Solution to the Problem of Zantema on a Persistent Property of Term Rewriting Systems

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Abstract. A property P of term rewriting systems is persistent if for any many-sorted term rewriting system R, R has the property P iff its underlying term rewriting system $\Theta(R)$, which results from R by omitting its sort information, has the property P. It is shown that termination is a persistent property of many-sorted term rewriting systems that contain only variables of the same sort. This is the positive solution to a problem of Zantema, which has been appeared as Rewriting Open Problem 60 in literature.

1 Introduction

Sort introduction technique in term rewriting has caught attention recently [2][8]. We prove in this paper a conjecture which opens up a possibility of new applications of this technique. The conjecture reads: for any terminating many-sorted term rewriting system $\mathcal{R}$, if $\mathcal{R}$ contains only variables of the same sort then $\Theta(\mathcal{R})$ is also terminating. Here, $\Theta(\mathcal{R})$—called the underlying term rewriting system of $\mathcal{R}$—is the term rewriting system obtained from $\mathcal{R}$ by omitting its sort information. The conjecture was raised by Zantema and adapted in [4] as Rewriting Open Problem 60.

A property $\mathcal{P}$ of TRSs is said to be *persistent* if $\mathcal{R}$ has the property $\mathcal{P}$ iff $\Theta(\mathcal{R})$ has the property $\mathcal{P}$; this notion is due to Zantema [10]. We say $\mathcal{P}$ *is persistent for a class* $\mathcal{C}$ *of many-sorted TRSs* if for any $\mathcal{R} \in \mathcal{C}$, $\mathcal{R}$ has the property $\mathcal{P}$ iff $\Theta(\mathcal{R})$ has the property $\mathcal{P}$. Thus the conjecture above equals persistency of termination for the class of many-sorted TRSs that contain only variables of the same sort. It is known that confluence is persistent for the class of many-sorted TRSs [3] and that termination is persistent for the class of many-sorted TRSs that do not contain both duplicating and collapsing rules [10]. Persistent properties in equational rewriting have been studied in [9].

A property ϕ of TRSs is said to be *modular for the direct sum* if $\phi(\mathcal{R}_1)$ and $\phi(\mathcal{R}_2)$ imply $\phi(\mathcal{R}_1 \cup \mathcal{R}_2)$ for any two TRSs $\mathcal{R}_1$ and $\mathcal{R}_2$ sharing no function symbols. For component closed properties ϕ, persistency of ϕ for the class of many-sorted TRSs implies modularity of ϕ for the direct sum of TRSs [10]. Similarly, persistency of property ϕ for a subclass $\mathcal{C}$ of many-sorted TRSs often implies modularity of ϕ for the direct sum of TRSs from the class $\{\Theta(\mathcal{R}) \mid \mathcal{R} \in \mathcal{C}\}$. Indeed, if we assign sorts $1 \times \cdots \times 1 \to 1$ to function symbols in $\mathcal{R}_1$ and

$2 \times \cdots \times 2 \to 2$ to those in $\mathcal{R}_2$, modularity of ϕ for the direct sum of $\mathcal{R}_1$ and $\mathcal{R}_2$ is a particular case of persistency of ϕ provided that $\mathcal{R}_1 \cup \mathcal{R}_2$ under this sort assignment is contained in $\mathcal{C}$. Thus, for example, persistency of termination for the class of many-sorted TRSs that do not contain both duplicating and collapsing rules implies the corresponding modularity result for the direct sum of TRSs. To the contrary, our requirement on the sorts of variables does not carry over for TRSs and therefore it is hard to formalize the corresponding modularity result. This contrasts sharply with other persistency results obtained so far.

Our result is useful to show that termination is preserved under suitable translations of TRSs. More precisely, it follows that coding a function symbol $f(,\ldots,)$ by a suitable term $t(,\ldots,)$ does not affect termination behavior of TRSs.

The rest of this paper is organized as follows. In Section 2, we fix notations on many-sorted term rewriting and its underlying term rewriting. Section 3 is devoted to the proof of the Zantema's conjecture. In Section 4, applications of our theorem and related works are discussed. We conclude our result in Section 5.

2 Preliminaries

We assume familiarity with basic notions in term rewriting. In what follows, we recall some less common definitions and fix notations used in this paper.

Let $\mathcal{S}$ be a set of sorts (denoted by $\alpha, \beta, \gamma, \ldots$), $\mathcal{F}$ a set of $\mathcal{S}$-sorted function symbols (denoted by $f, g, h, \ldots$), $\mathcal{V}$ a set of $\mathcal{S}$-sorted variables (denoted by $x, y, z, \ldots$). We write $f : \alpha_1 \times \cdots \times \alpha_n \to \beta$ when $f \in \mathcal{F}$ has sort $\alpha_1 \times \cdots \times \alpha_n \to \beta$. We assume that there are countably infinite variables of sort α for each sort $\alpha \in \mathcal{S}$.

We denote by $\mathcal{T}$ (and $\mathcal{T}^\alpha$) the set of terms (of sort α, respectively). For a term t, $\mathcal{V}(t)$ is the set of variables that appear in t. Syntactical equality is denoted by $\equiv$. We write $t \trianglelefteq s$ ($t \triangleleft s$) when t is a *(proper) subterm of* s, or equivalently s *(properly) contains* t.

For each sort α, the *hole* of sort α is written as $\Box^\alpha$. A *context* is a term possibly containing holes. We denote by $\mathcal{C}$ the set of contexts. We write $C : \alpha_1 \times \cdots \times \alpha_n \to \beta$ when $C \in \mathcal{C}$ has sort β (as a term) and has n holes $\Box^{\alpha_1}, \ldots, \Box^{\alpha_n}$ from left to right in it. If $C : \alpha_1 \times \cdots \times \alpha_n \to \beta$ and $t_1 \in \mathcal{T}^{\alpha_1}, \ldots, t_n \in \mathcal{T}^{\alpha_n}$ then $C[t_1, \ldots, t_n]$ is the term obtained from C by replacing holes with $t_1, \ldots, t_n$ from left to right. A context C is written as $C[\,]$ when C contains precisely one hole. A context is said to be *empty* when $C \equiv \Box^\alpha$ for some $\alpha \in \mathcal{S}$.

We denote by $\mathrm{Pos}(t)$ the set of *positions* of a term t; by t/p the subterm of t at a position $p \in \mathrm{Pos}(t)$. $\mathrm{Pos}_{\mathcal{V}}(t)$ stands for $\{p \in \mathrm{Pos}(t) \mid t/p \in \mathcal{V}\}$. The *empty* (or *root*) position is denoted by Λ. For $u, v \in \mathrm{Pos}(t)$, we write $u \leq v$ when u is a prefix of v and $u \mid v$ when u and v are incomparable. For a context $C[\,]$, we write $C[\,]_p$ when $C[\,]/p$ is a hole. When $s \equiv C[u]_p$, we denote by $s[p \leftarrow v]$ the term $C[v]$. When $p \notin \mathrm{Pos}(s)$, we define $s[p \leftarrow v] \equiv s$. We abbreviate $(\cdots(s[p_1 \leftarrow v_1]) \cdots [p_n \leftarrow v_n])$ to $s[p_i \leftarrow v_i \mid 1 \leq i \leq n]$.

A *substitution* σ is a mapping from $\mathcal{V}$ to $\mathcal{T}$ such that x and $\sigma(x)$ have the same sort. A substitution is extended to a homomorphism from $\mathcal{T}$ to $\mathcal{T}$ in the obvious way; $t\sigma$ stands for $\sigma(t)$ for a substitution σ and a term t.

