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NON-UNIQUENESS OF NORMAL PROOFS FOR MINIMAL FORMULAS IN IMPLICATION-CONJUNCTION FRAGMENT OF BCK *

Abstract

A minimal formula of a given logic L is a formula which is provable in L and is not a non-trivial substitution instance of other provable formulas in L. In [5], Y. Komori asked whether normal proofs of minimal formulas are unique in the implicational fragments of natural deduction systems for the intuitionistic logic and the logic BCK. It was already shown that the answer is positive for BCK, while it is negative for the intuitionistic logic ([3],[8],[9]). The present paper shows normal proofs for minimal formulas are not necessarily unique for the implication-conjunction fragment of BCK. This result contrasts sharply with the uniqueness of normal proofs of balanced formulas for the implication-conjunction fragment of the intuitionistic logic ([2],[6]).

1 Uniqueness of normal proofs

We will use $\rightarrow$ and $\wedge$ for the implication and conjunction throughout the present paper.

For a given logic L, we say that a formula A is minimal in L if and only if (1) A is provable in L and (2) if A is a substitution instance of other provable formula A' then the substitution is trivial, i.e. it is a renaming of propositional variables. Y. Komori put a question in [5] whether normal proofs of minimal formulas are unique in the natural deduction systems for the implicational fragments of the intuitionistic

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logic and the logic BCK. The logic BCK can be usually formulated as a sequent calculus LBCK obtained from the intuitionistic system LJ by eliminating the contraction rule. Corresponding to this, the natural deduction system NBCK for the implicational fragment of BCK can be defined as a system obtained from the intuitionistic system NJ by restricting the application of the $\rightarrow$ introduction in such a way that at most one occurrence of the assumption is cancelled.

Both G. E. Mints[8] and M. Tatsuta[9] solved independently Y. Komori's problem for the intuitionistic logic negatively. More precisely, there exists a minimal formula in the implicational intuitionistic logic which does not have unique $\beta\eta$-normal proofs. On the other hand Hirokawa[3] proved that any minimal formula in the implicational BCK has the unique β-normal proof. Notice here that any β-normal proof of a minimal formula in the implicational BCK is also $\beta\eta$-normal. It was pointed out also by A. Wroński that this result can be shown indirectly by combining the uniqueness result by G. E. Mints[6] and A. A. Babaev & S. V. Solovjev[2] on balanced formulas with S. Jaśkowski's result[4] which implies that any minimal formula in the implicational BCK is balanced.

It seems to be possible to extend the above uniqueness result of the implicational fragment to the implication-conjunction fragment. For, the uniqueness result on balanced formulas also holds for the implication-conjunction fragment. On the other hand, by a slight observation, it can be seen that S. Jaśkowski's result does not hold any more for the implication-conjunction fragment. In the following, we will show that there exists a minimal formula in the implication-conjunction fragment of BCK whose $\beta\eta$-normal proofs are not unique.

Finally, we remark that the above uniqueness result on balanced formulas in the implication-conjunction fragment of NJ by G. E. Mints and A. A. Babaev & S. V. Solovjov can be strengthen to a yet weaker condition (see [1] for details).

2 An example of minimal formula without unique normal proofs

Before giving an example of such a minimal formula, it is necessary to give a precise definition of the natural deduction system for the

implication-conjunction fragment of BCK. Certain complications will occur in the definition, since the implication is a multiplicative connective while the conjunction is additive.

The natural deduction system for the implication-conjunction fragment consists of following rules.

1. $\rightarrow$ introduction

$$\dfrac{\begin{array}{c}[A]^{(1)}\\ \vdots\\ B\end{array}}{A \rightarrow B}\ (1)$$

   Here $[A]^{(1)}$ means that the assumptions $[A]$ in $\mathcal{D}$ may be cancelled along with the application of this inference.

2. $\rightarrow$ elimination

$$\dfrac{A \rightarrow B \quad A}{B}$$

3. $\wedge$ introduction

$$\dfrac{A \quad B}{A \wedge B}$$

4. $\wedge$ eliminations

$$\dfrac{A \wedge B}{A} \qquad\qquad \dfrac{A \wedge B}{B}$$

Next, we will define proofs in NBCK from a multiset of formulas (called assumptions) Γ to a formula A (called the conclusion), inductively as follows.

