Unique normal proof property for implicational minimal formulas in the intuitionistic logic

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Abstract

A formula is said to have unique β-normal proof property and unique $\beta\eta$-normal proof property iff it has a unique β-normal proof and a $\beta\eta$-normal proof in NJ, respectively. In this report a condition of implicational minimal formulas in the intuitionistic logic for unique β-normal proof property is presented making use of unique $\beta\eta$-normal proof property for negatively non-duplicated formulas. It is shown that this condition properly includes both BCK-provability and depth ≤ 2 which are previously known conditions.

1 Introduction

In 1986, Y. Komori put a question whether normal proofs of minimal formulas are unique in the natural deduction systems for the implicational fragments of the intuitionistic logic and the logic BCK ([7]). S. Hirokawa proved that minimal formulas in implicational BCK have unique β-normal proofs ([5]). Notice here that any β-normal proof of a minimal formula in implicational BCK is also $\beta\eta$-normal. In the case for the implicational intuitionistic logic, both G. E. Mints ([10]) and M. Tatsuta ([12]) independently solved Y. Komori's problem negatively. More precisely, they found minimal formulas in the implicational intuitionistic logic which do not have unique $\beta\eta$-normal proofs. By S. Jaśkowski's result ([6]), any minimal formula in the implicational intuitionistic logic which is provable in implicational BCK is also minimal in it. So, "BCK-provability" is a condition of minimal formulas to have unique β-normal proofs. Another condition "depth ≤ 2" is also obtained by M. Tatsuta ([12]). Neither of these are included to the other.

Previous to S. Hirokawa's result ([5]), it was pointed out by A. Wroński that the uniqueness of $\beta\eta$-normal proofs for minimal formulas in implicational BCK can be shown indirectly by combining the uniqueness of $\beta\eta$-normal proofs for balanced formulas obtained by G. E. Mints ([8]) and A. A. Babaev & S. V. Solovjev ([3]) with S. Jaśkowski's result ([6]) which implies that any minimal formula in implicational BCK is balanced. We say a formula is BCK-minimal and J-minimal iff it is minimal in implicational BCK and in the implicational intuitionistic logic, respectively. We say a formula has unique β-normal proof property (written as β-UP) and unique $\beta\eta$-normal proof property (written as $\beta\eta$-UP) iff it has a unique β-normal proof and a $\beta\eta$-normal proof in NJ, respectively. Then above results on BCK-minimal formulas are surveyed as below.

$$\begin{array}{ccc} A \text{ is BCK-minimal} & \Longrightarrow & A \text{ has } \beta\text{-UP} \\ \Downarrow & & \Downarrow \\ A \text{ is balanced} & \Longrightarrow & A \text{ has } \beta\eta\text{-UP} \end{array}$$

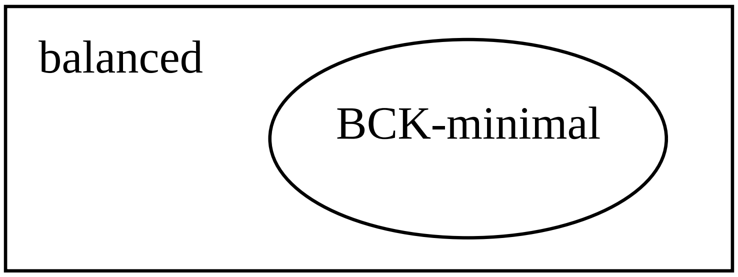

Figure 1: UP in implicational BCK

So, BCK-minimality is included in balancedness up to $\beta\eta$-UP (see figure 1[1]).

We mention here that an analogue for implication-conjunction fragments does not hold even in the case for BCK. More precisely, there exists a minimal formula in the implication-conjunction fragment of BCK whose $\beta\eta$-normal proofs are not unique ([1]).

Then how about an analogue for the implicational intuitionistic logic? Since J-minimal BCK-provable formulas are BCK-minimal, they are balanced. But the other condition, i.e. J-minimal depth ≤ 2 does not always implies balancedness as shown in the following example. (see figure 2).

Example. 1.1 $(P \to P \to Q) \to (P \to Q)$ is J-minimal formula whose depth is 2, but is not balanced.

In this report, we present a condition of J-minimal formulas for β-UP making use of $\beta\eta$-UP for negatively non-duplicated formulas appeared in [2]. It is shown that this condition properly includes both BCK-provability and depth ≤ 2 conditions. As a corollary, it is shown that the negatively non-duplicated condition includes all of these conditions up to $\beta\eta$-UP.

Let us describe brief contents of the present report. In section 2 we show that any formula provable in LJ without a non-prime contraction is negatively non-duplicated. Cut elimination theorem for LJ without a non-prime contraction is proved in section 3. A variant of NJ is presented in section 4. In section 5 these results are applied to $\beta\eta$-UP for implicational minimal formulas making use of $\beta\eta$-UP for negatively non-duplicated formulas and

[1] We illustrate conditions for $\beta\eta$-UP with rectangles and those for β-UP with ellipses.

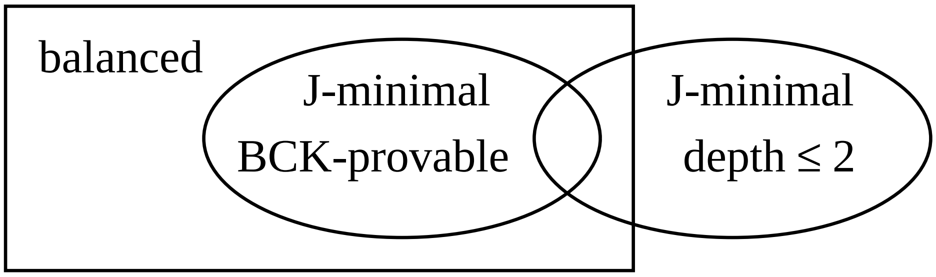

Figure 2: UP in the implicational intuitionistic logic

then strengthened to β-UP. It is shown that this condition includes both BCK-provability and depth ≤ 2 conditions in section 6. We end with reviews of the results and some remarks in section 7.

We will assume familiarity with LJ and NJ. Our terminology for LJ almost follows [11] and that for NJ almost follows [4], [13] and [14].

2 A characterization of minimal formulas

We will use $P, Q, R, \ldots$ for propositional variables and $\rightarrow$ for implication. Formulas are built from arbitrary propositional variables using the connective $\rightarrow$. We will use $A, B, C, \ldots$ for arbitrary formulas and $\equiv$ for syntactical equality. A formula A is said to be prime, if $A \equiv P$ for some propositional variable P, otherwise said to be non-prime. A sequent $A_1, \ldots, A_n \vdash B$ is built from a (finite) sequence of formulas $A_1, \ldots, A_n$ and a formula B. We will use $\Gamma, \Sigma, \Pi, \ldots$ for arbitrary (possibly empty) sequences of formulas and $S, T, U, \ldots$ for arbitrary sequents.

Implicational fragment of LJ (from now on, we refer to it as "the implicational LJ") is the sequent calculus specified by following an initial sequent and rules of inference.

1. initial sequent
$$A \vdash A$$
This formula A is call an ***initial*** formula.

2. cut
$$\frac{\Gamma \vdash A \quad A, \Sigma \vdash C}{\Gamma, \Sigma \vdash C}$$

3. exchange
$$\frac{\Gamma, A, B, \Sigma \vdash C}{\Gamma, B, A, \Sigma \vdash C}$$

4. weakening
$$\frac{\Gamma \vdash C}{A, \Gamma \vdash C}$$
This formula A is called a ***weakened*** formula.

5. contraction

$$\frac{A, A, \Gamma \vdash C}{A, \Gamma \vdash C}$$

Especially, contractions of form

$$\frac{P, P, \Gamma \vdash C}{P, \Gamma \vdash C}$$

are said to be ***prime***, otherwise said to be ***non-prime***.

6. implication left and right

$$\frac{\Gamma \vdash A \quad B, \Sigma \vdash C}{\Gamma, A \to B, \Sigma \vdash C} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$$

The following theorem is a standard fact concerning the implicational LJ, and we omit it's proof here.

THEOREM. 2.1 [cut elimination theorem] In the implicational LJ, if S is provable then it is also provable without a cut.

In the following, we only deal with the implicational fragment (of LJ and so on). So, formulas, provability and so on will be considered in the implicational fragment although we will not mention it explicitly.

A ***substitution*** θ is a mapping from the set of propositional variables $\mathcal{V}$ to the set of formulas $\mathcal{F}$. Any substitution θ is extended to mappings from $\mathcal{F}$ to $\mathcal{F}$ by

$$\theta(A \to B) \stackrel{\text{def}}{\equiv} (\theta(A) \to \theta(B)).$$

We write $\theta(A)$ as $A\theta$. A ***substitution order*** is a quasi-order on $\mathcal{F}$ such that $A \preceq B$ iff $B \equiv A\theta$ for some substitution θ. A substitution θ is said to be ***trivial to*** A iff $A\theta \preceq A$. We write $A \simeq B$ iff $B \equiv A\theta$ for some trivial substitution θ. It is easily seen that $A \simeq B$ iff $A \preceq B$ and $B \preceq A$, and $\simeq$ is an equivalence relation on $\mathcal{F}$.

DEFINITION. 2.2 For a given logic L, a formula A is said to be ***minimal in*** L iff (1) A is provable in L and (2) for any formula B that is provable in L, $B \preceq A$ implies $B \simeq A$.

A formula that is minimal in the intuitionistic logic is said to be ***J-minimal***. Same notion is adapted to sequents, i.e. $A_1, \ldots, A_n \vdash B$ is minimal iff $A_1 \to \cdots A_n \to B$ is minimal.