A *(many-sorted) rewrite rule* is a pair $l \to r$ of terms such that (1) l and r have the same sort, (2) $l \notin \mathcal{V}$, (3) $\mathcal{V}(r) \subseteq \mathcal{V}(l)$. A *many-sorted term rewriting system* (*STRS*, for short) is a set of rewrite rules. For an STRS $\mathcal{R}$, its reduction relation $s \to_{\mathcal{R}} t$ is defined as usual. Note that s and t have the same sort whenever $s \to_{\mathcal{R}} t$. The transitive-reflexive closure and the transitive closure of $\to_{\mathcal{R}}$ are denoted by $\overset{*}{\to}_{\mathcal{R}}$ and $\overset{+}{\to}_{\mathcal{R}}$, respectively.

When $\mathcal{S} = \{*\}$, an STRS is called a TRS. Given an arbitrary STRS $\mathcal{R}$, by identifying each sort with $*$, we obtain the TRS $\Theta(\mathcal{R})$—called the underlying TRS of $\mathcal{R}$. According to the definition of rewrite relation, $\Theta(\mathcal{R})$ acts on the set of terms possibly non-well-sorted with respect to the sorts of function symbols in $\mathcal{R}$. Fixing a STRS $\mathcal{R}$, we will denote by $\Theta(\mathcal{T})$ and $\Theta(\mathcal{C})$ the sets of terms and contexts possibly non-well-sorted with respect to the sorts of function symbols in $\mathcal{R}$ respectively. On the other hand, we preserve notation $C : \alpha_1 \times \cdots \times \alpha_n \to \beta$ only for well-sorted contexts; more precisely, we will write $C : \alpha_1 \times \cdots \times \alpha_n \to \beta$ only when C is well-sorted with respect to the sorts of function symbols in $\mathcal{R}$.

3 A Proof of the Conjecture

We prove in this paper the following theorem conjectured by Zantema.

Theorem 1. *Let $\mathcal{R}$ be an STRS that contains only variables of the same sort. Then $\mathcal{R}$ is terminating if and only if $\Theta(\mathcal{R})$ is terminating.*

For this purpose, *we assume in the sequel that there exists a special sort $0 \in \mathcal{S}$, and fix an STRS $\mathcal{R}$ such that every variable occurring in rules of $\mathcal{R}$ has sort* 0.

3.1 Sort introduction basics

Terms in $\Theta(\mathcal{T})$ can be partitioned into well-sorted components; and this partition yields a natural layered structure in each term. In this subsection, we present basic notion and notations related to this layered structure, and present a few key observations that follow from our assumption above.

From now on terms in $\Theta(\mathcal{T})$ are often referred to just terms.

Definition 2. 1. The *sort* of a term t is defined by

$$\mathrm{sort}(t) = \begin{cases} \alpha & \text{if } t \equiv x^{\alpha}, \\ \beta & \text{if } t \equiv f(t_1, \ldots, t_n) \text{ with } f : \alpha_1 \times \cdots \times \alpha_n \to \beta. \end{cases}$$

We extends the definition of sort for contexts in the obvious way by putting $\mathrm{sort}(\Box^{\alpha}) = \alpha$.

2. Let $t \equiv C[t_1, \ldots, t_n]$ $(n \geq 0)$ be a term with non-empty C. We write $t \equiv C[\![t_1, \ldots, t_n]\!]$ if (1) $C : \alpha_1 \times \cdots \times \alpha_n \to \beta$ and (2) $\mathrm{sort}(t_i) \neq \alpha_i$ for $i = 1, \ldots, n$. We call the terms $t_1, \ldots, t_n$ *principal subterms* of t. We denote by $\mathrm{cap}(t)$ the well-sorted context $C[\Box^{\alpha_1}, \ldots, \Box^{\alpha_n}]$. Clearly, every term t is uniquely written as $C[\![t_1, \ldots, t_n]\!]$ by some $C \in \mathcal{C}$ and terms $t_1, \ldots, t_n$.
3. The *rank* of a term t is defined by:

$$\mathrm{rank}(t) = \begin{cases} 1 & \text{if } t \in \mathcal{T}, \\ 1 + \max\{\mathrm{rank}(t_i) \mid 1 \leq i \leq n\} & \text{if } t \equiv C[\![t_1, \ldots, t_n]\!] \ (n > 0). \end{cases}$$

4. A subterm u of t is said to be *special* (in t) if either $u \equiv t$ or u is a principal subterm of a special subterm of t.

Convention 1. We denote $C[u]$ by $C[!u]$ if this occurrence of u is a proper special subterm of $C[u]$. We indicate $C[u_1, \ldots, u_n]$ as $C[u_1, \ldots, (u_i)_\gamma, \ldots, u_n]$ when $\Box^\gamma$ is not special in $C[u_1, \ldots, \Box^\gamma, \ldots, u_n]$. In the sequel, we assume every hole in non-empty $C \in \Theta(\mathcal{C})$ is not a special subterm of C; in this spirit, we often omit the superscript α of $\Box^\alpha$.

We next formalize the notion of peak and leaf, which are essential in our proof.

Definition 3. 1. For a non-empty position p in a term t, the sort of the position p in t is defined as: $\mathrm{sort}(t, p) = \gamma$ when $t \equiv C[(u)_\gamma]_p$ for some C and u.
2. For a term t, we define the set of *disconnections in* t as follows:

$$\mathrm{Dcn}(t) = \{p \in \mathrm{Pos}(t) \mid t/p \text{ is a special subterm of } t\}.$$

For a context C, $\mathrm{Dcn}(C)$ is defined similarly. It is clear that for every non-empty position p, we have $p \in \mathrm{Dcn}(t)$ iff $\mathrm{sort}(t, p) \neq \mathrm{sort}(t/p)$.
3. Let $t \equiv C[!u]_q$. The *peak of* u is the position p defined by

$$p = \max\{o \leq q \mid \mathrm{sort}(t, o) = 0 \text{ or } o \in \mathrm{Dcn}(t) \backslash \{q\}\}.$$

Clearly, C/p is again a context; we call it the *leaf* of C and denote it by $\mathrm{leaf}(C)$.

Let us illustrate our notation and concepts by an example.

Example 1. Let $\mathcal{F} = \{f : 0 \times 1 \to 0, g : 1 \to 1, h : 1 \to 0, a : 0, b : 0, c : 1\}$. Let $s \equiv f(h(f(h(a), g(b))), c)$. Let $C_1 \equiv f(h(\Box^1), c)$. Then we can write s as $C_1[(!f(h(a), g(b)))_1]$. We have $\mathrm{sort}(s, 1.1) = \mathrm{sort}(s, 1.1.2) = 1$ and $\mathrm{Dcn}(s) = \{\Lambda, 1.1, , 1.1.1.1, 1.1.2.1\}$. The peak of a in s is 1.1.1 and that of b in s is 1.1. Let $C_2 \equiv f(h(f(h(\Box^1), g(b))), c)$. Then $s \equiv C_2[!a]$ and $\mathrm{leaf}(C_2) \equiv h(\Box^1)$.

