(\mathcal{A}_1)$, (2) $\mathrm{CR}(\mathcal{A}_2)$, (3) $\mathrm{NF}(\mathcal{A}_1) \subseteq \mathrm{NF}(\mathcal{A}_2)$. Then $\leftrightarrow_1^* = \leftrightarrow_2^*$.

The third condition above is equivalent to $\mathrm{RED}(\mathcal{A}_2) \subseteq \mathrm{RED}(\mathcal{A}_1)$. It is often useful to consider a set $X \subseteq \mathrm{RED}(\mathcal{A}_1)$ of intermediate elements reachable from $\mathrm{RED}(\mathcal{A}_2)$.

Proposition 2 (inclusion of normal elements). Let $\mathcal{A}_1 = \langle A, \rightarrow_1 \rangle$ and $\mathcal{A}_2 = \langle A, \rightarrow_2 \rangle$ be ARSs such that $\rightarrow_1 \subseteq \rightarrow_2$. Suppose there exists a subset X of A such that (1) for any $a \in \mathrm{RED}(\mathcal{A}_2)$ there exists $b \in X$ such that $a \rightarrow_1^* b$, (2) $X \subseteq \mathrm{RED}(\mathcal{A}_1)$. Then $\mathrm{NF}(\mathcal{A}_1) \subseteq \mathrm{NF}(\mathcal{A}_2)$.

In this paper, we consider two instances of the proposition above in order to give criteria for the inclusion between two sets of normal forms. One is an abstraction of the use of coverset and the other corresponds sufficient completeness.

Proposition 3 (coverset in ARS). Let $\mathcal{A}_1 = \langle A, \rightarrow_1 \rangle$ and $\mathcal{A}_2 = \langle A, \rightarrow_2 \rangle$ be ARSs such that $\rightarrow_1 \subseteq \rightarrow_2$. Suppose there exists a collection $\mathcal{C}$ of subsets of A such that (1) for any $a \in \mathrm{RED}(\mathcal{A}_2)$ there exist $C \in \mathcal{C}$ and $b \in C$ such that $a \rightarrow_1^* b$, (2) $C \subseteq \mathrm{RED}(\mathcal{A}_1)$ for any $C \in \mathcal{C}$. Then $\mathrm{NF}(\mathcal{A}_1) \subseteq \mathrm{NF}(\mathcal{A}_2)$.

Proof. Take $X = \bigcup \mathcal{C}$ in Proposition 2. $\square$

Proposition 4 (sufficient completeness in ARS). Let $\mathcal{A}_1 = \langle A, \rightarrow_1 \rangle$ and $\mathcal{A}_2 = \langle A, \rightarrow_2 \rangle$ be ARSs such that $\rightarrow_1 \subseteq \rightarrow_2$. Suppose there exists a subset C of A such that (1) for any $a \in \mathrm{RED}(\mathcal{A}_2)$ there exists $b \in A \setminus C$ such that $a \rightarrow_1^* b$, (2) $A \setminus C \subseteq \mathrm{RED}(\mathcal{A}_1)$. Then $\mathrm{NF}(\mathcal{A}_1) \subseteq \mathrm{NF}(\mathcal{A}_2)$.

Proof. Take $X = A \setminus C$ in Proposition 2. $\square$

2.2 First-Order Inductive Theorem Proving

We denote by $\mathrm{T}(F, V)$ the set of *many-sorted terms* over a set F of sorted function symbols and the set V of sorted variables. An equation is a pair of terms of the same sort, written as $l \approx r$. The sets of variables occurring in a term t and in an equation $l \approx r$ are written as $\mathrm{V}(t)$ and $\mathrm{V}(l \approx r)$. A *ground* term is a term without variables. A substitution θ is said to be ground when $\theta(x)$ is ground for every variable in its *domain* $\mathrm{Dom}(\theta) = \{x \mid x \neq \theta(x)\}$. We put subscript $_g$ to denote ground terms, substitutions, and contexts, e.g. t_g, θ_g, and C_g.

Let $\mathcal{R} = \langle S, F, R \rangle$ be a *many-sorted term rewriting system* (*many-sorted TRS*, for short) where S is a set of sorts, F is a set of function symbols, and R is a set rewrite rules. An equation $l \approx r$ is an *inductive theorem* of $\mathcal{R}$ if $l\theta_g \leftrightarrow_{\mathcal{R}}^* r\theta_g$ holds for all ground substitutions θ_g such that $V(l \approx r) \subseteq \mathrm{Dom}(\theta_g)$. Obviously, $\langle \mathrm{T}(F, \emptyset), \rightarrow_{\mathcal{R}} \rangle$ forms an ARS. We denote $\mathrm{NF}(\langle \mathrm{T}(F, \emptyset), \rightarrow_{\mathcal{R}} \rangle)$, $\mathrm{WN}(\langle \mathrm{T}(F, \emptyset), \rightarrow_{\mathcal{R}} \rangle)$, etc. by $\mathrm{NF}_g(\mathcal{R})$, $\mathrm{WN}_g(\mathcal{R})$, etc., respectively. Proposition 1 gives a sufficient condition for an equation to be an inductive theorem of a many-sorted TRS. Let $\mathcal{R} = \langle S, F, R \rangle$ be a many-sorted TRS and $l \approx r$ an equation on $\mathrm{T}(F, V)$. In order to prove the equation $l \approx r$ to be an inductive theorem of $\mathcal{R}$, we often consider the extended TRS $\mathcal{R}' = \langle S, F, R \cup \{l \rightarrow r\} \rangle$, which we denote by $\mathcal{R} \cup \{l \rightarrow r\}$.

Proposition 5 (inductive theorem proving). Let $\mathcal{R}$ be a many-sorted TRS and $\mathcal{R}' = \mathcal{R} \cup \{l \rightarrow r\}$. Suppose the following conditions are satisfied: (1) $\mathrm{WN}_g(\mathcal{R})$, (2) $\mathrm{CR}_g(\mathcal{R}')$, (3) $\mathrm{NF}_g(\mathcal{R}) \subseteq \mathrm{NF}_g(\mathcal{R}')$. Then $l \approx r$ is an inductive theorem of $\mathcal{R}$.

Proof. Let $\mathcal{R} = \langle S, F, R \rangle$. Take $\mathcal{A}_1 = \langle\, \mathrm{T}(F, \emptyset), \rightarrow_{\mathcal{R}} \rangle$ and $\mathcal{A}_2 = \langle\, \mathrm{T}(F, \emptyset), \rightarrow_{\mathcal{R}'} \rangle$ in Proposition 1. ◻

To check condition (3) of the proposition efficiently, the notion of *coverset* is used. For a many-sorted TRS $\mathcal{R}$ and a term t, a set Θ of substitutions is a *coverset of substitutions* for t w.r.t. $\mathcal{R}$ if for any ground substitution θ_g satisfying $\mathrm{V}(t) \subseteq \mathrm{Dom}(\theta_g)$, there exists a substitution $\sigma \in \Theta$ and a ground substitution θ'_g such that $t\theta_g \rightarrow^*_{\mathcal{R}} t\sigma\theta'_g$.

Proposition 6 (inductive theorem proving with coverset). Let $\mathcal{R}$ be a many-sorted TRS and $\mathcal{R}' = \mathcal{R} \cup \{l \rightarrow r\}$. Suppose that the following conditions are satisfied: (1) $\mathrm{WN}_g(\mathcal{R})$, (2) $\mathrm{CR}_g(\mathcal{R}')$, (3) $l\sigma \in \mathrm{RED}(\mathcal{R})$ for all $\sigma \in \Theta$ where Θ is a coverset of substitutions for l w.r.t. $\mathcal{R}$. Then $l \approx r$ is an inductive theorem of $\mathcal{R}$.