In the following, $\Gamma \cup \Sigma$ and $\Gamma + \Sigma$ denote the multiset union and the multiset sum for each multiset Γ and Σ, respectively. More precisely, let B be any formula and suppose that the multiplicity of B in Γ and Σ are m and n, respectively. Then, the multiplicity of B in $\Gamma \cup \Sigma$ and $\Gamma + \Sigma$ are $max\{m, n\}$ and $m + n$, respectively. For instance, the multiset union and the multiset sum of $\{B, C, C, C\}$ and $\{B, B, C, D\}$ are $\{B, B, C, C, C, D\}$ and $\{B, B, B, C, C, C, C, D\}$, respectively.

1. The figure

$$[A]$$

is a proof from the multiset of assumption $\{A\}$ to the conclusion A.

2. Suppose that $\mathcal{D}$ is a proof from the multiset of assumption Γ to the conclusion B, which is expressed schematically as

$$\begin{array}{c}\mathcal{D}\\ B.\end{array}$$

Then the figure

$$\dfrac{\begin{array}{c}\mathcal{D}\\ B\end{array}}{A \to B}$$

is a proof from Γ' to $A \to B$. Here, Γ' is a multiset obtained from Γ by deleting at most one A, if any.

3. Suppose that $\mathcal{D}_1$ and $\mathcal{D}_2$ are proofs from Γ to $A \to B$ and from Σ to A, respectively. Then the figure

$$\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ A \to B & A\end{array}}{B}$$

is a proof from $\Gamma + \Sigma$ to B.

4. Suppose that $\mathcal{D}_1$ and $\mathcal{D}_2$ are proofs from Γ to A and from Σ to B, respectively. Then the figure

$$\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ A & B\end{array}}{A \wedge B}$$

is a proof from $\Gamma \cup \Sigma$ to $A \wedge B$.

5. Suppose that $\mathcal{D}$ is a proof from Γ to $A \wedge B$. Then the figures

$$\dfrac{\begin{array}{c}\mathcal{D}\\ A \wedge B\end{array}}{A} \qquad \text{and} \qquad \dfrac{\begin{array}{c}\mathcal{D}\\ A \wedge B\end{array}}{B}$$

are proofs from Γ to A and from Γ to B, respectively.

For a multiset of formulas Γ and a formula A, we say that A is derivable from Γ in NBCK ($\Gamma \vdash A$, in symbol) if and only if there exists a multiset of formulas Γ' which is a subset (as a multiset) of Γ and a proof $\mathcal{D}$ from Γ' to A. It is easy to see that $\Gamma \vdash A$ if and only if the sequent $\Gamma \to A$ is provable in the sequent calculus LBCK for BCK (see e.g. [7]).

Next, we will define $\beta\eta$-normal proofs. The followings are the reduction rules for the implication-conjunction fragment.

1. β-reductions

   (a) $\beta_{\to}$-contraction

$$\dfrac{\dfrac{\begin{array}{c}[A]^{(1)}\\ \mathcal{D}\\ B\end{array}}{A \to B}\,(1) \quad \begin{array}{c}\mathcal{D}'\\ A\end{array}}{B} \quad \Rightarrow \quad \begin{array}{c}\mathcal{D}'\\ A\\ \mathcal{D}\\ B\end{array}$$

   (b) $\beta_{\wedge}$-contractions

$$\dfrac{\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ A & B\end{array}}{A \wedge B}}{A} \quad \Rightarrow \quad \begin{array}{c}\mathcal{D}_1\\ A\end{array}$$

$$\dfrac{\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ A & B\end{array}}{A \wedge B}}{B} \quad \Rightarrow \quad \begin{array}{c}\mathcal{D}_2\\ B\end{array}$$

2. η-reductions

   (a) $\eta_{\to}$-contraction

$$\dfrac{\begin{array}{cc}\mathcal{D}\\ A \to B & [A]^{(1)}\end{array}}{\dfrac{B}{A \to B}\,(1)} \quad \Rightarrow \quad \begin{array}{c}\mathcal{D}\\ A \to B\end{array}$$

(b) $\eta_\wedge$-contraction

$$\dfrac{\dfrac{\overset{\mathcal{D}}{A \wedge B}}{A} \quad \dfrac{\overset{\mathcal{D}}{A \wedge B}}{B}}{A \wedge B} \qquad \Rightarrow \qquad \overset{\mathcal{D}}{A \wedge B}$$

A proof which is irreducible by any of β-reduction and η-reduction rules is said to be β-normal and η-normal, respectively. A proof which is both β-normal and η-normal is said to be $\beta\eta$-normal. As usual, we can show the strong normalizability and Church-Rosser property for these reduction rules. Moreover reduction rules are closed in NBCK. Therefore, each proof in NBCK has its unique $\beta\eta$-normal form.