A ***negative, positive occurrence*** of a propositional variables in A is defined by simultaneous induction as follows: (1) P has a positive occurrence in P; (2) P has a positive occurrence in $B \to C$ if P has a positive occurrence in C or has a negative occurrence in B; (3) P has a negative occurrence in $B \to C$ if P has a negative occurrence in C or has a positive occurrence in B. We often say that the ***sign*** of an occurrence of a propositional variable is negative or positive.

DEFINITION. 2.3 A formula A is said to be ***negatively non-duplicated*** iff every propositional variable has at most one negative occurrence in it.

Same notion is adapted to sequents, i.e. (4) P has a positive occurrence in $A_1, \ldots, A_n \vdash B$ if P has a positive occurrence in B or has a negative occurrence in A_i for some i; (5) P has a negative occurrence in $A_1, \ldots, A_n \vdash B$ if P has a negative occurrence in B or has a positive occurrence in A_i for some i. Moreover, $A_1, \ldots, A_n \vdash B$ is negatively non-duplicated iff $A_1 \to \cdots A_n \to B$ is negatively non-duplicated.

Now we can state our target to prove in this section:

THEOREM. 2.4 If J-minimal formula A is provable in LJ without a cut nor a non-prime contraction then it is negatively non-duplicated.

Let $\Box$ (we call it a ***hole***) be a fresh symbol. The set of ***contexts*** $\mathcal{C}$ is defined inductively as follows: (1) $\Box \in \mathcal{C}$; (2) $\mathcal{V} \subset \mathcal{C}$; (3) if $\alpha, \beta \in \mathcal{C}$ the $\alpha \to \beta \in \mathcal{C}$. $\alpha[, \ldots,]$ denotes a context in which n ($n \geq 0$) holes occur. If $A_1, \ldots, A_n$ are formulas, then $\alpha[A_1, \ldots, A_n]$ denotes the result of replacing the holes of $\alpha[, \ldots,]$ with $A_1, \ldots, A_n$ from left to right. Note that $\alpha[A_1, \ldots, A_n]$ specifies the subformula occurrences $A_1, \ldots, A_n$ in $\alpha[A_1, \ldots, A_n]$. We write $\alpha[\ ]$ to denote a context containing precisely one hole. The same notation is adapted to sequents, for example, $S[, \ldots,]$ denotes the sequent that contain n holes in it.

DEFINITION. 2.5 Given a cut-free poof figure $\mathcal{P}$,

1. each subformula occurrence of A of form

$$\alpha[A] \vdash \alpha[A]$$

appeared as an initial sequent in $\mathcal{P}$ is said to be ***reversely connected to*** the other;

2. the subformula occurrence of A and B in upper sequent and those in lower sequent of form

$$\frac{\Gamma, \alpha[A], \beta[B], \Sigma \vdash C}{\Gamma, \beta[B], \alpha[A], \Sigma \vdash C}$$

in $\mathcal{P}$ are said to be ***directly connected to*** the other, respectively;

3. each of the subformula occurrences of A in upper sequent and the subformula occurrences of A in lower sequent of form

$$\frac{\alpha[A], \alpha[A], \Gamma \vdash C}{\alpha[A], \Gamma \vdash C}$$

in $\mathcal{P}$ is said to be directly connected to the other;

4. the subformula occurrence of A and B in upper sequent and those in lower sequent of form

$$\frac{\Gamma \vdash \alpha[A] \quad \beta[B], \Sigma \vdash C}{\Gamma, \alpha[A] \to \beta[B], \Sigma \vdash C}$$

in $\mathcal{P}$ are said to be directly connected to the other, respectively;

5. the subformula occurrence of A and B in upper sequent and those in lower sequent of form
$$\frac{\Gamma, \alpha[A] \vdash \beta[B]}{\Gamma \vdash \alpha[A] \rightarrow \beta[B]}$$
in $\mathcal{P}$ are said to be directly connected to the other, respectively;

6. each subformula occurrence of corresponding formulas in upper sequents of an inference and that in lower sequent of it is said to be directly connected to the other.

A graph of all occurrences of a subformula that are directly connected is called a ***track***. It is easily seen that any subformula has the same sign if it is directly connected and has the inverse sign if it is reversely connected. If some occurrence of a subformula is specified, the track containing it is called the track of it. If (all) occurrences in a track are positive, then we call it a positive track and if negative, then a negative track. Negative tacks that are reversely connected to a positive track are called ***mates*** of it, and conversely, so are called positive tacks that are reversely connected to a negative track.

Note that by definition, for any cut-free proof $\mathcal{P}$ of $S[A_1, \ldots, A_n]$ with $A_1, \ldots, A_n$ actually occurs, all tracks of $A_1, \ldots, A_n$ in $\mathcal{P}$ are distinct.

A graph of all occurrences of a subformula that are directly or reversely connected is called a ***connection***. If some occurrence of a subformula is specified, the connection containing it is called the connection of it. Tracks are said to be ***connected*** iff they are in the same connection.

In this section, tracks and connections are only considered for propositional variables.

PROPOSITION. 2.6 Let P be some occurrence of propositional variable in a cut-free proof $\mathcal{P}$ and π be a connection of P in $\mathcal{P}$. Then replacing all occurrence of P in π with that of Q also yields a cut-free proof.

Proof.
Easy induction. □

LEMMA. 2.7 Let $\mathcal{P}$ be a cut-free proof of $S[P, \ldots, P]$ with all occurrences of P are indicated. If $S[P, \ldots, P]$ is J-minimal, then all tracks of $P, \ldots, P$ are connected.

Proof.
Let n be the number of occurrences of P in $S[P, \ldots, P]$. Assume that there are distinct connections π and π' in $\mathcal{P}$. Then by PROPOSITION.2.6, replacing all occurrence of P in π with that of Q yields a cut-free proof of $S[P_1, \ldots, P_n]$ where $P_i \equiv Q$ (if P_i is in π) or $P_i \equiv P$ (otherwise — including the case for P_i is in π'). Thus $S[P_1, \ldots, P_n]$ is provable and $S[P_1, \ldots, P_n] \preceq S[P, \ldots, P]$, hence $S[P, \ldots, P]$ can not be J-minimal. This is a contradiction. □

PROPOSITION. 2.8 Let P be some positive occurrence of propositional variable in a cut-free proof $\mathcal{P}$ and π be a positive track of P in $\mathcal{P}$. If $\mathcal{P}$ does not contain any non-prime contractions, then π is linear, i.e. it does not branch off. Moreover, π has only a single mate.

Proof .

Easy induction. □

LEMMA. 2.9 Let $\mathcal{P}$ be a cut-free proof of $S[P, \ldots, P]$ with all occurrences of P are indicated. If $\mathcal{P}$ does not contain any non-prime contractions and all tracks of $P, \ldots, P$ are connected, then there exists at most one negative occurrence of P in $S[P, \ldots, P]$.

Proof .

Assume there exist k ($k \geq 2$) negative occurrences in it. Since each track of $P, \ldots, P$ are distinct, we can assume their negative tracks be $\pi_1, \ldots, \pi_k$. They can be connected only through their mates. But by PROPOSITION.2.8, any positive tracks are linear, they have only a single mate for each. Thus $\pi_1, \ldots, \pi_k$ can not be connected. This is a contradiction. □

Proof of the THEOREM.2.4.

Let A be any J-minimal formula and P be any propositional variable that occurs in A. Assume that A is provable in LJ without a cut nor a non-prime contraction. Then there exists a cut-free proof $\mathcal{P}$ of LJ without a non-prime contraction. By LEMMA.2.7, all tracks of occurrences of P in A are connected. Thus by LEMMA.2.9, negative occurrence of P in A is at most one. □

3 Cut elimination

The following theorem will be proved in this section.

THEOREM. 3.1 In LJ, if S is provable without a non-prime contraction, then it is also provable without a cut nor a non-prime contraction.

For the later convention, we will prove the theorem indirectly as below.

Firstly let us introduce sequent calculus LBCK$_{pc}$ obtained from LJ by taking cut and non-prime contraction away, i.e. LBCK$_{pc}$ is the sequent calculus specified by following an initial sequent and rules of inference.

1. initial sequent

$$A \vdash A$$

2. exchange

$$\frac{\Gamma, A, B, \Sigma \vdash C}{\Gamma, B, A, \Sigma \vdash C}$$

3. weakening

$$\frac{\Gamma \vdash C}{A, \Gamma \vdash C}$$

4. prime contraction

$$\frac{P, P, \Gamma \vdash C}{P, \Gamma \vdash C}$$

5. implication left and right

$$\frac{\Gamma \vdash A \quad B, \Sigma \vdash C}{\Gamma, A \to B, \Sigma \vdash C} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$$

Now, let cut be the rule

$$\frac{\Gamma \vdash A \quad \Sigma, A, \Pi \vdash C}{\Sigma, \Gamma, \Pi \vdash C}$$

and LBCK_{pc} + cut be the system obtained by adding cut to the inference rules of LBCK_{pc}. It is easily seen that THEOREM.3.1 coincides with the following theorem.

THEOREM. 3.2 If S is provable in LBCK_{pc} + cut, then S is also provable in LBCK_{pc}.

It is also easily seen that it suffices to prove the following lemma.

LEMMA. 3.3 Let $\mathcal{P}$ be a proof of S in LBCK_{pc} + cut which contains only one cut as the last inference. Then S is also provable in LBCK_{pc}.

Proof.

Our proof proceeds by double induction on the grade g and the rank r of the proof $\mathcal{P}$ (, i.e. transfinite induction on $\omega \cdot g + r$).

1. Case of $r = 2$, viz. $rank_l(\mathcal{P}) = rank_r(\mathcal{P}) = 1$.

   (a) Case that the left upper sequent of cut is an initial sequent. Then $\mathcal{P}$ is in the form of

   $$\frac{A \vdash A \quad \overset{\vdots}{\Gamma, A, \Sigma \vdash C}}{\Gamma, A, \Sigma \vdash C.}$$

   Then the end-sequent is provable without a cut as

   $$\overset{\vdots}{\Gamma, A, \Sigma \vdash C.}$$

   (b) Case that the right upper sequent of cut is an initial sequent. Similarly to the case 1a.