Proof. We use Proposition 5. The inclusion $\mathrm{NF}_g(\mathcal{R}) \subseteq \mathrm{NF}_g(\mathcal{R}')$ follows from condition (3) by taking $\mathcal{C} = \{\mathrm{RED}_\sigma \mid \sigma \in \Theta\}$ in Proposition 3 where $\mathrm{RED}_\sigma = \{C_g[l\sigma\theta_g] \mid \mathrm{V}(l\sigma) \subseteq \mathrm{Dom}(\theta_g)\}$. ◻

Another condition is obtained by strengthening condition (1). Let us divide the set of function symbols into two disjoint sets as follows. We say a function symbol f is a *defined symbol* of a many-sorted TRS $\langle S, F, R \rangle$ if $f = \mathrm{root}(l)$ for some $l \rightarrow r \in R$; otherwise, f is a *constructor symbol*. The sets of defined and constructor symbols are denoted by F_{d} and F_{c}, respectively. A many-sorted TRS $\mathcal{R}$ is *sufficiently complete* ($\mathrm{SC}(\mathcal{R})$) if for any ground term s_g there exists a ground constructor term t_g such that $s_g \rightarrow^*_{\mathcal{R}} t_g$.

Proposition 7 (inductive theorem proving with SC). Let $\mathcal{R}$ be a many-sorted TRS and $\mathcal{R}' = \mathcal{R} \cup \{l \rightarrow r\}$. Suppose the following conditions are satisfied: (1) $\mathrm{SC}(\mathcal{R})$, (2) $\mathrm{CR}_g(\mathcal{R}')$, (3) l contains a defined symbol. Then $l \approx r$ is an inductive theorem of $\mathcal{R}$.

Proof. Let $\mathcal{R} = \langle S, F, R \rangle$. We use Proposition 5. By definition, $\mathrm{SC}(\mathcal{R})$ implies $\mathrm{WN}_g(\mathcal{R})$. By $\mathrm{SC}(\mathcal{R})$, every ground term containing a defined symbol is reducible. Hence, the inclusion $\mathrm{NF}_g(\mathcal{R}) \subseteq \mathrm{NF}_g(\mathcal{R}')$ follows from condition (3) by taking $C = \mathrm{T}(F_{\mathrm{c}}, \emptyset)$ in Proposition 4. ◻

It is known that when $\mathrm{WN}_g(\mathcal{R})$ holds, it is decidable whether $\mathrm{SC}(\mathcal{R})$ for possibly non-left-linear $\mathcal{R}$ (see, e.g., [7, 18]).

3 Higher-Order Inductive Theorems

In this section, we first recall the basic notions and terminology of simply typed term rewriting, which were introduced in [21]. Then we introduce the notion of higher-order inductive theorems, which is suitable for STTRSs.

3.1 Simply Typed Term Rewriting Systems

For a set B of *basic types*, the set of *simple types* is the smallest set $\mathrm{ST}(B)$ such that (1) $B \subseteq \mathrm{ST}(B)$, and (2) $\tau_1 \times \cdots \times \tau_n \to \tau_0 \in \mathrm{ST}(B)$ whenever $\tau_0, \tau_1, \ldots, \tau_n \in \mathrm{ST}(B)$. A non-basic type is called a *higher-order* type. Note that our definition allows multiple basic types whereas the original one in [21] is based on a single basic type. When clear, simple type is abbreviated as *type*.

Each *constant* or *variable* is associated with its type; the sets of constants and variables of type τ are denoted by Σ^τ and V^τ, respectively. Σ and V stand for the sets of all constants and variables, respectively. We assume that V^τ is countably infinite for each simple type τ. The sets of variables of basic types and of higher-order types are denoted by V^{b} and V^{h}, respectively.

The set $\mathrm{T}(\Sigma, V)^\tau$ of *simply typed terms* of type τ over Σ and V is defined as follows: (1) $\Sigma^\tau \cup V^\tau \subseteq \mathrm{T}(\Sigma, V)^\tau$, and (2) if $s \in \mathrm{T}(\Sigma, V)^{\tau_1 \times \cdots \times \tau_n \to \tau}$ $(n \geq 1)$ and $t_i \in \mathrm{T}(\Sigma, V)^{\tau_i}$ for all $i \in \{1, \ldots, n\}$ then $(s\ t_1 \cdots t_n) \in \mathrm{T}(\Sigma, V)^\tau$. A simply typed term t has type τ (denoted by t^τ) when $t \in \mathrm{T}(\Sigma, V)^\tau$. It is clear that each simply typed term has a unique type; thus τ is also referred to as the type of t (denoted by $\mathrm{type}(t)$). The set of all simply typed terms is denoted by $\mathrm{T}(\Sigma, V)$. The set of simply typed terms of basic types and of higher-order types are denoted by $\mathrm{T}^{\mathrm{b}}(\Sigma, V)$ and $\mathrm{T}^{\mathrm{h}}(\Sigma, V)$, respectively.

The *head symbol* of a simply typed term is defined as follows: (1) $\mathrm{head}(t) = t$ if $t \in \Sigma \cup V$, and (2) $\mathrm{head}((s\ t_1 \cdots t_n)) = \mathrm{head}(s)$. A simply typed term is *linear* if no variable occurs more than once in it. A *simply typed equation* is a pair of simply typed terms of the same type, written as $l \approx r$. The sets of variables occurring in a term t and in an equations $l \approx r$ are written as $\mathrm{V}(t)$ and $\mathrm{V}(l \approx r)$, respectively.

A context is a simply typed term with a special symbol $\Box$ (hole). If a context C has n holes and $t_1, \ldots, t_n$ are simply typed terms then $C[t_1, \ldots, t_n]$ is the simply typed term obtained from C by replacing holes with $t_1, \ldots, t_n$ from left to right. A term s is a *subterm* of a term t (denoted by $s \trianglelefteq t$) if $t = C[s]$ for some context C with precisely one hole. A *substitution* is a mapping $\sigma : V \to \mathrm{T}(\Sigma, V)$ that satisfies the following conditions: (1) $\mathrm{Dom}(\sigma) = \{x \mid \sigma(x) \neq x\}$ is finite, and (2) for every variable x, x and $\sigma(x)$ have the same type. The homomorphic extension of σ to $\mathrm{T}(\Sigma, V)$ is also denoted by σ. As usual, $\sigma(t)$ is written as $t\sigma$.

A simply typed equation $l \approx r$ is called a *simply typed rewrite rule* if $\mathrm{head}(l) \in \Sigma$ and $\mathrm{V}(r) \subseteq \mathrm{V}(l)$. A simply typed rewrite rule $l \approx r$ will be often written as $l \to r$. A triple $\langle B, \Sigma, R\rangle$ consisting of a set B of basic types, a set Σ of constants, and a set R of simply typed rewrite rules is called a simply typed term rewriting system (STTRS, for short). The *rewrite relation* $\to_{\mathcal{R}}$ induced by a simply typed term rewriting system $\mathcal{R} = \langle B, \Sigma, R\rangle$ is the smallest relation over $\mathrm{T}(\Sigma, V)$ satisfying the following conditions: (1) $l\sigma \to_{\mathcal{R}} r\sigma$ for all $l \to r \in R$ and for all substitutions σ, and (2) if $s \to_{\mathcal{R}} t$ then $C[s] \to_{\mathcal{R}} C[t]$ for all contexts C. A constant symbol f is a *defined symbol* of $\mathcal{R}$ if $f = \mathrm{head}(l)$ for some $l \to r \in R$; otherwise, f is a *constructor symbol*. The sets of defined and constructor symbols are denoted by Σ_{d} and Σ_{c}, respectively.