Definition 4. A rewrite step $s \to t$ is said to be *inner* (written as $s \to^i t$) if

$$s \equiv C[\![s_1, \ldots, C'[l\sigma], \ldots, s_n]\!] \to C[s_1, \ldots, C'[r\sigma], \ldots, s_n] \equiv t$$

for some terms $s_1, \ldots, s_n$, a substitution σ, $l \to r \in \mathcal{R}$, and $C' \in \Theta(\mathcal{C})$; otherwise it is *outer* (written as $s \to^o t$). The redex of an inner reduction is a *inner redex*; that of an outer reduction is an *outer redex*.

The next lemma is the first key observation arising from our assumption.

Lemma 5. *Let $l \to r \in \mathcal{R}$ and $l \equiv C_l[x_1, \ldots, x_k]_{o_1,\ldots,o_k}$ where C_l is a (non-empty) context containing no variables. Suppose $s \equiv C[l\sigma]_o$ and $l\sigma$ is an outer redex in s. Then,*

1. $\mathrm{sort}(s, o.o_i) = 0$ *for all* $i = 1, \ldots, k$;
2. *for the peak p of each principal subterm of s, we have either $o.o_i \leq p$ for some i, $p < o$ or $p \mid o$. (See Figure 1[1].)*

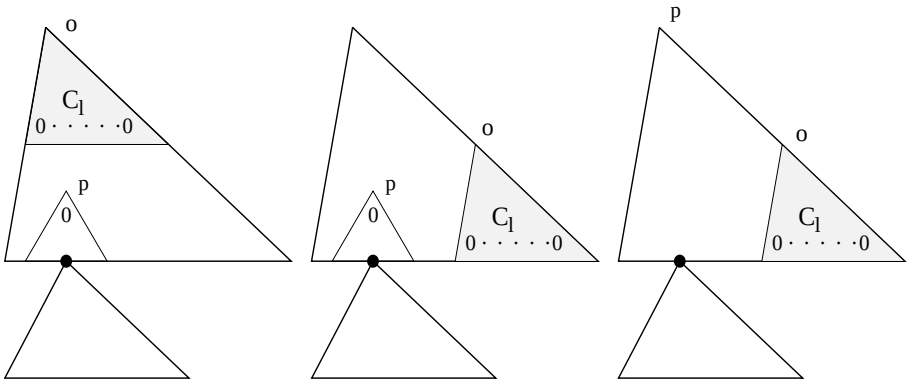

Fig. 1. The peak of a principal subterm and an outer redex

Definition 6. A rewrite step $s \to^o t$ is said to be *destructive at level* 1 if $\mathrm{sort}(s) \neq \mathrm{sort}(t)$; and $s \to^i t$ is destructive at level $k+1$ if

$$s \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!] \to^i C[s_1, \ldots, t_j, \ldots, s_n] \equiv t$$

with $s_j \to t_j$ destructive at level k.

The following lemma appears often (see e.g. [2]).

Lemma 7. *A rewrite step $s \to^o t$ is destructive if and only if $t \equiv \sigma(x)$ and $s \equiv C[\![s_1, \ldots, \sigma(x), \ldots, s_n]\!]$ for some terms $s_1, \ldots, s_n$, substitution σ, and $C \in \mathcal{C}$ such that $C[s_1, \ldots, \Box, \ldots, s_n] \equiv C'\sigma$ for some $C'[x] \to x \in \mathcal{R}$.*

The second key observation derived from our assumption concerns with destructive rewrite steps.

Lemma 8. *Let $s \equiv C[\![s_1, \ldots, s_n]\!]_{q_1,\ldots,q_n}$, and suppose that $s \to s_i$ is a destructive rewrite step of level 1, and p_j is the peak of s_j in s for each $j = 1, \ldots, n$. Then,*

1. $\mathrm{sort}(s) = 0$, $\mathrm{sort}(s, q_i) = 0$ *and* $\mathrm{sort}(s_i) \neq 0$;
2. $p_i = q_i$ *and p_i is minimal in* $\{p_1, \ldots, p_n\}$.

[1] In this and the succeeding figures, • denotes a non-empty disconnection.

3.2 Active subterms and $\rightrightarrows$-reduction

A crucial step of our proof is to define a particular mapping from terms to terms that maps a $\Theta(\mathcal{R})$-reduction to another with some "good" property. This kind of technique is popular in modularity proofs (e.g. pile and delete technique [6], top-down labelling technique [2]). Our mapping results from a confluent and terminating reduction (called $\rightrightarrows$-reduction), which we introduce in this subsection.

We first introduce a notion of active subterms.

Definition 9. 1. Let $s \equiv C[\![s_1, \ldots, s_n]\!]$, and suppose that p_i is the peak of s_i for each $i = 1, \ldots, n$. The term s is said to be *root-active* if $\mathrm{sort}(s) = 0$ and s_i is root-active whenever $p_i = \Lambda$.
2. A subterm of s is said to be *active (in s)* if (1) it is a proper special subterm of s and (2) it is root-active.

Definition 10. Let $s \equiv C[(!u)_\gamma]_q$ and $u \equiv C'[\![u_1, \ldots, u_n]\!]_{q_1,\ldots,q_n}$. Suppose p is the peak of u in s and p_i is the peak of u_i in s for each $i = 1, \ldots, n$. Then, this occurrence of u is said to be *properly active (in s)* if (1) $\mathrm{sort}(u) = 0$ and (2) $q < p_i$ for all i $(1 \leq i \leq n)$. When u is properly active, u_i $(1 \leq i \leq n)$ is called an *activator of u (in s)* if (1) p_i is minimal in $\{p_1, \ldots, p_n\}$ and (2) $\mathrm{sort}(u, q_i) = 0$; if further it satisfies (3) $\mathrm{sort}(u_i) = \gamma$, then u_i is called a *connectable activator of u (in s)*. Activators that are not connectable are *inconnectable activators*. (See Figure 2.)

It is readily checked that if $s \equiv C[(!u)_\gamma]_q$, $u \equiv C'[\![u_1, \ldots, u_n]\!]_{q_1,\ldots,q_n}$, p is the peak of u in s and p_i is the peak of u_i in s for each $i = 1, \ldots, n$, then u is active iff (1) $\mathrm{sort}(u) = 0$ and (2) u_i is active in s whenever $q = p_i$. Thus, every properly active subterm is active.

Intuitively, properly active subterms u of a term s are special subterms whose top layer might collapse by a reduction; activators are principal subterms of u that might come up when the top layer of u collapses. Active subterms of a term are special subterms whose descendants might become properly active.

Note that if u is an active subterm of $s \equiv C[!u]_q$ then, since $\mathrm{sort}(u) = 0$ and $q \in \mathrm{Dcn}(s)$, we have $\mathrm{sort}(s, q) \neq 0$, and hence $\mathrm{leaf}(C)$ is non-empty.

Example 2. In Example 1, subterms of s that are root-active are s, $h(f(h(a), g(b)))$ $f(h(a), g(b))$, $h(a)$, a and b; active subterms in s are $f(h(a), g(b))$, a and b; properly active subterms in s are a and b. Let t be the term obtained by replacing b in s with $f(c, c)$, i.e. $t \equiv s[1.1.2.1 \leftarrow f(c, c)]$. Then $f(c, c)$ is a properly active subterm of t, which has a connectable activator c.

The next lemma is an easy consequence of the definition above, Lemma 5 and Lemma 8.