   (c) Case that neither the left upper sequent of cut nor the right upper sequent of cut is an initial sequent. Since $rank_l(\mathcal{P}) = rank_r(\mathcal{P}) = 1$, left upper inference of cut is implication right, thus the cut formula is in the form of $A \to B$. Then $\mathcal{P}$ is in the form of

   $$\frac{\dfrac{\overset{\vdots}{\Gamma, A \vdash B}}{\Gamma \vdash A \to B} \quad \dfrac{\overset{\vdots}{\Sigma \vdash A} \quad \overset{\vdots}{B, \Pi \vdash C}}{\Sigma, A \to B, \Pi \vdash C}}{\Sigma, \Gamma, \Pi \vdash C.}$$

Then the end-sequent is also provable as

$$\frac{\dfrac{\dfrac{\vdots}{\Sigma \vdash A} \quad \dfrac{\vdots}{\Gamma, A \vdash B}}{\Gamma, \Sigma \vdash B} \qquad \dfrac{\vdots}{B, \Pi \vdash C}}{\Sigma, \Gamma, \Pi \vdash C.}$$

Since these two cuts have less degrees, the end-sequent is provable without a cut using induction hypothesis.

2. Case of $r > 2$.

   (a) $rank_l(\mathcal{P}) > 1$ and $rank_r(\mathcal{P}) = 1$. Since $rank_r(\mathcal{P}) = 1$, the right upper inference of cut is weakening or implication left.

      i. Case that the right upper inference of cut is weakening. Then $\mathcal{P}$ is in the form of

$$\frac{\dfrac{\dfrac{\vdots}{\Gamma \vdash A}}{\Gamma' \vdash A} \quad \dfrac{\dfrac{\vdots}{\Sigma \vdash C}}{A, \Sigma \vdash C}}{\Gamma', \Sigma \vdash C.}$$

      Then the end-sequent is provable without a cut as

$$\genfrac{}{}{1.5pt}{}{\dfrac{\vdots}{\Sigma \vdash C}}{\Gamma', \Sigma \vdash C.}$$

      ii. Case that the right upper inference of cut is implication left. Then $\mathcal{P}$ is in the form of

$$\frac{\dfrac{\dfrac{\vdots}{\Gamma \vdash A \to B}}{\Gamma' \vdash A \to B} \quad \dfrac{\dfrac{\vdots}{\Sigma \vdash A} \quad \dfrac{\vdots}{B, \Pi \vdash C}}{\Sigma, A \to B, \Pi \vdash C}}{\Sigma, \Gamma', \Pi \vdash C.}$$

      Then the end-sequent is also provable as

$$\genfrac{}{}{1.5pt}{}{\dfrac{\dfrac{\vdots}{\Gamma \vdash A \to B} \quad \dfrac{\dfrac{\vdots}{\Sigma \vdash A} \quad \dfrac{\vdots}{B, \Pi \vdash C}}{\Sigma, A \to B, \Pi \vdash C}}{\Sigma, \Gamma, \Pi \vdash C}}{\Sigma, \Gamma', \Pi \vdash C.}$$

      Since this cut has less rank, the end-sequent is provable without a cut using induction hypothesis.

   (b) $rank_l(\mathcal{P}) = 1$ and $rank_r(\mathcal{P}) > 1$. Since $rank_l(\mathcal{P}) = 1$, the left upper inference of cut is implication right.

i. Case that the right upper inference of cut is exchange or weakening or prime contraction or implication right. Then $\mathcal{P}$ is in the form of

$$\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma, A \vdash B$}
\UnaryInfC{$\Gamma \vdash A \to B$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Sigma, A \to B, \Pi \vdash C$}
\UnaryInfC{$\Sigma', A \to B, \Pi' \vdash C'$}
\BinaryInfC{$\Sigma, \Gamma, \Pi \vdash C'.$}
\end{prooftree}$$

Then the end-sequent is also provable as

$$\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma, A \vdash B$}
\UnaryInfC{$\Gamma \vdash A \to B$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Sigma, A \to B, \Pi \vdash C$}
\BinaryInfC{$\Sigma, \Gamma, \Pi \vdash C$}
\doubleLine
\UnaryInfC{$\Sigma', \Gamma, \Pi' \vdash C'.$}
\end{prooftree}$$

Since this cut has less rank, the end-sequent is provable without a cut using induction hypothesis.

ii. Case that the right upper inference of cut is implication left. Then $\mathcal{P}$ is in forms of

$$\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma, A \vdash B$}
\UnaryInfC{$\Gamma \vdash A \to B$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Sigma, A \to B, \Pi \vdash C$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Lambda \vdash D$}
\BinaryInfC{$\Sigma', A \to B, \Pi' \vdash C'$}
\BinaryInfC{$\Sigma', \Gamma, \Pi' \vdash C'$}
\end{prooftree}$$

or

$$\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma, A \vdash B$}
\UnaryInfC{$\Gamma \vdash A \to B$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Sigma \vdash C$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Pi, A \to B, \Lambda \vdash D$}
\BinaryInfC{$\Sigma', A \to B, \Pi' \vdash C'$}
\BinaryInfC{$\Sigma', \Gamma, \Pi' \vdash C'.$}
\end{prooftree}$$

Then the end-sequent is also provable as

$$\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma, A \vdash B$}
\UnaryInfC{$\Gamma \vdash A \to B$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Sigma, A \to B, \Pi \vdash C$}
\BinaryInfC{$\Sigma, \Gamma, \Pi \vdash C$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Lambda \vdash D$}
\doubleLine
\BinaryInfC{$\Sigma', \Gamma, \Pi' \vdash C'$}
\end{prooftree}$$

or

$$\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Sigma \vdash C$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma, A \vdash B$}
\UnaryInfC{$\Gamma \vdash A \to B$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Pi, A \to B, \Lambda \vdash D$}
\BinaryInfC{$\Pi, \Gamma, \Lambda \vdash D$}
\doubleLine
\BinaryInfC{$\Sigma', \Gamma, \Pi' \vdash C'.$}
\end{prooftree}$$

Since each of these cuts has less rank, each of the end-sequents is provable without a cut using induction hypothesis.

(c) $rank_l(\mathcal{P}) > 1$ and $rank_r(\mathcal{P}) > 1$.

i. Case that the right upper inference of cut is exchange or weakening or implication right. Then $\mathcal{P}$ is in the form of

$$
\begin{prooftree}
\AxiomC{$\vdots$}\noLine\UnaryInfC{$\Gamma \vdash A$}\UnaryInfC{$\Gamma' \vdash A$}
\AxiomC{$\vdots$}\noLine\UnaryInfC{$\Sigma, A, \Pi \vdash C$}\UnaryInfC{$\Sigma', A, \Pi' \vdash C'$}
\BinaryInfC{$\Sigma', \Gamma', \Pi' \vdash C'.$}
\end{prooftree}
$$

Then the end-sequent is also provable as

$$
\begin{prooftree}
\AxiomC{$\vdots$}\noLine\UnaryInfC{$\Gamma \vdash A$}
\AxiomC{$\vdots$}\noLine\UnaryInfC{$\Sigma, A, \Pi \vdash C$}
\BinaryInfC{$\Sigma, \Gamma, \Pi \vdash C$}
\doubleLine\UnaryInfC{$\Sigma', \Gamma', \Pi' \vdash C'.$}
\end{prooftree}
$$

Since this cut has less rank, the end-sequent is provable without a cut using induction hypothesis.

ii. Case that the right upper inference of cut is implication left. Then $\mathcal{P}$ is in forms of

$$
\begin{prooftree}
\AxiomC{$\vdots$}\noLine\UnaryInfC{$\Gamma \vdash A$}\UnaryInfC{$\Gamma' \vdash A$}
\AxiomC{$\vdots$}\noLine\UnaryInfC{$\Sigma, A, \Pi \vdash B$}
\AxiomC{$\vdots$}\noLine\UnaryInfC{$\Lambda \vdash C$}
\BinaryInfC{$\Sigma', A, \Pi' \vdash C'$}
\BinaryInfC{$\Sigma', \Gamma', \Pi' \vdash C'$}
\end{prooftree}
$$

or

$$
\begin{prooftree}
\AxiomC{$\vdots$}\noLine\UnaryInfC{$\Gamma \vdash A$}\UnaryInfC{$\Gamma' \vdash A$}
\AxiomC{$\vdots$}\noLine\UnaryInfC{$\Sigma \vdash B$}
\AxiomC{$\vdots$}\noLine\UnaryInfC{$\Pi, A, \Lambda \vdash C$}
\BinaryInfC{$\Sigma', A, \Pi' \vdash C'$}
\BinaryInfC{$\Sigma', \Gamma', \Pi' \vdash C'.$}
\end{prooftree}
$$

Then the end-sequent is also provable as

$$
\begin{prooftree}
\AxiomC{$\vdots$}\noLine\UnaryInfC{$\Gamma \vdash A$}
\AxiomC{$\vdots$}\noLine\UnaryInfC{$\Sigma, A, \Pi \vdash B$}
\BinaryInfC{$\Sigma, \Gamma, \Pi \vdash B$}
\AxiomC{$\vdots$}\noLine\UnaryInfC{$\Lambda \vdash C$}
\doubleLine\BinaryInfC{$\Sigma', \Gamma, \Pi' \vdash C'$}
\doubleLine\UnaryInfC{$\Sigma', \Gamma', \Pi' \vdash C'$}
\end{prooftree}
$$

or

$$
\begin{prooftree}
\AxiomC{$\vdots$}\noLine\UnaryInfC{$\Sigma \vdash B$}
\AxiomC{$\vdots$}\noLine\UnaryInfC{$\Gamma \vdash A$}
\AxiomC{$\vdots$}\noLine\UnaryInfC{$\Pi, A, \Lambda \vdash C$}
\BinaryInfC{$\Pi, \Gamma, \Lambda \vdash C$}
\doubleLine\BinaryInfC{$\Sigma', \Gamma, \Pi' \vdash C'$}
\doubleLine\UnaryInfC{$\Sigma', \Gamma', \Pi' \vdash C'.$}
\end{prooftree}
$$