Example 1 (simply typed term rewriting). Let $\mathcal{R} = \langle B, \Sigma, R \rangle$ be an STTRS where $B = \{\mathrm{N}, \mathrm{NList}\}$, $\Sigma = \{\ \mathsf{0}^{\mathrm{N}},\ \mathsf{s}^{\mathrm{N}\to\mathrm{N}},\ [\,]^{\mathrm{NList}},\ :^{\mathrm{N}\times\mathrm{NList}\to\mathrm{NList}},\ \mathsf{map}^{(\mathrm{N}\to\mathrm{N})\times\mathrm{NList}\to\mathrm{NList}},\ \circ^{(\mathrm{N}\to\mathrm{N})\times(\mathrm{N}\to\mathrm{N})\to(\mathrm{N}\to\mathrm{N})},\ \mathsf{twice}^{(\mathrm{N}\to\mathrm{N})\to(\mathrm{N}\to\mathrm{N})}\ \}$, and

$$R = \left\{ \begin{array}{lcl} \mathsf{map}\ F\ [\,] & \to & [\,] \\ \mathsf{map}\ F\ (x : xs) & \to & (F\ x) : (\mathsf{map}\ F\ xs) \\ (F \circ G)\ x & \to & F\ (G\ x) \\ \mathsf{twice}\ F & \to & F \circ F \end{array} \right\}.$$

Here is an example of rewrite sequence of $\mathcal{R}$:

$$\begin{array}{rcl} \mathsf{map}\ (\mathsf{twice}\ \mathsf{s})\ (\mathsf{0} : [\,]) & \to_{\mathcal{R}} & \mathsf{map}\ (\mathsf{s} \circ \mathsf{s})\ (\mathsf{0} : [\,]) \\ & \to_{\mathcal{R}} & ((\mathsf{s} \circ \mathsf{s})\ \mathsf{0}) : (\mathsf{map}\ (\mathsf{s} \circ \mathsf{s})\ [\,]) \\ & \to_{\mathcal{R}} & (\mathsf{s}\ (\mathsf{s}\ \mathsf{0})) : (\mathsf{map}\ (\mathsf{s} \circ \mathsf{s})\ [\,]) \\ & \to_{\mathcal{R}} & (\mathsf{s}\ (\mathsf{s}\ \mathsf{0})) : [\,]. \end{array}$$

3.2 Inductive Theorems of STTRSs

One may define the notion of inductive theorem of an STTRS exactly the same as that of a many-sorted STTRS, that is, an equation $l \approx r$ is an inductive theorem of an STTRS $\mathcal{R}$ if $l\theta_g \leftrightarrow^*_{\mathcal{R}} r\theta_g$ holds for any ground substitution θ_g such that $\mathrm{V}(l \approx r) \subseteq \mathrm{Dom}(\theta_g)$. Henceforth, we call this notion of inductive theorem *first-order*. Propositions 5 and 7 are extended to STTRSs in the obvious way.

Example 2 (first-order inductive theorem proving). Let $\mathcal{R} = \langle B, \Sigma, R\rangle$ be an STTRS where $B = \{\ \mathrm{N},\ \mathrm{NList}\ \}$, $\Sigma = \{\ \mathsf{0}^{\mathrm{N}},\ \mathsf{s}^{\mathrm{N}\to\mathrm{N}},\ [\,]^{\mathrm{NList}},\ :^{\mathrm{N}\times\mathrm{NList}\to\mathrm{NList}},\ \mathsf{append}^{\mathrm{NList}\times\mathrm{NList}\to\mathrm{NList}},\ \mathsf{map}^{(\mathrm{N}\to\mathrm{N})\times\mathrm{NList}\to\mathrm{NList}}\ \}$, and

$$R = \left\{ \begin{array}{lcl} \mathsf{map}\ F\ [\,] & \to & [\,] \\ \mathsf{map}\ F\ (x : xs) & \to & (F\ x) : (\mathsf{map}\ F\ xs) \\ \mathsf{append}\ [\,]\ ys & \to & ys \\ \mathsf{append}\ (x : xs)\ ys & \to & x : (\mathsf{append}\ xs\ ys) \end{array} \right\}.$$

Let us now show that the equation

$$e \quad = \quad \mathsf{map}\ F\ (\mathsf{append}\ xs\ ys) \ \approx\ \mathsf{append}\ (\mathsf{map}\ F\ xs)\ (\mathsf{map}\ F\ ys)$$

is a first-order inductive theorem of $\mathcal{R}$. Let $\mathcal{R}' = \mathcal{R} \cup \{e\}$. Using the method in [1] or [2] one can show that $\mathrm{SN}(\mathcal{R}')$ holds, and thus $\mathrm{WN}_g(\mathcal{R})$ follows. All critical pairs of $\mathcal{R}'$ are joinable. By $\mathrm{SN}(\mathcal{R}')$ and the Critical Pair Lemma for STTRS [3], it follows that $\mathcal{R}'$ is confluent. Hence $\mathrm{CR}_g(\mathcal{R}')$. One can show that $\Theta = \{\{xs \mapsto [\,]\}, \{xs \mapsto y : ys\}\}$ is a coverset of substitutions for the left-hand side of e w.r.t. $\mathcal{R}$. It is easy to check $(\mathsf{map}\ F\ (\mathsf{append}\ xs\ ys))\sigma \in \mathrm{RED}(\mathcal{R})$ for any $\sigma \in \Theta$. Therefore, e is a first-order inductive theorem of $\mathcal{R}$.

As seen from the next example, the notion of first-order inductive theorem is inadequate for equational reasoning in STTRS because of the presence of higher-order terms.

Example 3 (inadequateness of first-order inductive theorem). Let $\mathcal{R} = \langle B, \Sigma, R\rangle$ be an STTRS where $B = \{ \text{ N } \}$, $\Sigma = \{ \mathsf{0}^{\mathrm{N}}, \mathsf{s}^{\mathrm{N}\to\mathrm{N}}, \mathsf{id}_0^{\mathrm{N}\to\mathrm{N}}, \mathsf{id}_1^{\mathrm{N}\to\mathrm{N}} \}$, and

$$R = \left\{ \begin{array}{lcl} \mathsf{id}_0\ \mathsf{0} & \to & \mathsf{0} \\ \mathsf{id}_0\ (\mathsf{s}\ x) & \to & \mathsf{s}\ (\mathsf{id}_0\ x) \\ \mathsf{id}_1\ x & \to & x \end{array} \right\}.$$

Consider whether the equation $\mathsf{id}_0 \approx \mathsf{id}_1$ is an inductive theorem of $\mathcal{R}$. Since both id_0 and id_1 are ground, and $\mathsf{id}_0 \leftrightarrow^*_{\mathcal{R}_1} \mathsf{id}_1$ does not hold, the equation $\mathsf{id}_0 \approx \mathsf{id}_1$ is not a first-order inductive theorem of $\mathcal{R}$.

Since id_0 computes the same result as id_1 does for any argument of type N, it seems natural to regard the equation $\mathsf{id}_0 \approx \mathsf{id}_1$ inductively valid.

Inductive theorems are mostly useful to verify properties of functional programs, algebraic specification, etc. and from this point of view only "evaluation form" matters—since functional programs are evaluated in ground terms of basic types, it is reasonable to define inductive theorems in higher-order rewriting those equations that are valid on the set of ground terms of basic types.

Inductive theorems of a higher-order type are verified after turning them into an equation of basic type by appending fresh variables without changing its "inductive" meaning.

Definition 1 (extensional form).