Lemma 11. *Let $s \to t$ a rewrite step destructive at level $k > 1$. Then there exists a subterm u of s properly active in s and an activator u_i of u in s and a context C such that $u \to u_i$, $s \equiv C[u]$ and $t \equiv C[u_i]$.*

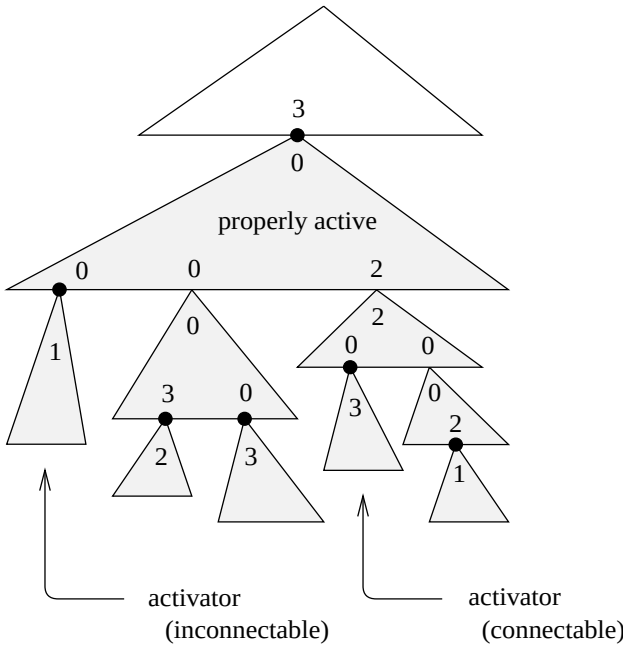

Fig. 2. A properly active subterm and activators

Now we are in a position to define the $\rightrightarrows$-reduction.

Definition 12. Let $s \equiv C[!u]$ and p the peak of u in s.

1. This occurrence of u in s is said to be a $\rightrightarrows$-*redex* if (1) u is properly active in s and (2) s/p is root-active.
2. The $\rightrightarrows$-redex u is a *left-most inner-most* $\rightrightarrows$-*redex* if (3) every proper special subterms of s/p with the peak different from p is not a $\rightrightarrows$-redex and (4) every principal subterms of $\mathrm{leaf}(C)$ placed on the left to the hole is not a $\rightrightarrows$-redex if its peak equals p.
3. Suppose that the $\rightrightarrows$-redex u is a left-most inner-most $\rightrightarrows$-redex. Let $\tilde{u} \equiv C'[\mathrm{leaf}(C)[u_1], \ldots, \mathrm{leaf}(C)[u_n]]$, where $u \equiv C'[u_1, \ldots, u_n]$ with all activators $u_1, \ldots, u_n$ of u displayed. Let $t \equiv s[p \leftarrow \tilde{u}]$. Then we say that s $\rightrightarrows$-*reduces* to t (symbolically $s \rightrightarrows t$) or t is a $\rightrightarrows$-*reduct* of s; we call $s \rightrightarrows t$ a $\rightrightarrows$-rewrite step. (See Figure 3.) The term u is called the $\rightrightarrows$-*redex* of this $\rightrightarrows$-rewrite step and we write $s \rightrightarrows_u t$ when this is the case. A $\rightrightarrows$-*normal* term is a term s such that $s \rightrightarrows t$ for no term t.

It is clear that every $\rightrightarrows$-normal term contains no $\rightrightarrows$-redex.

Example 3. In Example 2, $t \equiv f(h(f(h(a), g(f(c,c)))), c) \rightrightarrows f(h(f(a, g(f(c,c)))), c)$ $\rightrightarrows f(h(f(f(a, g(c)), c)), c) \rightrightarrows f(f(f(a, g(c)), c), c)$.

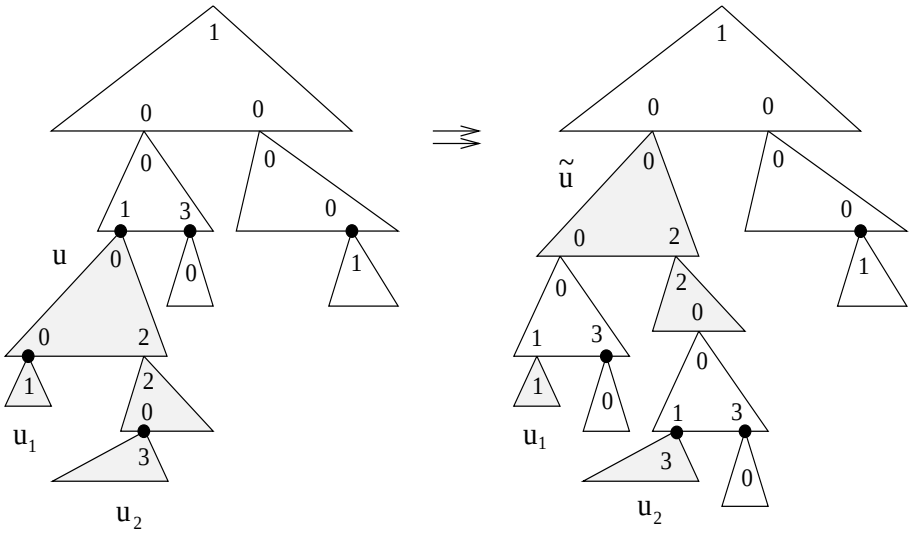

Fig. 3. A $\rightrightarrows$-rewrite step

We note that one can no longer expect a confluent $\rightrightarrows$-reduction if we define $\rightrightarrows$-reduction based on arbitrary $\rightrightarrows$-redexes instead of left-most inner-most $\rightrightarrows$-redexes.

We will need a notion of descendants of $\rightrightarrows$-reduction in the succeeding proofs.

Definition 13. Let $A : s \rightrightarrows_u t$ be a $\rightrightarrows$-rewrite step. Suppose $s \equiv C[u]_q$ and p is the peak of u in s. For $o \in \mathrm{Pos}(s)$, the set $A(o)$ of *descendants of* o by A is defined as follows: Let $u_1, \ldots, u_n$ $(n \geq 0)$ be activators of u with $u \equiv C[u_1, \ldots, u_n]_{q_1,\ldots,q_n}$. Then,

$$A(o) = \begin{cases} \{o\} & \text{if } o < p \text{ or } o \mid p, \\ \{p.q_i.(o\backslash p) \mid 1 \leq i \leq n\} & \text{if } p < o \text{ and } q \not\leq o, \\ \{p.(o\backslash q)\} & \text{if } q < o \text{ and } \forall i\ (q.q_i \not\leq o), \\ \{p.q_j.(p\backslash q).(o\backslash(q.q_j))\} & \text{if } q.q_j < o \text{ for some } j\ (1 \leq j \leq n). \\ \emptyset & \text{if } o \in \{p, q, q.q_1, \ldots, q.q_n\}. \end{cases}$$

For $u \equiv s/o$, the set of descendants of u by A is $\{t/o' \mid o' \in A(o)\}$.

One can readily check the next lemma.