Since each of these cuts has less rank, each of the end-sequents is provable without a cut using induction hypothesis.

iii. Case that the right upper inference of cut is prime contraction. Then $\mathcal{P}$ is in the form of

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash A\end{array}}{\Gamma' \vdash A} \quad \dfrac{\begin{array}{c}\vdots\\ \Sigma, A, \Pi, P, P \vdash C\end{array}}{\Sigma, A, \Pi, P \vdash C}}{\Sigma, \Gamma', \Pi, P \vdash C}$$

or

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash P\end{array}}{\Gamma' \vdash P} \quad \dfrac{\begin{array}{c}\vdots\\ \Sigma, P, P \vdash C\end{array}}{\Sigma, P \vdash C}}{\Sigma, \Gamma' \vdash C.}$$

In the first case, the end-sequent is provable as

$$\begin{array}{c} \begin{array}{c}\vdots\\ \Gamma \vdash A\end{array} \quad \begin{array}{c}\vdots\\ \Sigma, A, \Pi, P, P \vdash C\end{array} \\ \hline \Sigma, \Gamma, \Pi, P, P \vdash C \\ \hline \Sigma, \Gamma, \Pi, P \vdash C \\ \hline\hline \Sigma, \Gamma', \Pi, P \vdash C. \end{array}$$

Since this cut has less rank, the end-sequent is provable without a cut using induction hypothesis.

In the latter case, since the succedent of $\Gamma \vdash P$ is propositional variables, the upper most sequent of right most thread of

$$\begin{array}{c}\vdots\\ \Gamma \vdash P\end{array}$$

must be initial sequent $P \vdash P$.

Then replacing this $P \vdash P$ with $\Sigma, P \vdash C$ and any sequent $\Pi \vdash P$ in the thread with $\Sigma, \Pi \vdash C$ yields also a proof tree with end-sequent $\Sigma, \Gamma \vdash C$. Thus the end-sequent is also provable without a cut as

$$\begin{array}{c} \vdots \\ \Sigma, P, P \vdash C \\ \hline \Sigma, P \vdash C \\ \vdots \\ \Sigma, \Gamma \vdash C \\ \hline \Sigma, \Gamma' \vdash C. \end{array}$$

$\square$

4 Natural deduction system

In this section, we introduce a natural deduction system $\mathrm{NJ}_{\mathcal{M}}$ for the implicational intuitionistic logic. Since we want to be sensitive to the identification of open assumptions, we add a slight unusual modification to the system.

Proofs in $\mathrm{NJ}_{\mathcal{M}}$ are trees of formulas defined inductively as follows:

1. open assumption

   For any formula A,
   $$A^i$$
   where i is some natural number (we call it an ***assumption number***) is a proof with conclusion A. A subproof of form A^i is called an ***open assumption***.

2. contraction

   Assume $\mathcal{D}$ is a proof with conclusion C, and let $A^{i_1}, \ldots, A^{i_n}$ are open assumptions in $\mathcal{D}$. Then $\mathcal{D}'$ obtained by replacing all A^{i_k}s $(k = 1, \ldots, n)$ in $\mathcal{D}$ with A^js for some fresh[2] natural number j is a proof with the same conclusion.

3. implication introduction

   Assume $\mathcal{D}$ is a proof with conclusion B which we write
   $$\begin{array}{c}\mathcal{D}\\ B\end{array}$$
   and let $\mathcal{D}'$ a proof obtained by replacing all A^is in $\mathcal{D}$ with $[A]^i$s (A^i may not appear in $\mathcal{D}$). Then
   $$\dfrac{\begin{array}{c}\mathcal{D}'\\ B\end{array}}{A \to B}\ i$$
   is a proof with conclusion $A \to B$. A subproof of form $[A]^i$ is called a ***cancelled assumption***. (i is still referred as an assumption number.)

4. implication elimination

   Assume $\mathcal{D}$ is a proof with conclusion $A \to B$ and $\mathcal{D}'$ is a proof with conclusion A. If the sets of assumption numbers that appear in $\mathcal{D}$ and in $\mathcal{D}'$ are disjoint, then
   $$\dfrac{\begin{array}{c}\mathcal{D}\\ A \to B\end{array} \quad \begin{array}{c}\mathcal{D}'\\ A\end{array}}{B}$$
   is a proof with conclusion B.

Let $\mathcal{D}$ be a proof in $\mathrm{NJ}_{\mathcal{M}}$ and Δ be a set $\{A^i \mid A^i \text{ appears in } \mathcal{D}\}$. Then ***the multiset of all open assumptions in*** $\mathcal{D}$ is $|\Delta|$ where $|(\text{-})|$ is the operation of eliminating superscripts (, i.e. the multiplicity of A in $|\Delta|$ is the number of distinct A^is in Δ).

[2] We will use the word "fresh" to mean that this natural number does not yet appear in $\mathcal{D}$ as an assumption number.

Let Γ be a multiset of formulas and A be a formula. We say $\mathcal{D}$ is ***a proof of*** A ***from*** Γ in $\mathrm{NJ}_{\mathcal{M}}$ if conclusion of $\mathcal{D}$ is A and its multiset of all open assumptions is included in Γ. We say A ***is provable from*** Γ in $\mathrm{NJ}_{\mathcal{M}}$ if there exists a proof of A from Γ in $\mathrm{NJ}_{\mathcal{M}}$.

In $\mathrm{NJ}_{\mathcal{M}}$, contraction is said to be ***prime*** iff the identified assumption is a propositional variable, otherwise said to be ***non-prime***. We say a formula is ***provable in*** $\mathrm{NJ}_{\mathcal{M}}$ ***without a non-prime contraction*** iff there exist a proof that is constructed without using a non-prime contraction.

Readers will immediately agree that in the case when open assumptions do not remain, proofs in $\mathrm{NJ}_{\mathcal{M}}$ is nothing but proofs in usual NJ (for example in [4]). So, uniqueness of proofs in $\mathrm{NJ}_{\mathcal{M}}$ coincides with that of proofs in NJ. Moreover, call implication introduction in NJ non-prime contraction iff cancelled assumption appear more than one place and cancelled assumption is not a propositional variable. Then A is provable in NJ without a non-prime contraction iff A is provable in $\mathrm{NJ}_{\mathcal{M}}$ without a non-prime contraction.

Now, we prove equivalency of provability in LJ without a non-prime contraction and that in $\mathrm{NJ}_{\mathcal{M}}$ without a non-prime contraction.

NOTATION. 4.1

1. For each multiset of formulas Γ, a sequence of formulas in Γ (in an arbitrary order) is written as Γ^*;
2. For each sequence of formulas Γ, the multiset obtained by omitting the order of formulas in Γ is written as $\overline{\Gamma}$.

THEOREM. 4.2 $\vdash A$ is provable in LJ without a non-prime contraction iff A is provable in $\mathrm{NJ}_{\mathcal{M}}$ without a non-prime contraction.

Proof.
(if)-part. Let $\mathcal{D}$ be a proof in $\mathrm{NJ}_{\mathcal{M}}$ of A from multiset Γ which is constructed without using a non-prime contraction. Then we show that $\Gamma^* \vdash A$ is provable in LJ without a non-prime contraction.

Our proof proceeds by induction on the construction of $\mathcal{D}$.

1. Case that $\mathcal{D}$ is an open assumption. Then the proof is

$$A^i$$

and $A \in \Gamma$. Then $\Gamma^* \vdash A$ is provable as

$$\dfrac{A \vdash A}{\dfrac{}{\Gamma^* \vdash A.}}$$

2. Case that the last derivation-step is prime contraction. Assume that open assumptions $P^{i_1}, \ldots, P^{i_n}$ are replaced with P^i by this inference and let Σ be the multiset obtained from Γ by removing precisely one P.

Then by induction hypothesis, $\overbrace{P,\ldots,P}^{n \text{ times}},\Sigma^* \vdash A$ is provable. Then $\Gamma^* \vdash A$ is provable as

$$\dfrac{\dfrac{\dfrac{\vdots}{P,\ldots,P,\Sigma^* \vdash A}}{P,\Sigma^* \vdash A}}{\Gamma^* \vdash A.}$$

3. Case that the last derivation-step is implication introduction. Then $A \equiv B \to C$ for some B, C. Then $\Gamma^*, B \vdash C$ is provable by induction hypothesis. Then $\Gamma^* \vdash B \to C$ is provable as

$$\dfrac{\dfrac{\vdots}{\Gamma^*, B \vdash C}}{\Gamma^* \vdash B \to C}$$

4. Case that the last derivation-step is implication elimination. Let Σ and Π be the multisets of all open assumptions in proofs of $B \to A$ and B, respectively. Then by induction hypothesis, $\Sigma^* \vdash B \to A$ and $\Pi^* \vdash B$ are provable.

   Then $\Gamma^* \vdash A$ is provable as

$$\dfrac{\dfrac{\dfrac{\vdots}{\Pi^* \vdash B} \quad \dfrac{\dfrac{\vdots}{\Sigma^* \vdash B \to A} \quad \dfrac{B \vdash B \quad A \vdash A}{B, B \to A \vdash A}}{B, \Sigma^* \vdash A}}{\Pi^*, \Sigma^* \vdash A}}{\Gamma^* \vdash A.}$$

(only if)-part. Let $\mathcal{P}$ be a proof of $\Gamma \vdash A$ in LJ without a non-prime contraction. By THEOREM.3.1, we can assume $\mathcal{P}$ does not contain any cuts. We show that A is provable from $\overline{\Gamma}$ without a non-prime contraction.