1. Let t be a simply typed term of type $\tau_{11} \times \cdots \times \tau_{1n_1} \to (\tau_{21} \times \cdots \times \tau_{2n_2} \to \cdots \to (\tau_{m1} \times \cdots \times \tau_{mn_m} \to \rho) \cdots)$ such that ρ is a basic type. An *extensional form* of t is a term $((\cdots((t\ x_{11} \cdots x_{1n_1})\ x_{21} \cdots x_{2n_2}) \cdots)\ x_{m1} \cdots x_{mn_m})$ where $x_{11}, \ldots, x_{mn_m}$ are mutually distinct fresh variables. Clearly, any extensional form of a simply typed term has a basic type. The extensional form of t with fresh variables $x_{11}, \ldots, x_{mn_m}$ is written as $t{\uparrow}_{x_{11},\ldots,x_{mn_m}}$.
2. Let $\phi(s_1, \ldots, s_n)$ be an expression containing simply typed terms $s_1, \ldots, s_n$ of the same type. The extensional form $\phi(s_1{\uparrow}_{x_{11},\ldots,x_{mn_m}}, \ldots, s_n{\uparrow}_{x_{11},\ldots,x_{mn_m}})$ of $\phi(s_1, \ldots, s_n)$ is obtained by replacing each simply typed term s_i in the expression by $s_i{\uparrow}_{x_{11},\ldots,x_{mn_m}}$, where $x_{11}, \ldots, x_{mn_m}$ are mutually distinct fresh variables.
3. When the variables $x_{11}, \ldots, x_{mn_m}$ are not important, we omit them and simply write, e.g., $t{\uparrow}$ and $\phi(s_1{\uparrow}, \ldots, s_n{\uparrow})$.

Example 4 (extensional form). An extensional form of map is $\mathsf{map}\ F\ xs$, and an extensional form of $\circ$ is $(F \circ G)\ xs$. An extensional form of the equation $\mathsf{id}_0 \approx \mathsf{id}_1$ is $\mathsf{id}_0\ x \approx \mathsf{id}_1\ x$.

Definition 2 (higher-order inductive theorem). An equation $l \approx r$ is a *higher-order inductive theorem* of an STTRS $\mathcal{R}$ if $l{\uparrow}\theta_g \leftrightarrow^*_{\mathcal{R}} r{\uparrow}\theta_g$ holds for all ground substitutions θ_g such that $\mathrm{V}(l{\uparrow} \approx r{\uparrow}) \subseteq \mathrm{Dom}(\theta_g)$.

The next theorem follows immediately from the definition of higher-order inductive theorem and Proposition 1.

Theorem 1 (higher-order inductive theorem proving). Let $\mathcal{R}$ be an STTRS and $\mathcal{R}' = \mathcal{R} \cup \{l\!\uparrow \to r\!\uparrow\}$. Suppose the following conditions are satisfied: (1) $\mathrm{WN}^{\mathrm{b}}_g(\mathcal{R})$, (2) $\mathrm{CR}^{\mathrm{b}}_g(\mathcal{R}')$, (3) $\mathrm{NF}^{\mathrm{b}}_g(\mathcal{R}) \subseteq \mathrm{NF}^{\mathrm{b}}_g(\mathcal{R}')$. Then $l \approx r$ is a higher-order inductive theorem of $\mathcal{R}$.

The notion of coverset of substitutions is extended to STTRS in an obvious way as follows. For an STTRS $\mathcal{R}$ and a simply typed term t, a set Θ of substitutions is a *coverset of substitutions for t w.r.t. $\mathcal{R}$* if for any ground substitution θ_g satisfying $\mathrm{V}(t) \subseteq \mathrm{Dom}(\theta_g)$, there exists a substitution $\sigma \in \Theta$ and a ground substitution θ'_g such that $t\theta_g \to^*_{\mathcal{R}} t\sigma\theta'_g$.

Theorem 2 (higher-order inductive theorem proving with coverset). Let $\mathcal{R}$ be an STTRS and $\mathcal{R}' = \mathcal{R} \cup \{l\!\uparrow \to r\!\uparrow\}$. Suppose the following conditions are satisfied: (1) $\mathrm{WN}^{\mathrm{b}}_g(\mathcal{R})$, (2) $\mathrm{CR}^{\mathrm{b}}_g(\mathcal{R}')$, (3) $l\!\uparrow\!\sigma \in \mathrm{RED}(\mathcal{R})$ for all $\sigma \in \Theta$, where Θ is a coverset of substitutions for $l\!\uparrow$ w.r.t. $\mathcal{R}$. Then $l \approx r$ is a higher-order inductive theorem of $\mathcal{R}$.

Example 5 (higher-order inductive theorem proving). Let us consider Example 3 again. By taking an extensional form of the equation $\mathsf{id}_0 \approx \mathsf{id}_1$, we obtain the equation $\mathsf{id}_0\ x \approx \mathsf{id}_1\ x$. Let $\mathcal{R}' = \mathcal{R} \cup \{\mathsf{id}_0\ x \to \mathsf{id}_1\ x\}$. Then $\mathrm{WN}^{\mathrm{b}}_g(\mathcal{R})$ and $\mathrm{CR}^{\mathrm{b}}_g(\mathcal{R}')$ are proved in the same way as Example 2. By taking the coverset of substitutions $\Theta = \{\{x \mapsto \mathsf{0}\}, \{x \mapsto \mathsf{s}\ y\}\}$, condition (3) is easily verified. Therefore, the equation $\mathsf{id}_0 \approx \mathsf{id}_1$ is a higher-order inductive theorem of $\mathcal{R}$.

The notion of sufficient completeness is extended to STTRS as follows. An STTRS $\mathcal{R}$ is *sufficiently complete* ($\mathrm{SC}(\mathcal{R})$) if for any ground term s_g of basic type there exists a ground constructor term t_g of basic type such that $s_g \to^*_{\mathcal{R}} t_g$.

Theorem 3 (higher-order inductive theorem proving with SC). Let $\mathcal{R}$ be an STTRS and $\mathcal{R}' = \mathcal{R} \cup \{l\!\uparrow \to r\!\uparrow\}$. Suppose the following conditions are satisfied: (1) $\mathrm{SC}(\mathcal{R})$, (2) $\mathrm{CR}^{\mathrm{b}}_g(\mathcal{R}')$, (3) l contains a defined symbol. Then $l \approx r$ is a higher-order inductive theorem of $\mathcal{R}$.

4 Monotone Higher-Order Inductive Theorem

In contrast to the first-order inductive theorems, higher-order inductive theorems introduced in the previous section may not be monotone, that is, even if $l \approx r$ is a higher-order inductive theorem of an STTRS, $C[l] \approx C[r]$ may not be a higher-order inductive theorem depending on the choice of the context C—this observation is due to K. Kusakari. In this section, we study monotonicity of higher-order inductive theorems.

Definition 3 (monotone higher-order inductive theorem). Let $\mathcal{R}$ be an STTRS. A higher-order inductive theorem $l \approx r$ of $\mathcal{R}$ is *monotone* if $C[l] \approx C[r]$ is a higher-order inductive theorem of $\mathcal{R}$ for any context C.

First, let us examine examples of higher-order inductive theorems that are not monotone.