Note that any β-normal proof of a minimal formula in the implication-conjunction fragment of BCK is not necessarily $\beta\eta$-normal, while it is in the implicational BCK ([3]). This is shown by following example.

$$\dfrac{\dfrac{\dfrac{\dfrac{[P \wedge Q]^{(1)}}{P} \quad \dfrac{[P \wedge Q]^{(1)}}{Q}}{P \wedge Q} \quad \dfrac{[P \wedge Q]^{(1)}}{P}}{(P \wedge Q) \wedge P}}{(P \wedge Q) \to ((P \wedge Q) \wedge P)}\ (1)$$

Clearly, this is β-normal but is not η-normal. Moreover we can easily check that $(P \wedge Q) \to ((P \wedge Q) \wedge P)$ is minimal even in the classical logic. Hence it is minimal in BCK.

Now we will give an example of a minimal formula in the implication-conjunction fragment of BCK whose $\beta\eta$-normal proofs are not unique. It is

$$\begin{aligned}(Q \to P) \to (P \to S) \to (S \to P \to V) \\ \to (T \to P) \to (U \to S) \to (R \to P) \\ \to (T \to Q \to V) \wedge (R \to U \to V) \wedge (R \to S).\end{aligned}$$

There are two $\beta\eta$-normal proofs of this formula in NBCK. One of the $\beta\eta$-normal proofs for the above formula is as follows:

$$\dfrac{\dfrac{\dfrac{\overset{\mathcal{D}_1}{T \to Q \to V} \quad \overset{\mathcal{D}_2}{R \to U \to V}}{(T \to Q \to V) \wedge (R \to U \to V)} \quad \overset{\mathcal{D}_3}{R \to S}}{(T \to Q \to V) \wedge (R \to U \to V) \wedge (R \to S)}}{(Q \to P) \to \cdots \to (T \to Q \to V) \wedge (R \to U \to V) \wedge (R \to S)}\ (3) \ldots (8)$$

where $\mathcal{D}_1$ is

$$\dfrac{\dfrac{[S \to P \to V]^{(6)} \quad \dfrac{[P \to S]^{(7)} \quad \dfrac{[Q \to P]^{(8)} \quad [Q]^{(1)}}{P}}{S}}{P \to V} \qquad \dfrac{[T \to P]^{(5)} \quad [T]^{(2)}}{P}}{\dfrac{\dfrac{V}{Q \to V}\,(1)}{T \to Q \to V,}\,(2)}$$

$\mathcal{D}_2$ is

$$\dfrac{\dfrac{[S \to P \to V]^{(6)} \quad \dfrac{[U \to S]^{(4)} \quad [U]^{(1)}}{S}}{P \to V} \qquad \dfrac{[R \to P]^{(3)} \quad [R]^{(2)}}{P}}{\dfrac{\dfrac{V}{U \to V}\,(1)}{R \to U \to V}\,(2)}$$

and $\mathcal{D}_3$ is

$$\dfrac{[P \to S]^{(7)} \quad \dfrac{[R \to P]^{(3)} \quad [R]^{(1)}}{P}}{\dfrac{S}{R \to S.}\,(1)}$$

Another proof is obtained by replacing $\mathcal{D}_1$ with

$$\dfrac{\dfrac{[S \to P \to V]^{(6)} \quad \dfrac{[P \to S]^{(7)} \quad \dfrac{[T \to P]^{(5)} \quad [T]^{(2)}}{P}}{S}}{P \to V} \qquad \dfrac{[Q \to P]^{(8)} \quad [Q]^{(1)}}{P}}{\dfrac{\dfrac{V}{Q \to V}\,(1)}{T \to Q \to V.}\,(2)}$$

It remains to show that the above formula is minimal in BCK. This can be done by detailed (and tedious) examinations of the fact that it can not be a non-trivial substitution instance of a provable formula in BCK, using the decidability of BCK.

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