Our proof proceeds by induction on the construction of $\mathcal{P}$.

1. Case that the end-sequent is an initial sequent. Then $\mathcal{P}$ is in the form of

$$A \vdash A.$$

   Then A is provable from $\overline{A}$ as

$$A^i.$$

2. Case that the last inference is exchange. Then $\mathcal{P}$ is in the form of

$$\dfrac{\dfrac{\vdots}{\Sigma, B, C, \Pi \vdash A}}{\Sigma, C, B, \Pi \vdash A.}$$

   By induction hypothesis, A is provable from $\overline{\Sigma, B, C, \Pi}$, hence from $\overline{\Sigma, C, B, \Pi}$.

3. Case that the last inference is weakening. Then $\mathcal{P}$ is in the form of

$$\frac{\begin{array}{c}\vdots\\ \Sigma \vdash A\end{array}}{B, \Sigma \vdash A.}$$

By induction hypothesis, there exists a proof with conclusion A and whose multiset of all open assumptions is included in $\overline{\Sigma}$, hence in $\overline{B, \Sigma}$. Therefore A is provable from $\overline{B, \Sigma}$.

4. Case that the last inference is prime contraction. Then $\mathcal{P}$ is in the form of

$$\frac{\begin{array}{c}\vdots\\ P, P, \Sigma \vdash A\end{array}}{P, \Sigma \vdash A.}$$

By induction hypothesis, there exists a proof with conclusion A and whose multiset of all open assumptions is included in $\overline{P, P, \Sigma}$. By construction of the multiset of open assumptions of a proof, P^is and P^js might appear in the proof for some $i \neq j$. Then applying prime contraction, we can replace all P^is and P^js with P^ks for some fresh k. This gives a proof with conclusion A and whose the multiset of open assumptions is included in $\overline{P, \Sigma}$. Otherwise if such P^is and P^js does not appear in the proof, then the multiset of open assumptions in the proof is also included in $\overline{P, \Sigma}$, hence is a proof of A from $\overline{P, \Sigma}$.

5. Case that the last inference is implication left. Then $\mathcal{P}$ is in the form of

$$\frac{\begin{array}{c}\vdots\\ \Sigma \vdash B\end{array} \quad \begin{array}{c}\vdots\\ C, \Pi \vdash A\end{array}}{\Sigma, B \to C, \Pi \vdash A.}$$

By induction hypothesis, there exists a proof $\mathcal{D}_1$ with conclusion B and whose multiset of all open assumptions is included in $\overline{\Sigma}$ and a proof $\mathcal{D}_2$ with conclusion A and whose multiset of all open assumptions is included in $\overline{C, \Pi}$. We may assume that the sets of all assumption numbers in $\mathcal{D}_1$ and $\mathcal{D}_2$ are disjoint, since otherwise we can rename the duplicated numbers with another fresh numbers. By definition, C^i might appear in $\mathcal{D}_2$ for some i.

If C is non-prime, then C^i must appear at most once. Then replacing C^i in $\mathcal{D}_2$ with

$$\frac{B \to C^k \quad \begin{array}{c}\mathcal{D}_1\\ B\end{array}}{C}$$

(k for fresh assumption number) gives a proof of A form $\overline{\Sigma, B \to C, \Pi}$ that is constructed without using a non-prime contraction.

Otherwise $C \equiv P$ for some prime P, then P^i might appear more than once in $\mathcal{D}_2$ for some i. So, we can not execute the same construction as above, otherwise we shall need non-prime contractions in some cases[3]. Let $\mathcal{D}_2'$ be a proof obtained from $\mathcal{D}_2$ by replacing all P^is with $[P]^i$s for that i. Then

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_2'\\ A\end{matrix}}{P \to A}\, i \qquad \dfrac{B \to P^k \quad \begin{matrix}\mathcal{D}_1\\ B\end{matrix}}{P}}{A}$$

where k is a fresh assumption number is a proof of A from $\overline{\Sigma, B \to C, \Pi}$ that is constructed without using a non-prime contraction.

6. Case that the last inference is implication right. Then $A \equiv B \to C$ for some B, C and $\mathcal{P}$ is in the form of

$$\dfrac{\begin{matrix}\vdots\\ \Sigma, B \vdash C\end{matrix}}{\Sigma \vdash B \to C.}$$

By induction hypothesis, there exists a proof $\mathcal{D}$ with conclusion C and whose assumption is included in $\overline{\Sigma, B}$. Then B^i might appear in $\mathcal{D}$ for some i. Then applying implication introduction gives a proof of A from $\overline{\Sigma}$. $\square$

5 Application to unique normal proof property

Following result appears implicitly in [9]. (A close examination of the proof is presented in [2].)

Theorem. 5.1 Any provable formula A has a unique $\beta\eta$-normal proof in NJ if it is negatively non-duplicated.

This theorem is proved for the implication-conjunction fragment of NJ, but here we deal with the implicational fragment only.

Combining this with Theorem.2.4, Theorem.3.1 and Theorem.4.2, we have the following corollary.

Corollary. 5.2 Any J-minimal formula A has a unique $\beta\eta$-normal proof in NJ if it is provable in NJ without a non-prime contraction.

Example. 5.3 The following is a unique $\beta\eta$-normal proof in NJ of a J-minimal formula.

$$\dfrac{\dfrac{\dfrac{[(P \to R) \to S]^3 \qquad \dfrac{\dfrac{\dfrac{[P \to Q \to R]^4 \quad [P]^1}{Q \to R} \qquad \dfrac{[P \to Q]^2 \quad [P]^1}{Q}}{R}}{P \to R}\,1}{\dfrac{S}{(P \to Q) \to S}\,2}}{((P \to R) \to S) \to (P \to Q) \to S}\,3}{(P \to Q \to R) \to ((P \to R) \to S) \to (P \to Q) \to S}\,4$$

[3] It might be help for readers to see the example in section 7 in which the trace of this construction appears.

In the remaining of this section, we will show that COROLLARY.5.2 is in fact strengthened to the following theorem:

THEOREM. 5.4 Any J-minimal formula A has a unique β-normal proof in NJ if it is provable in NJ without a non-prime contraction.

Proofs in LBCK$_{pc}$ is nothing but proofs in LJ without a cut nor a non-prime contraction. Thus we incorporate discussions in section 2 to LBCK$_{pc}$ without explicitly mentioning it.

DEFINITION. 5.5 Given a poof figure $\mathcal{P}$ in LBCK$_{pc}$, a connection of formula B that contains a occurrence of B as an initial formula is called an arch.

It is easily seen that an arch always contains an occurrence in the end-sequent.

PROPOSITION. 5.6 Let $\mathcal{P}$ be a proof of S in LBCK$_{pc}$. Assume S is J-minimal.

1. For any sequent $T[B]$ in $\mathcal{P}$, if connection of B in $\mathcal{P}$ form an arch, then B is prime.
2. For any sequent $\Sigma \vdash B$ in $\mathcal{P}$, if $B \in \Sigma$ then B is prime.

Proof.

1. Assume that the connection of B form an arch. If B is not prime, it consists of two linear tracks both are starting at an initial sequent $B \vdash B$ and ending in the end-sequent $S \equiv U[B]$. So, replacing all occurrences of B in the connection with P for some new propositional variable P yields a proof of $U[P]$. Since B is not prime, $U[P] \prec U[B]$. Therefore S can not be J-minimal. This is a contradiction.
2. Assume $B \in \Sigma$ for some sequent $\Sigma \vdash B$ in $\mathcal{P}$. Then we can replace subproof of $\Sigma \vdash B$ with
$$\frac{B \vdash B}{\Sigma \vdash B.}$$
(double inference line)

   Then B form an arch and B is prime by 1. $\square$

DEFINITION. 5.7 Let $\mathcal{P}$ be a proof of $\Gamma \vdash A$ in LBCK$_{pc}$. For any sequent $B_1, \ldots, B_n \vdash C$ in $\mathcal{P}$, we inductively associate assumption number for each B_i $(i = 1, \ldots, n)$. The assumption numbers are is 0 or $\langle i_1, \ldots, i_n \rangle$ for some natural number $i_1, \ldots, i_n$. Here $\langle i_1, \ldots, i_n \rangle$ denote a natural number $\rho_1^{i_1} \cdots \rho_n^{i_n}$ where ρ_i is the i-th prime number.

1. Case that the end-sequent is an initial sequent. Then $\mathcal{P}$ is in the form of
$$A \vdash A.$$
   We choose some natural number i, and attach $\langle i \rangle$ as the assumption number of A.

2. Case that the last inference is exchange. Then $\mathcal{P}$ is in the form of

$$\dfrac{\begin{array}{c}\vdots\\ \Sigma, B, C, \Pi \vdash A\end{array}}{\Sigma, C, B, \Pi \vdash A.}$$

   Assumption numbers of B, C and those of formulas in Σ or Π do not change.

3. Case that the last inference is weakening. Then $\mathcal{P}$ is in the form of

$$\dfrac{\begin{array}{c}\vdots\\ \Sigma \vdash A\end{array}}{B, \Sigma \vdash A.}$$

   The assumption number of B is 0. Those of formulas in Σ do not change.

4. Case that the last inference is prime contraction. Then $\mathcal{P}$ in the form of

$$\dfrac{\begin{array}{c}\vdots\\ P, P, \Sigma \vdash A\end{array}}{P, \Sigma \vdash A.}$$

   (a) In the case that either one of the assumption numbers of the first two Ps in the sequence P, P, Σ is 0. The assumption number of P in the sequence P, Σ is the same as that of the other. (Therefore if both are 0, then it is 0.)