Example 6. Let $\mathcal{R} = \langle B, \Sigma, R\rangle$ be an STTRS such that $B = \{ \mathrm{N} \}$, $\Sigma = \{ \mathsf{0}^{\mathrm{N}}, \mathsf{s}^{\mathrm{N}\to\mathrm{N}}, +^{\mathrm{N}\to\mathrm{N}\to\mathrm{N}}, \mathsf{f}^{(\mathrm{N}\to\mathrm{N})\to\mathrm{N}} \}$, and

$$R = \left\{ \begin{array}{lcl} (+\ x)\ \mathsf{0} & \to & x \\ (+\ x)\ (\mathsf{s}\ y) & \to & \mathsf{s}\ ((+\ x)\ y) \\ \mathsf{f}\ \mathsf{s} & \to & \mathsf{0} \\ \mathsf{f}\ (+\ x) & \to & \mathsf{s}\ \mathsf{0} \end{array} \right\}.$$

The equation $+\ (\mathsf{s}\ \mathsf{0}) \approx \mathsf{s}$ is an inductive theorem of $\mathcal{R}$. However, $\mathsf{f}\ (+\ (\mathsf{s}\ \mathsf{0})) \approx \mathsf{f}\ \mathsf{s}$ is not, since $\mathsf{f}\ (+\ (\mathsf{s}\ \mathsf{0})) \to_{\mathcal{R}} \mathsf{s}\ \mathsf{0} \neq \mathsf{0}\ {}_{\mathcal{R}}{\leftarrow}\ \mathsf{f}\ \mathsf{s}$. Thus the equation $+\ (\mathsf{s}\ \mathsf{0}) \approx \mathsf{s}$ is not monotone.

The key observation on what leads to non-monotonicity of higher-order inductive theorems is that $l \approx r$ is a higher-order inductive theorem implies only $C[l{\uparrow}\theta_g] \leftrightarrow^*_{\mathcal{R}} C[r{\uparrow}\theta_g]$ but not necessarily $(C[l]){\uparrow}\sigma_g \leftrightarrow^*_{\mathcal{R}} (C[r]){\uparrow}\sigma_g$. Let us examine an example where such a desired implication seems satisfied.

Example 7 (inductive theorem within a context). Consider the STTRS defined by $\mathcal{R} = \langle B, \Sigma, R\rangle$ where $B = \{ \mathrm{N}, \mathrm{NList} \}$, $\Sigma = \{ \mathsf{0}^{\mathrm{N}}, \mathsf{s}^{\mathrm{N}\to\mathrm{N}}, [\,]^{\mathrm{NList}}, :^{\mathrm{N}\times\mathrm{NList}\to\mathrm{NList}}, +^{\mathrm{N}\to\mathrm{N}\to\mathrm{N}}, \mathsf{map}^{(\mathrm{N}\to\mathrm{N})\times\mathrm{NList}\to\mathrm{NList}} \}$, and

$$R = \left\{ \begin{array}{lcl} (+\ \mathsf{0})\ y & \to & y \\ (+\ (\mathsf{s}\ x))\ y & \to & \mathsf{s}\ ((+\ x)\ y) \\ \mathsf{map}\ F\ [\,] & \to & [\,] \\ \mathsf{map}\ F\ (x : xs) & \to & (F\ x) : (\mathsf{map}\ F\ xs) \end{array} \right\}.$$

The equation $+\ (\mathsf{s}\ \mathsf{0}) \approx \mathsf{s}$ is a higher-order inductive theorem of $\mathcal{R}$. Consider the context $C = \mathsf{map}\ \Box\ (\mathsf{0} : [\,])$. We are going to verify the equation $C[+\ (\mathsf{s}\ \mathsf{0})] \approx C[\mathsf{s}]$ also is a higher-order inductive theorem.

$$\begin{array}{rl} \mathsf{map}\ (+\ (\mathsf{s}\ \mathsf{0}))\ (\mathsf{0} : [\,]) & \to_{\mathcal{R}} ((+\ (\mathsf{s}\ \mathsf{0}))\ \mathsf{0}) : (\mathsf{map}\ (+\ (\mathsf{s}\ \mathsf{0}))\ [\,]) \\ & \to_{\mathcal{R}} (\mathsf{s}\ ((+\ \mathsf{0})\ \mathsf{0})) : (\mathsf{map}\ (+\ (\mathsf{s}\ \mathsf{0}))\ [\,]) \\ & \to_{\mathcal{R}} (\mathsf{s}\ \mathsf{0}) : (\mathsf{map}\ (+\ (\mathsf{s}\ \mathsf{0}))\ [\,]) \\ & \to_{\mathcal{R}} (\mathsf{s}\ \mathsf{0}) : [\,] \\ & {}_{\mathcal{R}}{\leftarrow}\ (\mathsf{s}\ \mathsf{0}) : \mathsf{map}\ \mathsf{s}\ [\,] \\ & {}_{\mathcal{R}}{\leftarrow}\ \mathsf{map}\ \mathsf{s}\ (\mathsf{0} : [\,]) \end{array}$$

This rewrite sequence can be reconstructed from $(+\ (\mathsf{s}\ \mathsf{0}))\ \mathsf{0} \leftrightarrow^*_{\mathcal{R}} \mathsf{s}\ \mathsf{0}$ and

$$\mathsf{map}\ F\ (\mathsf{0} : [\,]) \quad \to_{\mathcal{R}} \quad (F\ \mathsf{0}) : \mathsf{map}\ F\ [\,] \quad \to_{\mathcal{R}} \quad (F\ \mathsf{0}) : [\,].$$

The observation above motivates the notion of higher-order sufficient completeness.

Definition 4 (higher-order sufficient completeness).

1. Let t be a simply typed term such that $t = C[F]$ for some $F \in V^{\mathrm{h}}$. The occurrence of F is said to be *expanded* if the occurrence of F in t has the form $t = C'[F{\uparrow}\theta]$ for some context C' and a substitution θ.

2. An STTRS $\mathcal{R} = \langle B, \Sigma, R\rangle$ is said to be *higher-order sufficiently complete* (HSC($\mathcal{R}$)) if for any simply typed term $s \in \mathrm{T}^{\mathrm{b}}(\Sigma, V^{\mathrm{h}})$ there exists $t \in \mathrm{T}^{\mathrm{b}}(\Sigma, V^{\mathrm{h}})$ such that $s \rightarrow^{*}_{\mathcal{R}} t$ and that either t has an expanded variable occurrence or t is a ground constructor term.

Example 8 (higher-order sufficient completeness). Let $\mathcal{R}$ be the STTRS in Example 7, $s = \mathsf{map}\ F\ (\mathsf{0} : [\,])$, and $t = (F\ \mathsf{0}) : [\,]$. The occurrence of variable F in s is non-expanded, while that of F in t is expanded. Using a result in Section 5, it will turn out that $\mathcal{R}$ is higher-order sufficiently complete. Indeed, we have $s \rightarrow^{*}_{\mathcal{R}} t \in \mathrm{T}^{\mathrm{b}}(\Sigma, V^{\mathrm{h}})$ and t has an expanded variable occurrence.

Note that HSC($\mathcal{R}$) implies SC($\mathcal{R}$) since reduction preserves groundness. Hence the notion of higher-order sufficient completeness extends the notion of (first-order) sufficient completeness.

Lemma 1 (sufficient condition for monotonicity). If an STTRS $\mathcal{R}$ satisfies $\mathrm{SN}^{\mathrm{b}}_{g}(\mathcal{R})$ and HSC($\mathcal{R}$), then every higher-order inductive theorem of $\mathcal{R}$ is monotone.