Lemma 14. *Let $A : s \rightrightarrows_u t$ be a $\rightrightarrows$-rewrite step. Suppose $s \equiv C[u]_q$ and p is the peak of u in s. Let $u_1, \ldots, u_n$ $(n \geq 0)$ be activators of u with $u \equiv C'[u_1, \ldots, u_n]_{q_1,\ldots,q_n}$. Let $P = \bigcup\{A(o) \mid o \in \mathrm{Dcn}(s)\}$ and $Q = \{p.(q_i\backslash q).(q\backslash p) \mid u_i$ is inconnectable, $1 \leq i \leq n\}$. Then,*

$$\mathrm{Dcn}(t) = \begin{cases} P \cup Q & \text{if } p \notin \mathrm{Dcn}(s), \\ \{p\} \cup P \cup Q & \text{if } p \in \mathrm{Dcn}(s). \end{cases}$$

Lemma 15. *Let $A : s \rightrightarrows_u t$ be a $\rightrightarrows$-rewrite step, and p the peak of u in s. Then,*

1. *s/p is active iff t/p is active;*
2. *every descendant of an active subterm is active;*
3. *every active subterm of t different from t/p is a descendant of an active subterm different from u and s/p.*

Proof. By Lemma 14 and definition, it suffices to prove that s/o is active iff t/o is active, where $o = \max\{p' \leq p \mid s/p' \text{ is a special subterm of } s\}$. We distinguish two cases:

1. $o = p$. The case where $p = \Lambda$ is trivial. Otherwise, suppose that s/p is a proper special subterm of s. We show s/p and t/p are active in s and t respectively. Since s/p is root-active, s/p is active in s. We know hence $\mathrm{sort}(s/p) = 0, \mathrm{sort}(s,p) \neq 0$ and every principal subterm of s/p with the peak p is active. If u has no activator, then $t/p \equiv u$, which is active in t. Otherwise suppose u has activators $u_1, \ldots, u_n$ $(n > 0)$ with $u \equiv C'[u_1, \ldots, u_n]$. First, $\mathrm{sort}(t/p) = \mathrm{sort}(u) = 0$. By definition, each principal subterm of t/p is either a descendant of a principal subterm of u that is not an activator, or a principal subterm of $\mathrm{leaf}(C')[u_i]$ $(1 \leq i \leq n)$. Principal subterms from the former do not have the peak p in t, because u is properly active in s. Principal subterms from the latter also do not have the peak p, because $\mathrm{sort}(t, p.q_i) = 0$. Thus, t/p is active.
2. $o < p$. Then s/p is not special. Thus, $\mathrm{sort}(s,p) = 0$ and so the principal subterms of s/o contained in s/p can not have the peak o. Also, the principal subterms of s/o not contained in s/p coincide those of t/o not contained in t/p. Therefore, s/o is active in s iff t/o is active in t.

Definition 16. For each special subterm u of t, we define $\mathrm{depth}(u)$ inductively as follows: (1) $\mathrm{depth}(t) = 0$. (2) Suppose u is a special subterm of $\mathrm{depth}(u) = k$ and $u \equiv C[\![(u_1)_{\gamma_1}, \ldots, (u_n)_{\gamma_n}]\!]$. Then for $i = 1, \ldots, n$

$$\mathrm{depth}(u_i) = \begin{cases} k \dot{-} 1 & \text{if } \gamma_i = 0, \\ k + 1 & \text{if } \mathrm{sort}(u_i) = 0, \\ k & \text{otherwise.} \end{cases}$$

Lemma 17. *The relation $\rightrightarrows$ is well-founded and confluent.*

Proof. For each $\rightrightarrows$-redex v in s, let $\mathrm{weight}(v) = \langle \mathrm{depth}(v), \mathrm{rank}(v), M \rangle$ where $M = [\langle \mathrm{depth}(v'), \mathrm{rank}(v') \rangle \mid v' \text{ is an active subterm of } s \text{ different from } v \text{ yet has the same peak as } v\ ]$. And, for each term s, let $\|s\| = [\mathrm{weight}(u) \mid u \text{ is a } \rightrightarrows\text{-redex in } s\ ]$.

Let $\gtrdot$ be an order defined by: $\langle i_1, j_1 \rangle \gtrdot \langle i_2, j_2 \rangle$ iff either (1) $i_1 > i_2$ or (2) $i_1 = i_2$ and $j_1 > j_2$; and $\ggg$ the multiset extension of $\gtrdot$. Let $\succ$ be an order defined by: $\langle i_1, i_2, M_1 \rangle \succ \langle j_1, j_2, M_1 \rangle$ iff either (1) $i_1 > i_2$, (2) $i_1 = i_2$ and $j_1 > j_2$ or (3) $i_1 = i_2$, $j_1 = j_2$ and $M_1 \ggg M_2$; and $\succ\!\!\succ$ the multiset extension of $\succ$. We are going to verify that $s \rightrightarrows t$ implies $\|s\| \succ\!\!\succ \|t\|$.

Suppose $A : s \rightrightarrows_u t$, and let p be the peak of u in s. By the definition of $\rightrightarrows$-reduction, if v is a $\rightrightarrows$-redex in t then either (1) $v \equiv t/p$, (2) v is a descendant of a $\rightrightarrows$-redex in s different from u and s/p, or (3) v is a descendant of a properly active subterm of s with the peak $q.q_i$ for some i.

If t/p is a $\rightrightarrows$-redex then s/p is special by Lemma 14 and so $\mathrm{depth}(s/p) = \mathrm{depth}(t/p)$. Therefore, we have $\mathrm{weight}(u) \succ \mathrm{weight}(t/p)$, because $\mathrm{depth}(u) = \mathrm{depth}(s/p) + 1$. If v is a descendant of a properly active subterm of s with the peak $q.q_i$ for some i, then we have $\mathrm{weight}(u) \succ \mathrm{weight}(v)$, because $\mathrm{depth}(u) = \mathrm{depth}(v)$ and $\mathrm{rank}(u) > \mathrm{rank}(v)$. Thus, because u has no descendant, it suffices to show that $[\mathrm{weight}(v)] \succcurlyeq\!\succcurlyeq [\mathrm{weight}(t/o') \mid o' \in A(v)]$ for any $\rightrightarrows$-redex v of s different from u and s/p.

Let v be a $\rightrightarrows$-redex in s and o the position of v in s. We distinguish two cases:

1. $o < p$ or $o \mid p$. Then $A(o) = \{o\}$ and therefore it suffices to show $\mathrm{weight}(v) \geq \mathrm{weight}(t/o)$. For any subterm s/o' satisfying $o' \leq p$ or $o' \mid p$, the following hold: (1) s/o' is active iff t/o' is active (by Lemma 15); (2) $\mathrm{rank}(s/o') \geq \mathrm{rank}(t/o')$ (for, $\mathrm{rank}(s/p) \geq \mathrm{rank}(t/p)$); (3) $\mathrm{depth}(s/o') = \mathrm{depth}(t/o')$. Therefore, we have $\mathrm{weight}(v) \geq \mathrm{weight}(t/o)$.
2. $p < o$. By definition, $\rightrightarrows$-redexes contained in s/p have the peak p. Thus, for any descendant v' of v, $\mathrm{depth}(v) = \mathrm{depth}(v')$ and $\mathrm{rank}(v) = \mathrm{rank}(v')$. Further we have $\pi_3(\mathrm{weight}(v)) \backslash \pi_3(\mathrm{weight}(v')) = [\langle \mathrm{depth}(u), \mathrm{rank}(u)\rangle]$, and $\mathrm{depth}(u) = \mathrm{depth}(w)$ and $\mathrm{rank}(u) > \mathrm{rank}(w)$ for any $\langle \mathrm{depth}(w), \mathrm{rank}(w)\rangle \in \pi_3(\mathrm{weight}(v')) \backslash \pi_3(\mathrm{weight}(v))$[2]. Thus, we get $\pi_3(\mathrm{weight}(v)) > \pi_3(\mathrm{weight}(v'))$, and so $\mathrm{weight}(v) > \mathrm{weight}(v')$.