   (b) In the case that both of the assumption numbers of the first two Ps in the sequence P, P, Σ are not 0. Let them be $\langle i_1, \ldots, i_k\rangle$ and $\langle j_1, \ldots, j_l\rangle$. The assumption number of P in the sequence P, Σ is $\langle i_1, \ldots, i_k, j_1, \ldots, j_l\rangle$.

   Assumption numbers of formulas in Σ do not change.

5. Case that the last inference is implication left. Then $\mathcal{P}$ is in the form of

$$\dfrac{\begin{array}{c}\mathcal{P}_1\\ \Sigma \vdash B\end{array} \quad \begin{array}{c}\mathcal{P}_2\\ C, \Pi \vdash A\end{array}}{\Sigma, B \to C, \Pi \vdash A.}$$

   Firstly, we assume that the set of assumption numbers of formulas in the antecedents of sequents in $\mathcal{P}_1$ and that in $\mathcal{P}_2$ are disjoint except 0 (Call it "***disjointness assumption***"). To satisfy this assumption, it suffices to choose assumption numbers for the antecedent of initial sequents and for the assumptions introduced by implication left not to overlap between $\mathcal{P}_1$ and $\mathcal{P}_2$.

   (a) In the case that the assumption number of C is 0. The assumption number of $B \to C$ and those of formulas in Σ are now defined to be 0. Those of formulas in Π do not change.

   (b) In the case that the assumption number of C not 0. The assumption number of $B \to C$ in the sequence $\Sigma, B \to C, \Pi$ is $\langle i\rangle$ where i is a fresh natural number. Those of formulas in Σ or Π do not change.

6. Case that the last inference is implication right. Then $A \equiv B \to C$ for some B, C and $\mathcal{P}$ is in the form of

$$\frac{\begin{array}{c}\vdots\\ \Sigma, B \vdash C\end{array}}{\Sigma \vdash B \to C.}$$

   Assumption numbers of formulas in Σ do not change.

NOTATION. 5.8 Let $\mathcal{P}$ be a proof in LBCK$_{pc}$ with some assumption numbers are associated. We attach superscripts to the formulas in the antecedent as $A_1^{i_1}, \ldots, A_n^{i_n} \vdash C$ to denote the assumption number of A_j is i_j $(j = 1, \ldots, n)$.

PROPOSITION. 5.9 Let $\mathcal{P}$ be a poof in LBCK$_{pc}$ with some assumption numbers are associated. Let $\Pi \vdash A$ be any sequent in $\mathcal{P}$ and $D, E \in \Pi$. If non-zero assumption numbers $\langle i_1, \ldots, i_k \rangle$ and $\langle j_1, \ldots, j_l \rangle$ are attached to D, E, respectively then all $i_1, \ldots, i_k, j_1, \ldots, j_l$ are distinct.

Proof.

Our proof proceeds by induction on the construction of $\mathcal{P}$.

1. Case that the end-sequent is an initial sequent. Then obviously follows from definition.

2. Case that the last inference is exchange. Immediately follows from induction hypothesis.

3. Case that the last inference is weakening. Since the assumption number of weakened formula is 0, it immediately follows from induction hypothesis.

4. Case that the last inference is prime contraction. Then $\mathcal{P}$ in the form of

$$\frac{\begin{array}{c}\vdots\\ P, P, \Sigma \vdash A\end{array}}{P, \Sigma \vdash A.}$$

   The only non-obvious case is in the case that one of D, E is P and both of the assumption numbers of the first two Ps in the sequence P, P, Σ are not 0. Assume $D \equiv P$ and let the assumption numbers of the first two Ps in the sequence P, P, Σ be $\langle i_1, \ldots, i_m \rangle$ and $\langle i_{m+1}, \ldots, i_k \rangle$ and the assumption numbers of $E \in \Sigma$ be $\langle j_1, \ldots, j_l \rangle$. By induction hypothesis $i_1, \ldots, i_m, j_1, \ldots, j_l$ and $i_{m+1}, \ldots, i_k, j_1, \ldots, j_l$ are distinct. Therefore $i_1, \ldots, i_k, j_1, \ldots, j_l$ are distinct.

5. Case that the last inference is implication left. Then $\mathcal{P}$ is in the form of

$$\frac{\begin{array}{c}\mathcal{P}_1\\ \Sigma \vdash B\end{array} \quad \begin{array}{c}\mathcal{P}_2\\ C, \Pi \vdash A\end{array}}{\Sigma, B \to C, \Pi \vdash A.}$$

   If the assumption number of C is 0, it easily follows from induction hypothesis. So, let us assume the assumption number of C is not 0. If one of the D, E is $B \to C$, then follows from the way we choose i for the assumption number $\langle i\rangle$ of $B \to C$. The remaining non-trivial case is that $D \in \Sigma$ and $E \in \Pi$. Then $i_m = i_n$ or $j_m = j_n$ for some m, n contradicts with induction hypothesis. Also since $\langle i_1\rangle, \dots, \langle i_k\rangle$ appears in $\mathcal{P}_1$ and $\langle j_1\rangle, \dots, \langle j_l\rangle$ appears in $\mathcal{P}_2$ by definition, $i_m = j_n$ for some m, n contradicts with disjointness assumption.

6. Case that the last inference is implication right. Immediately follows from induction hypothesis. $\square$

PROPOSITION. 5.10 Let $\mathcal{P}$ be a proof of $\Gamma \vdash A$ in LBCK$_{pc}$ with some assumption numbers are associated. For any propositional variable $P \in \Gamma$, if the assumption number of P is non-zero $\langle i_1, \dots, i_k\rangle$, then $P^{\langle i_j\rangle} \vdash P$ uniquely appears in $\mathcal{P}$ for each $j = 1, \dots, k$.

Proof.

Our proof proceeds by induction on the construction of $\mathcal{P}$.

1. Case that the end-sequent is an initial sequent. Obvious by definition.
2. Case that the last inference is exchange. Immediately follows from induction hypothesis.
3. Case that the last inference is weakening. Since weakened formula has the assumption number 0.
4. Case that the last inference is prime contraction. Then $\mathcal{P}$ in the form of

$$\dfrac{\begin{matrix}\mathcal{P}_1\\ P, P, \Sigma \vdash A\end{matrix}}{P, \Sigma \vdash A.}$$

   The non-trivial case is in the case that both of the assumption number of the first two Ps in the sequence P, P, Σ are not 0. Let them be $\langle i_1, \dots, i_k\rangle$ and $\langle j_1, \dots, j_l\rangle$. Then since $i_1, \dots, i_k, j_1, \dots, j_l$ are distinct by PROPOSITION.5.9, it follows from induction hypothesis.

5. Case that the last inference is implication left. Since $B \to C$ is not prime, it immediately follows from induction hypothesis and disjointness assumption.
6. Case that the last inference is implication right. Immediately follows from induction hypothesis. $\square$

PROPOSITION. 5.11 Let $\mathcal{P}$ be a proof of $\Gamma \vdash A$ in LBCK$_{pc}$ with some assumption numbers are associated. Then we can construct a proof $\mathcal{P}^\circ$ of A from $\overline{\Gamma}$ in NJ$_{\mathcal{M}}$ inductively as follows, and it satisfies

1. the set $\{B^j \mid B^j \text{ appears in } \mathcal{P}^\circ\}$ is equal to the set

$$\{B^j \mid B \in \Gamma,\ \text{the assumption number of } B \text{ is non-zero } j\};$$

2. for any D in Γ, if assumption number of D is non-zero number $\langle j_1, \dots, j_l\rangle$ then $D^{\langle j_1, \dots, j_l\rangle}$ appears in $\mathcal{P}^\circ$ at least l times:

Proof.

1. Case that the end-sequent is an initial sequent. Then $\mathcal{P}$ is in the form of
$$A^{\langle i\rangle} \vdash A$$
for some i. Then it suffices to take
$$A^{\langle i\rangle}$$
as $\mathcal{P}^\circ$.

2. Case that the last inference is exchange. Then $\mathcal{P}$ is in the form of
$$\dfrac{\begin{array}{c}\mathcal{P}_1\\ \Sigma, B, C, \Pi \vdash A\end{array}}{\Sigma, C, B, \Pi \vdash A.}$$
Then take the proof $\mathcal{P}_1^\circ$ of A from $\overline{\Sigma, B, C, \Pi}$ as $\mathcal{P}^\circ$. Then 1 and 2 immediately follows from induction hypothesis.

3. Case that the last inference is weakening. Then $\mathcal{P}$ is in the form of
$$\dfrac{\begin{array}{c}\mathcal{P}_1\\ \Sigma \vdash A\end{array}}{B, \Sigma \vdash A.}$$
Then take the proof $\mathcal{P}_1^\circ$ of A from $\overline{\Sigma}$ as $\mathcal{P}^\circ$. Since the assumption number of B is 0, 1 and 2 immediately follow from induction hypothesis.

4. Case that the last inference is prime contraction. Then $\mathcal{P}$ in the form of
$$\dfrac{\begin{array}{c}\mathcal{P}_1\\ P, P, \Sigma \vdash A\end{array}}{P, \Sigma \vdash A.}$$
   (a) In the case that either one of the assumption numbers of the first two Ps in the sequence P, P, Σ is 0. Then take $\mathcal{P}_1^\circ$ as $\mathcal{P}^\circ$. This is also a proof of A from $\overline{P, \Sigma}$, since 1 holds for $\mathcal{P}_1$. Then 1 and 2 immediately follow from induction hypothesis.