Proof. Suppose $l \approx r$ is a higher-order inductive theorem. We have to show that $C[l] \approx C[r]$ also is a higher-order inductive theorem. Let $s = C_g[l\theta_g, \ldots, l\theta_g]$ and $t = C_g[r\theta_g, \ldots, r\theta_g]$. We prove $s \leftrightarrow^{*}_{\mathcal{R}} t$ for all ground contexts C_g of basic type and ground substitutions θ_g, which implies the desired result as a special case when C_g contains precisely one hole. The proof proceeds by well-founded induction on s w.r.t. $\rightarrow_{\mathcal{R}}$. If l and r have a basic type, then $s \leftrightarrow^{*}_{\mathcal{R}} t$ immediately follows from the assumption that $l \approx r$ is an inductive theorem. Consider the case that l and r have a non-basic type. Let C' be a context obtained from C_g by replacing all (possibly zero) expanded occurrences of holes with an arbitrary variable F whose type is the same as that of l and r. Let σ and τ be substitutions satisfying $\sigma(F) = l\theta_g$ and $\tau(F) = r\theta_g$. We have $s = C'\sigma[l\theta_g, \ldots, l\theta_g]$ and $t = C'\tau[r\theta_g, \ldots, r\theta_g]$. Since all occurrences of F in C' are expanded, the assumption that $l \approx r$ is a higher-order inductive theorem implies $C'\sigma[F, \ldots, F] \leftrightarrow^{*}_{\mathcal{R}} C'\tau[F, \ldots, F]$. By HSC($\mathcal{R}$), there exists a term $u = D_g[F, \ldots, F]$ with some ground context D_g such that $C'\sigma[F, \ldots, F] \rightarrow^{*}_{\mathcal{R}} u$ and either u is a ground constructor term or there is an expanded occurrence of F in u. In the former case, D_g has no holes and hence both $s = C'\sigma[l\theta_g, \ldots, l\theta_g] \rightarrow^{*}_{\mathcal{R}} u\sigma = D_g$ and $t = C'\tau[r\theta_g, \ldots, r\theta_g] \leftrightarrow^{*}_{\mathcal{R}} C'\sigma[r\theta_g, \ldots, r\theta_g] \rightarrow^{*}_{\mathcal{R}} u\tau = D_g$ are satisfied. In the latter case, the reduction sequence from $C'\sigma[F, \ldots, F]$ to u consists of at least one step, which implies the non-emptiness of the corresponding rewrite sequences from s to $u\sigma$. By the induction hypothesis, we obtain $u\sigma = D_g[l\theta_g, \ldots, l\theta_g] \leftrightarrow^{*}_{\mathcal{R}} D_g[r\theta_g, \ldots, r\theta_g] = u\tau$. Hence, both $s \rightarrow^{*}_{\mathcal{R}} u\sigma \leftrightarrow^{*}_{\mathcal{R}} u\tau$ and $t = C'\tau[r\theta_g, \ldots, r\theta_g] \leftrightarrow^{*}_{\mathcal{R}} C'\sigma[r\theta_g, \ldots, r\theta_g] \rightarrow^{*}_{\mathcal{R}} u\tau$ are satisfied. In both cases, we obtained $s \leftrightarrow^{*}_{\mathcal{R}} t$. $\square$

Combining Theorem 3 and Lemma 1, we obtain the following theorem.

Theorem 4 (monotone higher-order inductive theorem proving with HSC). Let $\mathcal{R}$ be an STTRS and $\mathcal{R}' = \mathcal{R} \cup \{l\!\uparrow\ \rightarrow r\!\uparrow\}$. Suppose the following conditions are satisfied: (1) $\mathrm{SN}^{\mathrm{b}}_{g}(\mathcal{R})$, (2) $\mathrm{HSC}(\mathcal{R})$, (3) $\mathrm{CR}^{\mathrm{b}}_{g}(\mathcal{R}')$, (4) l contains a defined symbol. Then $l \approx r$ is a monotone higher-order inductive theorem of $\mathcal{R}$.

Example 9 (monotone higher-order inductive theorem proving). Let $\mathcal{R} = \langle B, \Sigma, R\rangle$ be an STTRS where $B = \{\ \mathrm{N}, \mathrm{NList}\ \}$, $\Sigma_{\mathrm{d}} = \{\ \mathsf{map}^{(\mathrm{N}\to\mathrm{N})\to(\mathrm{NList}\to\mathrm{NList})}, \circ^{(\mathrm{N}\to\mathrm{N})\times(\mathrm{N}\to\mathrm{N})\to(\mathrm{N}\to\mathrm{N})}, \bullet^{(\mathrm{NList}\to\mathrm{NList})\times(\mathrm{NList}\to\mathrm{NList})\to(\mathrm{NList}\to\mathrm{NList})}\ \}$, $\Sigma_{\mathrm{c}} = \{\ \mathsf{0}^{\mathrm{N}}, \mathsf{s}^{\mathrm{N}\to\mathrm{N}}, [\,]^{\mathrm{NList}}, :^{\ \mathrm{N}\times\mathrm{NList}\to\mathrm{NList}}\ \}$, and

$$R = \left\{ \begin{array}{lcl} (\mathsf{map}\ F)\ [\,] & \to & [\,] \\ (\mathsf{map}\ F)\ (x : xs) & \to & (F\ x) : ((\mathsf{map}\ F)\ xs) \\ (F \circ G)\ x & \to & F\ (G\ x) \\ (X \bullet Y)\ xs & \to & X\ (Y\ xs) \end{array} \right\}.$$

$\mathrm{HSC}(\mathcal{R})$ will be shown in Example 11. So, for the moment, suppose $\mathrm{HSC}(\mathcal{R})$. Let us consider the equation $e\ =\ \mathsf{map}\ (F \circ G) \approx (\mathsf{map}\ F) \bullet (\mathsf{map}\ G)$. By taking an extensional form of this equation, we obtain $e'\ =\ (\mathsf{map}\ (F \circ G))\ xs \approx ((\mathsf{map}\ F) \bullet (\mathsf{map}\ G))\ xs$. Let $\mathcal{R}' = \mathcal{R} \cup \{e'\}$. In the same way as Example 2, one can show $\mathrm{SN}^{\mathrm{b}}_{g}(\mathcal{R})$ and $\mathrm{CR}^{\mathrm{b}}_{g}(\mathcal{R}')$. The term $\mathsf{map}\ (F \circ G)$ contains defined symbols map and $\circ$. Therefore, by Theorem 4, e is a monotone higher-order inductive theorem of $\mathcal{R}$.

5 Decidability of Higher-Order Sufficient Completeness

It is undecidable, in general, whether higher-order sufficient completeness holds for a given STTRS. In first-order term rewriting, the related property called quasi-reducibility plays an important role in proving the decidability of sufficient completeness for TRSs satisfying certain conditions [7, 18]. In order to present a class of STTRSs where the higher-order sufficient completeness is decidable, we need to extend the notion of quasi-reducibility.

Definition 5 (argument subterms). Every constant and variable has no *argument subterms*. The *argument subterms* of a simply typed term $(s\ t_1 \cdots t_n)$ are argument subterms of s together with the terms $t_1, \ldots, t_n$. The set of all argument subterms in t is denoted by $\mathrm{Arg}(t)$.

Example 10 (argument subterms). A simply typed term $\mathsf{map}\ F$ has a unique argument subterm F. The argument subterms of $(\mathsf{map}\ (\mathsf{s} \circ \mathsf{s}))\ (\mathsf{0} : [\,])$ are $\mathsf{s} \circ \mathsf{s}$ and $\mathsf{0} : [\,]$.

Definition 6 (higher-order quasi-reducibility). An STTRS $\mathcal{R} = \langle B, \Sigma, R\rangle$ is *higher-order quasi-reducible* ($\mathrm{HQR}(\mathcal{R})$) when for any term t such that $\mathrm{head}(t) \in \Sigma_{\mathrm{d}}$, if every argument subterms of t of basic type is a ground constructor term and of higher-order type is a variable, then t is reducible.

We are going to show, under some conditions, that the equivalence of $\mathrm{HQR}(\mathcal{R})$ and $\mathrm{HSC}(\mathcal{R})$ holds and that $\mathrm{HQR}(\mathcal{R})$ is decidable. Consequently, $\mathrm{HSC}(\mathcal{R})$ is decidable for that class of STTRSs. We first present a condition by which $\mathrm{HQR}(\mathcal{R})$ implies $\mathrm{HSC}(\mathcal{R})$.