Thus $s \rightrightarrows t$ implies $\|s\| \succ\!\succ \|t\|$. It is readily checked $\succ\!\succ$ is well-founded; hence so is $\rightrightarrows$.

Next, $\rightrightarrows$ is locally confluent; for, if two left-most inner-most $\rightrightarrows$-redexes occur in a term then their peaks are disjoint. Therefore $\rightrightarrows$ is confluent.

Thus every term s has the unique $\rightrightarrows$-normal form; we denote it by $s{\downarrow\downarrow}$.

3.3 Simulating $\Theta(\mathcal{R})$-reductions in $\rightrightarrows$-normal terms

To show termination of $\mathcal{R}$ implies that of $\Theta(\mathcal{R})$, we derive a contradiction assuming that there exists an infinite $\Theta(\mathcal{R})$-reduction sequence and that $\mathcal{R}$ is terminating. To this end, we next map infinite $\Theta(\mathcal{R})$-reductions to infinite $\Theta(\mathcal{R})$-reductions consisting of $\rightrightarrows$-normal terms.

We write $s \to_{[p, l\to r]} t$ when $s \equiv C[l\sigma]_p$ and $t \equiv C[r\sigma]_p$ for some context C, substitution σ and $l \to r \in \mathcal{R}$.

Definition 18. Let $A : s \to_{[p, l\to r]} t$ be a rewrite step. For $o \in \mathrm{Pos}(s)$, the set

[2] Here, $\pi_i\langle a_1, a_2, a_3\rangle = a_i$.

$A(o)$ of *descendants of* o by A is defined as follows:

$$A(o) = \begin{cases} \{o\} & \text{if } o < p \text{ or } o \mid p, \\ \{p.p_3.p_2 \mid r/p_3 = l/p_1\} & \text{if } o = p.p_1.p_2 \text{ with } p_1 \in \mathrm{Pos}_{\mathcal{V}}(l), \\ \emptyset & \text{otherwise.} \end{cases}$$

For $u \equiv s/q$, the set of descendants of u by A is the set $\{t/q' \mid q' \in A(q)\}$.

We first need the following lemma.

Lemma 19. *If $s \to t$ and s is $\rightrightarrows$-normal, then t is also $\rightrightarrows$-normal.*

Proof. We distinguish two cases:

1. $s \to t$ is not destructive. Let $s \equiv C[!u]_q$, $t \equiv C[!u']_q$ and $u \to^o u'$. By the definition of the $\rightrightarrows$-redex, it suffices to show that (a) no principal subterm of u' is a $\rightrightarrows$-redex. (b) u' itself is not a $\rightrightarrows$-redex.
   (a) Suppose that there exists a principal subterm v' of u' that is a $\rightrightarrows$-redex. Then there exists a principal subterm v of u with descendant v'. Let the peak of v in s be p and that of v' in t p'. Since $u \to u'$ is outer, v' is properly active iff so is v. Thus, by our assumption, we know t/p' is root-active while s/p is not. But this can not happen by Lemma 5.
   (b) Suppose u' is a $\rightrightarrows$-redex. By our assumption, this implies that u' is properly active while u is not. Thus, since $\mathrm{sort}(u) = \mathrm{sort}(u') = 0$, there exists a principal subterm v of u with peak q. But then, by Lemma 5, there exists a descendant of v with the peak q. This is a contradiction.
2. $s \to t$ is destructive. If $s \to t$ is destructive at level 1 then the statement holds obviously. Suppose $s \to t$ is destructive at level $k > 1$. Then, by Lemma 11, we have $s \equiv C[(!u)_\gamma]$, $t \equiv C[u_i]$ and $u \to u_i$ for some u properly active in s and u_i an activator of u in s. Let p the peak of u in s.
   If $\gamma \neq \mathrm{sort}(u_i)$, then the only possible $\rightrightarrows$-redex in t is u_i, but, since $\mathrm{sort}(u_i) \neq 0$, this can not be a new $\rightrightarrows$-redex. Therefore, suppose otherwise, i.e. $\gamma = \mathrm{sort}(u_i)$. Then the only possible $\rightrightarrows$-redex in t is (a) a principal subterm of u_i and (b) the minimal special subterm of t that contains t/p.
   (a) Suppose that there exists a principal subterm w of u_i that is a $\rightrightarrows$-redex. Then the peak of the descendant of w in t equals p. For otherwise, the peak of w in s is contained in u_i, and therefore it follows that w is a $\rightrightarrows$-redex. Thus, by our assumption, we know that t/p is root-active. But this implies s/p is root-active since u is properly active. Therefore, it follows that u is a $\rightrightarrows$-redex, which is not the case.
   (b) Suppose that w is the minimal special subterm of t that contains t/p and w is a $\rightrightarrows$-redex. Let v be the minimal special subterm of s that contains s/p. From the fact that t/p is properly active while s/p is not, we know $t/p \equiv w$, because u is the only principal subterm of v that has no descendant that is a principal subterm of w. But this implies, since u and w are properly active, s/p is root-active, and hence u is a $\rightrightarrows$-redex. This is a contradiction.

We also need in our proof the fact that our mapping from terms to $\rightrightarrows$-normal terms preserves infiniteness of $\Theta(\mathcal{R})$-reductions. A notion of head reductions is required to show this.

Definition 20. Let $t \equiv C[\![t_1, \ldots, t_n]\!]$, and p_i the peak of t_i for each $i = 1, \ldots, n$. The *head of* t is defined as: $\mathrm{head}(t) = t[p_i \leftarrow \Box \mid 1 \leq i \leq n]$. We write $t \equiv C'\langle\!\langle u_1, \ldots, u_k \rangle\!\rangle$ when $t \equiv C'[u_1, \ldots, u_k]$ and C' is the head of t.

Definition 21. A rewrite step $s \to t$ is said to be *a body reduction* (written as $s \to^{bd} t$) if

$$s \equiv C[\![s_1, \ldots, C'[l\sigma], \ldots, s_n\rangle\!] \to C[s_1, \ldots, C'[r\sigma], \ldots, s_n] \equiv t$$

for some terms $s_1, \ldots, s_n$, a substitution σ, $l \to r \in \mathcal{R}$, and $C' \in \Theta(\mathcal{C})$; otherwise it is *a head reduction* (written as $s \to^{hd} t$). The redex of a body reduction is a *body redex*; that of a head reduction a *head redex*.

The following commutativity lemma is a basis of our reduction simulation.