   (b) In the case that both of the assumption number of the first two Ps in the sequence P, P, Σ are not 0. Let them be $\langle i_1, \ldots, i_k\rangle$ and $\langle j_1, \ldots, j_l\rangle$. Firstly, replacing $P^{\langle i_1,\ldots,i_k\rangle}$ and $P^{\langle j_1,\ldots,j_l\rangle}$ in $\mathcal{P}_1^\circ$ with $P^{\langle i_1,\ldots,i_k,j_1,\ldots,j_l\rangle}$ yields a proof of A from $\overline{P, \Sigma}$. This is because $P^{\langle i_1,\ldots,i_k\rangle}$ and $P^{\langle j_1,\ldots,j_l\rangle}$ actually appear in $\mathcal{P}_1$ since 1 holds for $\mathcal{P}_1$, and moreover, they are distinct and $\langle i_1, \ldots, i_k, j_1, \ldots, j_l\rangle$ is fresh natural number by PROPOSITION.5.9. Let it be $\mathcal{P}^\circ$. Then 1 follows from induction hypothesis and the construction. Since $P^{\langle i_1,\ldots,i_k\rangle}$ and $P^{\langle j_1,\ldots,j_l\rangle}$ are distinct and appear in $\mathcal{P}_1$ at least $k > 0$ and $l > 0$ times, respectively, 2 also holds by induction hypothesis.

5. Case that the last inference is implication left. Then $\mathcal{P}$ in the form of
$$\dfrac{\begin{array}{c}\mathcal{P}_1\\ \Sigma \vdash B\end{array} \quad \begin{array}{c}\mathcal{P}_2\\ C, \Pi \vdash A\end{array}}{\Sigma, B \to C, \Pi \vdash A.}$$
Since 1 holds for $\mathcal{P}_1$ and $\mathcal{P}_2$, the sets of assumption numbers in $\mathcal{P}_1^\circ$ and $\mathcal{P}_2^\circ$ are disjoint from disjointness assumption.

(a) In the case that the assumption number of C in the sequence C, Π is 0. Then take $\mathcal{P}_2^\circ$ as $\mathcal{P}^\circ$. Since 1 holds for $\mathcal{P}_2$, this indeed is a proof of A from $\overline{\Sigma, B \to C, \Pi}$. Then 1 and 2 immediately follow from induction hypothesis.

(b) In the case that the assumption number of C in the sequence C, Π is not 0. Let it be $\langle j_1, \ldots, j_l \rangle$. Then replace all open assumption $C^{\langle j_1, \ldots, j_l \rangle}$ in $\mathcal{P}_2$ with

$$\frac{B \to C^{\langle i \rangle} \quad \begin{array}{c}\mathcal{P}_1^\circ \\ B\end{array}}{C}$$

where $\langle i \rangle$ is a fresh assumption number and let it be $\mathcal{P}^\circ$. This indeed is a proof of A from $\overline{\Sigma, B \to C, \Pi}$. Since 1 holds for $\mathcal{P}_2$, $C^{\langle j_1, \ldots, j_l \rangle}$ indeed appears $\mathcal{P}_2^\circ$. Therefore 1 and 2 follows from induction hypothesis.

6. Case that the last inference is implication right. Then $A \equiv B \to C$ for some B, C and $\mathcal{P}$ is in the form of

$$\frac{\begin{array}{c}\mathcal{P}_1 \\ \Sigma, B \vdash C\end{array}}{\Sigma \vdash B \to C.}$$

Let the assumption number of B in Σ, B be j. Then $\mathcal{P}^\circ$ is

$$\frac{\begin{array}{c}\mathcal{D} \\ C\end{array}}{B \to C}\ j$$

where $\mathcal{D}$ is the proof obtained from $\mathcal{P}_1^\circ$ by replacing all open assumption B^j with $[B]^j$. Since 1 holds for $\mathcal{P}_1$, this is a proof of $B \to C$ from $\overline{\Sigma}$. Then 1 and 2 immediately follows from induction hypothesis. □

PROPOSITION. 5.12 Let $\mathcal{P}$ be a proof of $\Gamma \vdash A$ in LBCK$_{pc}$ with some assumption numbers are associated. Then we can translate $\mathcal{P}$ to a proof $\mathcal{P}'$ of $[\![\Gamma]\!] \vdash A$ in LBCK$_{pc}$ with some assumption numbers are associated such that $\mathcal{P}^\circ \equiv \mathcal{P}'^\circ$. Here $[\![\ ]\!]$ is the operation of removing all formulas whose assumption number is 0.

Proof.

Our proof proceeds by induction on the construction of $\mathcal{P}$.

1. Case that the end-sequent is an initial sequent. Let $\mathcal{P}$ be

$$A \vdash A.$$

Since assumption number of A is not 0, this is also a proof of $[\![A]\!] \vdash A$ in LBCK$_{pc}$.

2. Case that the last inference is exchange. Then $\mathcal{P}$ is in the form of

$$\frac{\begin{array}{c}\mathcal{P}_1 \\ \Sigma, B, C, \Pi \vdash A\end{array}}{\Sigma, C, B, \Pi \vdash A.}$$

Let $\mathcal{P}_1'$ be the proof of $[\![\Sigma, B, C, \Pi]\!] \vdash A$ obtained by induction hypothesis.

If both B and C remains in $[\![\Sigma, B, C, \Pi]\!]$ then applying similar exchange to $\mathcal{P}_1'$ yields the one we want. Let it be $\mathcal{P}'$. Then $\mathcal{P}'^{\circ} \equiv \mathcal{P}_1'^{\circ} \equiv \mathcal{P}_1{}^{\circ} \equiv \mathcal{P}^{\circ}$ by definition and induction hypothesis.

Otherwise either B or C does not appear in $[\![\Sigma, B, C, \Pi]\!]$. Then obviously, $\mathcal{P}_1'$ suffices. Similarly as above, $\mathcal{P}'^{\circ} \equiv \mathcal{P}^{\circ}$.

3. Case that the last inference is weakening. Then $\mathcal{P}$ is in the form of

$$\dfrac{\begin{matrix}\mathcal{P}_1 \\ \Sigma \vdash A\end{matrix}}{B, \Sigma \vdash A.}$$

Let $\mathcal{P}_1'$ be the proof of $[\![\Sigma]\!] \vdash A$ obtained by induction hypothesis. Since the assumption number of B is 0, $\mathcal{P}_1'$ the one is we want. Then $\mathcal{P}^{\circ} \equiv \mathcal{P}_1{}^{\circ} \equiv \mathcal{P}_1'^{\circ}$ by definition and induction hypothesis.

4. Case that the last inference is prime contraction. Then $\mathcal{P}$ in the form of

$$\dfrac{\begin{matrix}\mathcal{P}_1 \\ P, P, \Sigma \vdash A\end{matrix}}{P, \Sigma \vdash A.}$$

Let $\mathcal{P}_1'$ be the proof of $[\![P, P, \Sigma]\!] \vdash A$ obtained by induction hypothesis.

   (a) In the case that the assumption number of P in the sequence of P, Σ is 0. Then both of the assumption number of the first Ps in the sequence P, P, Σ are 0 by definition. Thus $\mathcal{P}_1'$ suffices in this case. Then $\mathcal{P}^{\circ} \equiv \mathcal{P}_1{}^{\circ} \equiv \mathcal{P}_1'^{\circ}$ by definition and induction hypothesis.

   (b) In the case the assumption number of P in the sequence of P, Σ is not 0, and either one of the assumption number of the first Ps in the sequence P, P, Σ is 0. Then $[\![P, P, \Sigma]\!] \equiv [\![P, \Sigma]\!]$. Thus $\mathcal{P}_1'$ suffices. Then $\mathcal{P} \equiv \mathcal{P}_1 \equiv \mathcal{P}_1'$ by definition and induction hypothesis.

   (c) In the case that the assumption number of P in the sequence of P, Σ is not 0, and both of the assumption number of the first Ps in the sequence P, P, Σ is not 0. Then $[\![P, P, \Sigma]\!] \equiv P, P, [\![\Sigma]\!]$ and $[\![P, \Sigma]\!] \equiv P, [\![\Sigma]\!]$. So, applying similar process to $\mathcal{P}_1'$ yields a proof of $[\![P, \Sigma]\!] \vdash A$. Let it be $\mathcal{P}'$. Since $\mathcal{P}_1{}^{\circ} \equiv \mathcal{P}_1'^{\circ}$ by induction hypothesis, $\mathcal{P}^{\circ} \equiv \mathcal{P}'^{\circ}$.

5. Case that the last inference is implication left. Then $\mathcal{P}$ in the form of

$$\dfrac{\begin{matrix}\mathcal{P}_1 \\ \Sigma \vdash B\end{matrix} \quad \begin{matrix}\mathcal{P}_2 \\ C, \Pi \vdash A\end{matrix}}{\Sigma, B \to C, \Pi \vdash A.}$$

Let $\mathcal{P}_1'$ be the proof of $[\![\Sigma]\!] \vdash B$ and $\mathcal{P}_2'$ be the proof of $[\![C, \Pi]\!] \vdash A$ obtained by induction hypothesis.

   (a) In the case that the assumption number of C in the sequence C, Π is 0. Then $[\![\Sigma, B \to C, \Pi]\!] \equiv [\![C, \Pi]\!]$ by definition. Thus $\mathcal{P}_2'$ suffices. In this case, $\mathcal{P}^{\circ} \equiv \mathcal{P}_2{}^{\circ} \equiv \mathcal{P}_2'^{\circ}$ by definition and induction hypothesis.

(b) In the case that the assumption number of C in C, Π is not 0. Then C is in $[\![C, \Pi]\!]$ and the assumption number of $B \to C$ is not 0. Therefore applying similar implication left to $\mathcal{P}_1'$ and $\mathcal{P}_2'$ yields the one we want. Let it be $\mathcal{P}'$. By induction hypothesis, $\mathcal{P}_1{}^\circ \equiv \mathcal{P}_1'{}^\circ$ and $\mathcal{P}_2{}^\circ \equiv \mathcal{P}_2'{}^\circ$. Therefore $\mathcal{P} \equiv \mathcal{P}'$.