Definition 7 (elementary STTRS). The set $\mathrm{ST}_{\mathrm{el}}$ of *elementary types* is defined inductively as: (1) $B \subseteq \mathrm{ST}_{\mathrm{el}}$, (2) if $\tau_1, \ldots, \tau_n \in B$ $(n \geq 1)$ and $\tau_0 \in \mathrm{ST}_{\mathrm{el}}$ then $\tau_1 \times \cdots \times \tau_n \to \tau_0 \in \mathrm{ST}_{\mathrm{el}}$. Let $\Sigma_{\mathrm{el}} = \bigcup_{\tau \in \mathrm{ST}_{\mathrm{el}}} \Sigma^\tau$. An STTRS $\mathcal{R} = \langle B, \Sigma, R \rangle$ is said to be *elementary* ($\mathrm{EL}(\mathcal{R})$) if $\Sigma_{\mathrm{c}} \subseteq \Sigma_{\mathrm{el}}$ is satisfied.

Observe that if a set Σ of function symbols satisfies $\Sigma_{\mathrm{c}} \subseteq \Sigma_{\mathrm{el}}$, then all argument subterms of a simply typed term t are of basic type whenever $\mathrm{head}(t)$ is a constructor symbol.

Lemma 2 (key property for ensuring HSC). Let $\mathcal{R}$ be an STTRS satisfying $\mathrm{HQR}(\mathcal{R})$ and $\mathrm{EL}(\mathcal{R})$. If t is a normal form of basic type and all variables in t are of higher-order type, then t either has some expanded variable occurrence or is a ground constructor term.

Proof. The proof proceeds by structural induction on t. Distinguish cases by $\mathrm{head}(t)$. $\square$

The notions of higher-order quasi-reducibility and elementarity appear in [12] to give a sufficient condition of $\mathrm{SC}(\mathcal{R})$.

Theorem 5 (equivalence condition for HQR and HSC). Let $\mathcal{R}$ be an STTRS satisfying $\mathrm{WN}^{\mathrm{b}}(\mathcal{R})$ and $\mathrm{EL}(\mathcal{R})$. Then, $\mathrm{HQR}(\mathcal{R})$ iff $\mathrm{HSC}(\mathcal{R})$.

Proof. The "if" direction is obvious by definition. For a proof of the "only if" direction, let $\mathcal{R} = \langle B, \Sigma, R \rangle$ and $s \in \mathrm{T}^{\mathrm{b}}(\Sigma, V^{\mathrm{h}})$. By $\mathrm{WN}^{\mathrm{b}}(\mathcal{R})$, there exists a normal form t of s such that $t \in \mathrm{T}^{\mathrm{b}}(\Sigma, V^{\mathrm{h}})$. Hence the result follows from Lemma 2 using $\mathrm{EL}(\mathcal{R})$. $\square$

Example 11 (proving higher-order sufficient completeness). We now show that STTRS in Example 9 is higher-order sufficiently complete. In that example, it has been shown $\mathrm{SN}^{\mathrm{b}}(\mathcal{R})$ holds and hence so does $\mathrm{WN}^{\mathrm{b}}(\mathcal{R})$. Clearly, $\mathrm{EL}(\mathcal{R})$ and $\mathrm{HQR}(\mathcal{R})$ hold. Thus by Theorem 5 it follows that $\mathcal{R}$ is higher-order sufficiently complete.

The condition $\mathrm{EL}(\mathcal{R})$ is essential, because $\mathrm{WN}^{\mathrm{b}}(\mathcal{R})$ and $\mathrm{HQR}(\mathcal{R})$ do not necessarily guarantee $\mathrm{HSC}(\mathcal{R})$ as the following example shows.

Example 12. Let $\mathcal{R} = \langle B, \Sigma, R \rangle$ be an STTRS where $B = \{\ \mathrm{N}, \mathrm{NPair}\ \}$, $\Sigma = \{\ \mathsf{0}^{\mathrm{N}}, \mathsf{s}^{\mathrm{N} \to \mathrm{N}}, +^{\mathrm{N} \to \mathrm{N} \to \mathrm{N}}, \mathsf{f}^{(\mathrm{N} \to \mathrm{N}) \to \mathrm{NPair}}, \langle -, - \rangle^{(\mathrm{N} \to \mathrm{N}) \times (\mathrm{N} \to \mathrm{N}) \to \mathrm{NPair}}\ \}$, and

$$R = \left\{ \begin{array}{lcl} \mathsf{f}\ F & \to & \langle F, F \rangle \\ (+\ x)\ \mathsf{0} & \to & x \\ (+\ x)\ (\mathsf{s}\ y) & \to & \mathsf{s}\ ((+\ x)\ y) \end{array} \right\}.$$

Then, both $\mathrm{WN}^{\mathrm{b}}(\mathcal{R})$ and $\mathrm{HQR}(\mathcal{R})$ are satisfied. The term $\langle (+\ \mathsf{0}), (+\ \mathsf{0}) \rangle$ is a normal form and contains defined symbols, while it has no expanded variables.

We next show that higher-order sufficient completeness is decidable for a class of elementary STTRSs. In our decidability proof, we need to distinguish occurences of argument subterms. The notions of argument depth and argument position are introduced for that purpose.

Definition 8 (argument depth). The *argument depth*, or simply the depth, $\mathrm{d}(t)$ of a simply typed term t is defined by $\mathrm{d}(t) = 1 + \max\{\mathrm{d}(s) \mid s \in \mathrm{Arg}(s)\}$, where $\max \emptyset = 0$. The argument depth $\mathrm{d}(\mathcal{R})$ of an STTRS $\mathcal{R} = \langle B, \Sigma, R\rangle$ is defined by $\mathrm{d}(\mathcal{R}) = \max\{\mathrm{d}(l) \mid l \to r \in R\}$.

Definition 9 (argument position). A finite sequence of pairs of positive integers is called an *argument position*. The set $\mathrm{APos}(t)$ of argument positions in a simply typed term t and the subterm $t_{|p}$ of t at $p \in \mathrm{APos}(t)$ are defined as follows:

$$\begin{aligned} \mathrm{APos}(t) &= \{\varepsilon\} \cup \textstyle\bigcup_{ij}\{ij.p \mid p \in \mathrm{APos}(t_{ij})\} \\ t_{|\varepsilon} &= t \\ t_{|ij.p} &= t_{ij}{}_{|p} \end{aligned}$$

where $t = ((\cdots((a\ t_{11}\cdots t_{1n_1})\ t_{21}\cdots\ t_{2n_2})\cdots)\ t_{m1}\cdots t_{mn_m})$ with $a \in \Sigma \cup V$. The prefix order $\geq$ on argument positions is defined as follows: $p \geq q$ iff $p = q.q'$ for some q'. We write $p \parallel q$ when neither $p \geq q$ nor $p \leq q$.

Example 13 (argument depth/argument position). Let $s = (\mathsf{map}\ (\mathsf{s} \circ \mathsf{s}))\ (\mathsf{0} : [\,])$. Then the $\mathrm{d}(s) = 3$ and $\mathrm{APos}(s) = \{\varepsilon, 11, 21, 11.11, 11.12, 21.11, 21.12\}$. We have $s_{|p} = \mathsf{0}$ when $p = 21.11$.

Let $B^{\mathrm{f}}, B^{\mathrm{i}} \subseteq B$ be the sets of basic types τ such that $\mathrm{T}^{\tau}(\Sigma_{\mathrm{c}}, \emptyset)$ is finite and infinite, respectively. Without loss of generality, we assume that[1] there exists a constant of type τ for each $\tau \in B$ and that[2] for any $\tau \in B^{\mathrm{f}}$ all ground constructors of type τ are constants. Clearly, for each type in $\tau \in B^{\mathrm{i}}$ and each positive integer k, there exists a ground constructor of type τ whose depth is k; so, let δ_k^{τ} be a simply typed term in $\mathrm{T}^{\tau}(\Sigma_{\mathrm{c}}, \emptyset)$ such that $\mathrm{d}(\delta_k) = k$, for each $\tau \in B^{\mathrm{i}}$ and $k > 0$. We will omit superscript $^{\tau}$ of δ_k^{τ} when it is clear from its context.