Lemma 22. *If $s \to t$ and $s \rightrightarrows s_1$, then there exists s_2, t_1 such that $s_1 \overset{*}{\rightrightarrows} s_2 \overset{*}{\to} t_1 \overset{*}{\leftleftarrows} t$; further if $s \to^{hd} t$ then $s_2 \to^{hd} t_1$.*

Proof. Suppose $A : s \equiv \hat{C}[l\sigma]_o \to \hat{C}[r\sigma] \equiv t$ and $B : s \rightrightarrows_u s_1$. Let $s \equiv C[u]_q$, p the peak of u. We distinguish the cases according to the position of the redex of $s \to t$:

1. $o \mid p$. Then, we have $s_1 \equiv s_2 \to_{[o, l\to r]} t_1 \leftleftarrows_{t/q} t$ for some t_1.
2. $o < p$. Then $o.o_x \leq p$ for some $o_x \in \mathrm{Pos}_{\mathcal{V}}(l)$ by Lemma 5. Let $\{o_1, \ldots, o_l\} = \{o' \in \mathrm{Pos}_{\mathcal{V}}(l) \mid l/o' = l/o_x, o' \neq o_x\}$ and $\{v_1, \ldots, v_k\} = A(s/o_x)$. Then v_i is a $\rightrightarrows$-redex for all $i = 1, \ldots, k$ and so is $s/(o.o_i.(q \backslash o_x))$ for all $i = 1, \ldots, l$. Thus, we have $s_1 \rightrightarrows_{o_1} \cdots \rightrightarrows_{o_l} s_2 \to_{[o, l\to r]} t_1 \leftleftarrows_{v_k} \cdots \leftleftarrows_{v_1} t$ for some t_1.
3. $p \leq o$. Let $u_1, \ldots, u_n$ $(n \geq 0)$ are activators of u and $u \equiv C'[u_1, \ldots, u_n]$.
   (a) $q \mid o$. Then $C[\,]_q \to_{[o, l\to r]} C^*[\,]_q$ and $\mathrm{leaf}(C) \to_{[o \backslash p, l\to r]} \mathrm{leaf}(C^*)$ for some C^*. Let $B(o) = \{o_1, \ldots, o_n\}$. Then s_1/o_i is a redex for any $i = 1, \ldots, n$, and so we have $s_1 \equiv s_2 \to_{[o_1, l\to r]} \cdots \to_{[o_n, l\to r]} t_1 \leftleftarrows_{t/q} t$.
   (b) $q.q_i \not\leq o$ for all i $(1 \leq i \leq n)$. If $o = q$ and $s \to t$ is destructive, then $r\sigma$ is an activator by Lemma 11, and so we have $s_1 \to_{[p, l\to r]} t_1 \equiv t$. (See Figure 4.) Otherwise, for any $i = 1, \ldots, n$, either $o \mid q.q_i$ or $o.o_x \leq q.q_i$ for some $o_x \in \mathrm{Pos}_{\mathcal{V}}(l)$. Hence, t/q' is an activator of t/p for any $q' \in \bigcup_i A(q.q_i)$. Also, $s_1/p.(q \backslash o)$ is a ($\mathcal{R}$-)redex, because $u_i \equiv u_j$ implies $\mathrm{leaf}(C')[u_i] \equiv \mathrm{leaf}(C')[u_j]$. Thus, we have $s_1 \to_{[p.(o \backslash q), l\to r]} t_1 \leftleftarrows_{t/q} t$. (Note that there may be the case that $\bigcup_i B(q.q_i) = \emptyset$; however, by the definition of $\rightrightarrows$-reduction, we have $t_1 \leftleftarrows_{t/q} t$ also in this case.)
   (c) $q.q_i \leq o$ for some i $(1 \leq i \leq n)$. Let $u_j \to u'_j$ with $s \equiv C'[u_j]$ and $t \equiv C'[u'_j]$. Since $\mathrm{sort}(u_j) \neq 0$, $u_j \to u'_j$ is not destructive at level 1 by Lemma 8. Hence $\mathrm{sort}(u_j) = \mathrm{sort}(u'_j)$ and so u'_j is an activator of the descendants of u. Thus, we have $s_1 \to_{[p.q_j.(q \backslash p).(o \backslash (q.q_j)), l\to r]} t_1 \leftleftarrows_{t/q} t$.

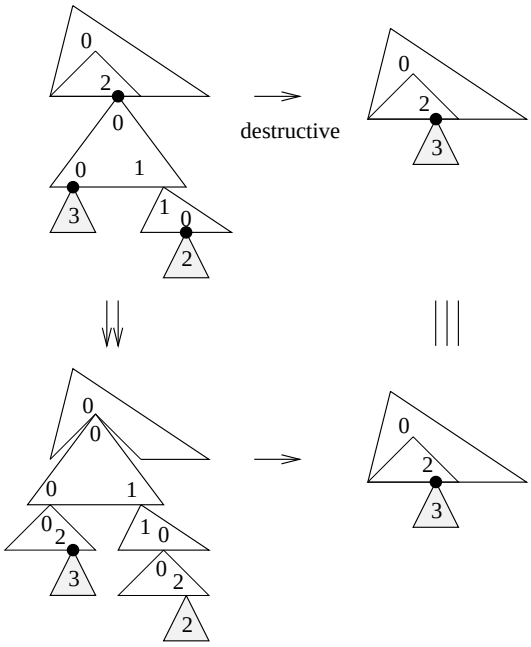

Fig. 4. Destructive step from an active subterm

Finally, observe that $s \to^{hd} t$ only in the cases 1 or 2.

Lemma 23. *If $s \to t$, then $s{\downarrow\downarrow} \stackrel{*}{\to} t{\downarrow\downarrow}$; further, if $s \to^{hd} t$, then $s{\downarrow\downarrow} \to^{hd} t{\downarrow\downarrow}$.*

Proof. For any term u, let $\sharp u = \max\{n \mid n$ is the length of $u \stackrel{*}{\rightrightarrows} u{\downarrow\downarrow}\}$. Note $\sharp u$ is well-defined since $\stackrel{*}{\rightrightarrows}$ is terminating (Lemma 17) and every term contains only finitely many $\rightrightarrows$-redexes. Using Lemma 19 and Lemma 22, it is not difficult to show $s \stackrel{*}{\to} t$ implies $s{\downarrow\downarrow} \stackrel{*}{\to} t{\downarrow\downarrow}$ by induction on $\sharp s$.

Definition 24. Let $s \to t$ be a destructive rewrite step at level $k > 1$. Then by Lemma 11 we have $s \equiv C[!u]$, $t \equiv C[u_i]$ and $u \to u_i$ for some u properly active in s and u_i an activator of u in s. Let p be the peak of u in s. We say $s \to t$ *critically destructive* if s/p is root-active.

The two lemmata below are proved in a straightforward way.

Lemma 25. *Suppose $\mathcal{R}$ is terminating. Every infinite $\Theta(\mathcal{R})$-reduction sequence containing infinitely many head reductions contains infinitely many rewrite steps that are critically destructive.*

Lemma 26. *If $s \to t$ and s is $\rightrightarrows$-normal then $s \to t$ is not critically destructive.*

Lemma 27. *If $\mathcal{R}$ is terminating then so is $\Theta(\mathcal{R})$.*

Proof. Suppose $\mathcal{R}$ is terminating. Let $s_0 \to s_1 \to \cdots$ be an infinite $\Theta(\mathcal{R})$-reduction sequence. Without loss of generality, we can assume that there are infinitely many head reductions in it. Then by Lemma 23, we have a reduction sequence $s_0{\downarrow\downarrow} \xrightarrow{*} s_1{\downarrow\downarrow} \xrightarrow{*} \cdots$, which contains infinitely many head reductions. By Lemma 25, it must contain infinitely many rewrite steps that are critically destructive. As $s_i{\downarrow\downarrow}$ is $\rightrightarrows$-normal, this contradicts Lemma 26.

Now, Theorem 1 immediately follows from Lemma 27.

4 Applications and Related Results

Sort introduction technique based on our result, as well as that based on other known persistency results, is useful to detect a property of TRSs.