Definition 10 (top of a term). Suppose $\Sigma_{\mathrm{c}} \subseteq \Sigma_{\mathrm{el}}$. Let k be a positive integer and t be a ground constructor term of basic type. Let $p_1, \ldots, p_n$ be the list of all argument positions p in t satisfying $|p| \geq k - 1$. Then the *top* of t of depth k is the term obtained from t by replacing each subterm $t_{|p_i}$ by x_i, which is denoted by $\mathrm{top}_k(t)$. Finally, let $\mathrm{Top}_k^{\tau} = \{\mathrm{top}_k(t) \mid t \in \mathrm{T}^{\tau}(\Sigma_{\mathrm{c}}, \emptyset)\}$ for all $\tau \in B$. We will omit superscript $^{\tau}$ of Top_k^{τ} whenever it is clear from the context.

Example 14 (top of a term). Let $B = \{\ \mathrm{N}, \mathrm{NList}\ \}$ and $\Sigma_{\mathrm{c}} = \{\ \mathsf{0}^{\mathrm{N}}, \mathsf{s}^{\mathrm{N}\to\mathrm{N}}, [\,]^{\mathrm{NList}}, :^{\mathrm{N}\times\mathrm{NList}\to\mathrm{NList}}\ \}$. Then $\mathrm{Top}_1^{\mathrm{N}} = \{x\}$, $\mathrm{Top}_2^{\mathrm{N}} = \{\mathsf{0},\ \mathsf{s}\ x\}$, $\mathrm{Top}_2^{\mathrm{NList}} = \{[\,],\ y : ys\}$, and $\mathrm{Top}_3^{\mathrm{NList}} = \{[\,],\ \mathsf{0} : [\,],\ \mathsf{0} : (y : ys),\ (\mathsf{s}\ x) : [\,],\ (\mathsf{s}\ z) : (w : ws)\}$.

Lemma 3 (finiteness of tops). Suppose $\Sigma_{\mathrm{c}} \subseteq \Sigma_{\mathrm{el}}$. If Σ_{c} is finite, then Top_k^{τ} is finite (modulo variable renaming) for any positive integer k and $\tau \in B$.

[1] If there is no ground constructor term of type τ then just ignore that type; otherwise take a minimal ground constructor term of type τ and regard it as a constant.

[2] Regard each ground constructor term of type τ as a constant.

To present a class of STTRSs for which higher-order sufficient completeness is decidable, we need the notion of higher-order scheme.

Definition 11 (higher-order scheme). An STTRS $\mathcal{R} = \langle B, \Sigma, R \rangle$ is said to be a *higher-order scheme* ($\mathrm{HS}(\mathcal{R})$) if every rewrite rule $l \to r \in R$ satisfies the properties that no higher-order variable occurs more than once in l and that for each subterm $(s\ t_1 \cdots t_n)$ in l and for any $i \in \{1, \ldots, n\}$, t_i is a variable whenever it is of higher-order type.

Example 15 (higher-order scheme). STTRSs in Examples 1, 2, 3, 7, 9, and 12 are all higher-order schemes. The STTRS in Example 6 is not.

Lemma 4. Let $\mathcal{R} = \langle B, \Sigma, R \rangle$ be an STTRS satisfying $\mathrm{EL}(\mathcal{R})$ and $\mathrm{HS}(\mathcal{R})$. Let $k = \mathrm{d}(\mathcal{R})$ and $f \in \Sigma_{\mathrm{d}}$. Then the following two statements are equivalent:

1. There exists a linear simply typed term t in normal form with $\mathrm{head}(t) = f$ such that for every argument subterm s of t, we have $s \in \mathrm{Top}_k$ if s has a basic type, otherwise s is a variable.
2. There exists a simply typed term t in normal form with $\mathrm{head}(t) = f$ such that every argument subterm s of t is a ground constructor term if s has a basic type and is a variable otherwise.

Combining Lemmata 3 and 4, we obtain the following theorem, which yields our main decidability result as a corollary with help of Theorem 5.

Theorem 6 (decidability of HQR). Let $\mathcal{R}$ be an STTRS which satisfies $\mathrm{HS}(\mathcal{R})$ and $\mathrm{EL}(\mathcal{R})$. Then it is decidable whether $\mathcal{R}$ is higher-order quasi-reducible.

Corollary 1 (decidability of HSC). Let $\mathcal{R}$ be an STTRS which satisfies $\mathrm{WN}^{\mathrm{b}}(\mathcal{R})$, $\mathrm{HS}(\mathcal{R})$ and $\mathrm{EL}(\mathcal{R})$. Then it is decidable whether $\mathcal{R}$ is higher-order sufficiently complete.

6 Related Works

Linestad et al. [14] extended Huet and Hullot's inductive theorem proving procedure [5] for Nipkow's higher-order rewriting framework [18]. They introduced the notion of first-order substitution—substitution that instantiates only first-order variables—and showed that when their inductive theorem proving procedure terminates with Proof, then the given equation $s \approx t$ is initially consistent with a convergent HRS $\mathcal{R}$, that is, the normal form of $s\sigma$ equals to that of $t\sigma$ for any first-order ground substitution σ. Clearly, the notion of first-order substitution is closely related to the way that we extend the notion of quasi-reducibility to higher-order. The notion of initial consistency is similar to the first-order inductive validity, but in our framework initial consistency is not easy to work with, since our notion of inductive validity starts from more abstract setting.

Kusakari et al. [13] introduced a higher-order equational logic and its extensional algebraic semantics based on higher-order term rewriting introduced by Kusakari [11]—regardless of difference of underlying formulation, Kusakari's higher-order term rewriting framework is included to simply typed term rewriting. They showed the quotient ground term algebra is an initial extensional model and introduced two kinds of inductive validity which correspond to first-order inductive theorem and monotone higher-order inductive theorem; they also presented inductive theorem proving framework similar to the one given in Example 2. In [12], Kusakari extends these results to S-expression rewriting systems (SRS, for short); there he also claims that any SRS $\mathcal{R}$ satisfies SC($\mathcal{R}$) whenever WN($\mathcal{R}$), EL($\mathcal{R}$), and HQR($\mathcal{R}$) hold.

Apart from the one mentioned above, there is a number of works on higher-order extensional algebraic structures and its equational proof theory. In a survey [16], higher-order extensional algebraic structures and its sound and complete equational calculus have been given based on a higher-order language with product and functional types. The equational calculus given there, however, seems difficult to relate with simply typed term rewriting.

7 Conclusion

We have proposed the notion of higher-order inductive theorem in simply typed term rewriting, which enables us to reason about equational specification in the presence of higher-order terms. It turned out that for the proof of an equation $l \approx r$ to be a higher-order inductive theorem of a given STTRS $\mathcal{R}$, it suffices to show the following three properties on ground terms: weak normalisation of $\mathcal{R}$, confluence of the extended system $\mathcal{R} \cup \{l\!\uparrow\, \to r\!\uparrow\}$, and the coincidence of reducibility in these two systems. There are two ways to guarantee the the third condition. One is to use the notion of coverset of substitutions, and the other is by sufficient completeness.

Since higher-order inductive theorems are not necessarily monotone, we have provided a method to guarantee this property. Higher-order sufficient completeness together with termination guarantees monotonicity. We have also proved that higher-order sufficient completeness is decidable for the class of weakly normalizing elementary higher-order schemes.

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