Example 4. Let $\mathcal{R} = \{f(g(a), g(b), x) \to f(x, x, x),\ g(x) \to x\}$. To show termination of $\mathcal{R}$, we assume the following sort assignment: $\{f : 0 \times 0 \times 0 \to 1,\ g : 0 \to 0,\ a : 0,\ b : 0\}$. It is clear that $\mathcal{R}$ under this sort assignment is a STRS with only variables of sort 0. Thus, by Theorem 1, it suffices to show termination of $\mathcal{R}$ under this sort assignment. Terms of sort 0 are terminating and confluent, since only applicable rule is $g(x) \to x$. Terms of sort 1 have form $f(t_1, t_2, t_3)$ where t_1, t_2, t_3 are terms of sort 0. Suppose contrary that there exists an infinite reduction sequence of terms of sort 1. Then since terms of sort 0 are terminating, it must contain a reduction at the root position, which has the form $f(g(a), g(b), t) \to f(t, t, t)$ for some term t of sort 0. Since $g(a)$ and $g(b)$ have distinct normal forms and terms of sort 0 are confluent, we never have $g(a) \xleftarrow{*} t \xrightarrow{*} g(b)$. Thus we know that every reduction sequence starting from $f(t, t, t)$ never has a reduction at the root position. Since t is terminating, this implies $f(t, t, t)$ is terminating, which is a contradiction.

It is known that termination is persistent for the class of many-sorted TRSs that do not contain both collapsing and duplicating rules [10]. However, the argument above does no work with this result, because $\mathcal{R}$ is collapsing and duplicating.

Another kind of applications of our result is to prove that termination is preserved under suitable translations of TRSs.

Example 5 [4]. For each n-ary function symbol $f \in \mathcal{F}$, let $f_1, \ldots, f_{n-1}$ be new binary function symbols, and let $\hat{\mathcal{F}}$ be the collection of such new function symbols. Define a transformation $\hat{}$ from terms on $\mathcal{F}$ to those on $\hat{\mathcal{F}}$ by:

$$\hat{t} = \begin{cases} t & \text{if } t \in \mathcal{V}, \\ f_1(\hat{t}_1, f_2(\hat{t}_2, f_3(\cdots, f_{n-1}(\hat{t}_{n-1}, \hat{t}_n)\cdots))) & \text{if } t \equiv f(t_1, \ldots, t_n). \end{cases}$$

And, finally let $\hat{\mathcal{R}} = \{\hat{l} \to \hat{r} \mid l \to r \in \mathcal{R}\}$. Using Theorem 1, we can show that $\mathcal{R}$ is terminating if and only if $\hat{\mathcal{R}}$ is terminating.

($\Leftarrow$)-part is trivial. To show ($\Rightarrow$)-part, we introduce a set of sorts by $\mathcal{S} = \{0\} \cup \bigcup\{\delta_f^1, \ldots, \delta_f^{n-2} \mid f \in \mathcal{F},\ f \text{ is } n\text{-ary}\}$, and sort assignment on $\hat{\mathcal{F}}$ as: $f_1 : 0 \times \delta_f^1 \to 0$, $f_i : 0 \times \delta_f^i \to \delta_f^{i-1}$ $(i = 2, \ldots, n-2)$, and $f_{n-1} : 0 \times 0 \to \delta_f^{n-2}$ for each $f \in \mathcal{F}$. It is clear that $\hat{\mathcal{R}}$ is a STRS under this sort assignment with only variables of sort 0. Thus, by Theorem 1, $\hat{\mathcal{R}}$ is terminating if and only if $\hat{\mathcal{R}}$ is terminating on this sort assignment. Suppose $\hat{\mathcal{R}}$ is not terminating. Then there also exists an infinite reduction sequence consisted of terms well-sorted under this sort assignment. Without loss of generality, we assume terms in this reduction sequence have sort 0. It is clear from the sort assignment that if once a function symbol f_i occurs in a well-sorted term of sort 0, it occurs in a form of $f_1(s_1, f_2(s_2, \ldots, f_{n-2}(s_{n-2}, f_{n-1}(s_{n-1}, s_n)) \ldots))$ for some terms $s_1, \ldots, s_n$. Thus, all these terms have form $\hat{t}$ for some t and so we can reversely translate this infinite reduction sequence to that of terms on $\mathcal{F}$. Thus, we know $\mathcal{R}$ is not terminating.

Note that, in the example above, function symbols in $\hat{\mathcal{F}}$ has arity of at most 2, and therefore terms in $\mathcal{T}(\hat{\mathcal{F}}, \mathcal{V})$ have simple structures. Instead, $\mathcal{T}(\hat{\mathcal{F}}, \mathcal{V})$ in general contains much more terms than $\mathcal{T}(\mathcal{F}, \mathcal{V})$, and not all terms in $\mathcal{T}(\hat{\mathcal{F}}, \mathcal{V})$ are images of terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$. We believe, however, that this kind of coding that preserves termination behavior would be helpful for studying properties of complicated systems by interpreting them in simpler systems.

Another way of simulating a TRS by another TRS containing simpler function symbols is "currying": each function symbol is coded by a constant and a new binary function symbol for "application" is added. Again new terms have simple structures and not all new terms are the images of original terms. This transformation also does not affect termination behavior of TRSs [5] [7].

5 Concluding Remarks

We have proved that for any terminating many-sorted TRS $\mathcal{R}$, if $\mathcal{R}$ contains only variables of the same sort then its underlying TRS is also terminating. This is the positive solution to the problem of Zantema that has been appeared as Rewriting Open Problem 60 in [4]. We have also presented some applications of this result.

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References

1. T. Aoto. A proof of the conjecture of Zantema on a persistent property of term rewriting systems. Research Report IS-RR-98-0008F, School of Information Science, JAIST, 1998.
2. T. Aoto and Y. Toyama. On composable properties of term rewriting systems. In *Proceedings of the 6th International Conference on Algebraic and Logic Programming (ALP'97), Southampton, UK*, volume 1298 of *Lecture Notes in Computer Science*, pages 114–128. Springer-Verlag, 1997.
3. T. Aoto and Y. Toyama. Persistency of confluence. *Journal of Universal Computer Science*, 3(11):1134–1147, 1997.
4. N. Dershowitz, J.-P. Jouannaud, and J. W. Klop. More problems in rewriting. In *Proceedings of the 5th International Conference on Rewriting Techniques and Applications (RTA-93)*, volume 690 of *Lecture Notes in Computer Science*, pages 468–487. Springer-Verlag, 1993.
5. J. R. Kennaway, J. W. Klop, M. R. Sleep, and F. J. de Vries. Comparing curried and uncurried rewriting. *Journal of Symbolic Computation*, 21:15–39, 1996.
6. M. Marchiori. On the modularity of normal forms in rewriting. *Journal of Symbolic Computation*, 22:143–154, 1996.
7. A. Middeldorp, H. Ohsaki, and H. Zantema. Transforming termination by self-labelling. In *Proceedings of the 13th International Conference on Automated Deduction (CADE-13)*, volume 1104 of *Lecture Notes in Artificial Intelligence*, pages 373–387. Springer-Verlag, 1996.
8. H. Ohsaki. *Termination of Term Rewriting Systems: Transformation and Persistence*. PhD thesis, University of Tsukuba, March 1998.
9. H. Ohsaki and A. Middeldorp. Type introduction for equational rewriting. In *Proceedings of the 4th International Symposium on Logical Foundations of Computer Science*, volume 1234 of *Lecture Notes in Computer Science*, pages 283–293. Springer-Verlag, 1997.
10. H. Zantema. Termination of term rewriting: interpretation and type elimination. *Journal of Symbolic Computation*, 17:23–50, 1994.

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