6. Case that the last inference is implication right. Then $A \equiv B \to C$ for some B, C and $\mathcal{P}$ is in the form of

$$\dfrac{\begin{matrix}\mathcal{P}_1 \\ \Sigma, B \vdash C\end{matrix}}{\Sigma \vdash B \to C.}$$

Let $\mathcal{P}_1'$ be the proof of $[\![\Sigma, B]\!] \vdash C$ obtained by induction hypothesis. If $[\![\Sigma, B]\!]$ contains B, applying similar implication right to $\mathcal{P}_1'$ yields the one we want. Let it be $\mathcal{P}'$. Since $\mathcal{P}_1{}^\circ \equiv \mathcal{P}_1'{}^\circ$ by induction hypothesis, $\mathcal{P}^\circ \equiv \mathcal{P}'^\circ$.

Otherwise

$$\dfrac{\dfrac{\begin{matrix}\mathcal{P}_1' \\ [\![\Sigma, B]\!] \vdash C\end{matrix}}{[\![\Sigma]\!], B \vdash C}}{\Sigma \vdash B \to C}$$

yields the one we want. Let it be $\mathcal{P}'$. Firstly, $\mathcal{P}_1{}^\circ \equiv \mathcal{P}_1'{}^\circ$ by induction hypothesis. Since weakening does not change the corresponding proofs, $\mathcal{P}'^\circ$ is

$$\dfrac{\dfrac{\mathcal{P}_1'^\circ}{C}}{B \to C.}$$

Therefore $\mathcal{P}'^\circ \equiv \mathcal{P}^\circ$. □

COROLLARY. 5.13 Let $\mathcal{P}$ be a proof of $\Gamma, D, \Sigma \vdash A$ in LBCK$_{pc}$ with some assumption numbers are associated. If the assumption number of D is 0, there exists a proof $\mathcal{P}'$ of $\Gamma, \Sigma \vdash A$ in LBCK$_{pc}$ such that $\mathcal{P}^\circ \equiv \mathcal{P}'^\circ$.

Proof.

If the assumption number of D in the sequence Γ, D, Σ is 0, $[\![\Gamma, D, \Sigma]\!] \equiv [\![\Gamma, \Sigma]\!]$. Then there exists a proof $\mathcal{P}''$ of $[\![\Gamma, \Sigma]\!] \vdash A$ in LBCK$_{pc}$ such that $\mathcal{P}^\circ \equiv \mathcal{P}''^\circ$ by PROPOSITION.5.12. Then applying weakening to this proof yields a proof $\mathcal{P}'$ of $\Gamma, \Sigma \vdash A$ in LBCK$_{pc}$. Since weakening does not change corresponding proof, $\mathcal{P}^\circ \equiv \mathcal{P}'^\circ$. □

LEMMA. 5.14 Let $\mathcal{P}$ be a poof in LBCK$_{pc}$. For any association of assumption number to $\mathcal{P}$, $\mathcal{P}^\circ$ in NJ$_{\mathcal{M}}$ is β-normal.

Proof.

By induction on the construction of $\mathcal{P}$. □

Recall that the following part of a proof in $\mathrm{NJ}_{\mathcal{M}}$ is called η-redex,

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A \to B\end{array} \quad [A]^i}{B}}{A \to B}\, i \\ \vdots$$

if subproof of form $[A]^i$ does not appear in $\mathcal{D}$ at all.

LEMMA. 5.15 Let $\mathcal{P}$ be a poof of $\vdash A$ in LBCK_{pc}. Assume A is J-minimal. Then for any association of assumption number to $\mathcal{P}$, there exists no η-redex in proof $\mathcal{P}^\circ$ of A in $\mathrm{NJ}_{\mathcal{M}}$.

Proof.

Suppose contrarily that there exists a η-redex in $\mathcal{P}^\circ$. By construction, in the process of constructing η-redex, there must be a process of

$$\dfrac{B \to C^{\langle j\rangle} \quad B^{\langle i\rangle}}{C} \quad \Rightarrow \cdots \Rightarrow \quad \dfrac{\begin{array}{c}\vdots\\ B \to C\end{array} \quad B^{\langle i\rangle}}{C} \quad \Rightarrow \cdots \Rightarrow \quad \dfrac{\dfrac{\begin{array}{c}\vdots\\ B \to C\end{array} \quad [B]^{\langle i\rangle}}{C}}{B \to C.}\,\langle i\rangle$$

Note that $[B]^{\langle i\rangle}$ appears at one place in the proof by definition of η-redex. Therefore we can assume the assumption number of B is in the form of $\langle i\rangle$ by PROPOSITION.5.11.

We can assume by PROPOSITION.5.12. that the corresponding steps in LBCK_{pc} is

$$\dfrac{\dfrac{\dfrac{B^{\langle i\rangle} \vdash B \quad C \vdash C}{B^{\langle i\rangle}, B \to C^{\langle j\rangle} \vdash C}}{\Pi, B^{\langle i\rangle} \vdash C}}{\Pi \vdash B \to C,}$$

where all B and all C displayed are connected, respectively.

We show that we can assume that the track of B is linear. Suppose contrarily that the track of B is not linear. Then the last confluence step is in the form

$$\dfrac{\begin{array}{c}\vdots\\ B, B, \Delta \vdash E\end{array}}{B^{\langle i\rangle}, \Delta \vdash E.}$$

Then by definition, one of the assumption numbers of two Bs in the sequence B, B, Δ is 0. So, by COROLLARY.5.13 we can transform it to a proof of $B^{\langle i\rangle}, \Delta \vdash E$ which does not have this contraction step, and which has the same corresponding proof in $\mathrm{NJ}_{\mathcal{M}}$. Thus we can eliminate this ramification. Also, this transformation does not eliminate the subproof

$$\dfrac{B^{\langle i\rangle} \vdash B \quad C \vdash C}{B^{\langle i\rangle}, B \to C^{\langle j\rangle} \vdash C.}$$

Suppose contrarily that the above subproof is eliminated by this transformation. Since B is prime by PROPOSITION.5.6, we can apply PROPOSITION.5.10, and $B^{\langle i\rangle} \vdash B$ always uniquely appears in the proof of $\Pi, B \vdash C$. This is a contradiction. Repeatedly applying this transformation, we finally get a proof of $\Pi, B \vdash C$ in which the track of B is linear.

Now, since the track of B is linear,

$$\dfrac{\dfrac{\dfrac{\dfrac{B^{\langle i\rangle} \vdash B \quad C \vdash C}{B^{\langle i\rangle}, B \to C^{\langle j\rangle} \vdash C}}{B \to C^{\langle j\rangle}, B^{\langle i\rangle} \vdash C}}{B \to C^{\langle j\rangle} \vdash B \to C}}{\Pi \vdash B \to C,}$$

also form a proof.

Now, this proof contains sequents of form $B \to C \vdash B \to C$. But this contradicts with J-minimality of A by PROPOSITION.5.6.

$\square$

Recall that the following part of a proof in $\mathrm{NJ}_{\mathcal{M}}$ is called η^{-1}-redex,

$$\dfrac{\vdots}{A \to B} \atop \vdots$$

if $A \to B$ does not appear neither as a left upper formula of implication elimination nor as a lower formula of implication introduction.

Note this condition ensures that any η^{-1}-reduction never create β-redex. Thus β-normality is preserved by any of η-reductions and η^{-1}-reductions. Moreover if β-normal proof $\mathcal{D}$ η-reduces to $\mathcal{D}'$, then $\mathcal{D}'$ η^{-1}-reduces to $\mathcal{D}$, and conversely, if β-normal proof $\mathcal{D}'$ η^{-1}-reduces to $\mathcal{D}$, then $\mathcal{D}$ η-reduces to $\mathcal{D}'$.

PROPOSITION. 5.16 Let $\mathcal{P}$ be a poof in LBCK_{pc}. If A does not appear as an initial formula in $\mathcal{P}$, then A does not occur as η^{-1}-redex.

Proof.

By induction on construction of $\mathcal{P}$. $\square$

LEMMA. 5.17 Let $\mathcal{P}$ be a poof of $\vdash A$ in LBCK_{pc}. Assume A is J-minimal. Then for any association of assumption number to $\mathcal{P}$, there exists no η^{-1}-redex in proof $\mathcal{P}^\circ$ of A in $\mathrm{NJ}_{\mathcal{M}}$.

Proof.

If $B \to C$ is a η^{-1}-redex in $\mathcal{P}^\circ$, then $B \to C$ appears as an initial formula in $\mathcal{P}$ by PROPOSITION.5.16. This contradicts with J-minimality of A by PROPOSITION.5.6. $\square$

Proof of the THEOREM.5.4.

Let A be a J-minimal formula that is provable in NJ without a non-prime contraction. By definition of $\mathrm{NJ}_{\mathcal{M}}$, it is provable in $\mathrm{NJ}_{\mathcal{M}}$ without a non-prime contraction. Then by

THEOREM.4.2 and THEOREM.3.2, $\vdash A$ is provable in LBCK_{pc}. Let its proof be $\mathcal{P}$. Then proof $\mathcal{P}^\circ$ of A in $\mathrm{NJ}_{\mathcal{M}}$ is β-normal by LEMMA.5.14 and has no η-redex by LEMMA.5.15. Therefore $\mathcal{P}^\circ$ is $\beta\eta$-normal.

Now, let $\mathcal{D}$ be any β-normal proof of A in $\mathrm{NJ}_{\mathcal{M}}$. Let it's $\beta\eta$-normal form be $\mathcal{D}'$. By COROLLARY.5.2, $\mathcal{D}' \equiv \mathcal{P}^\circ$. But since $\mathcal{P}^\circ$ has no η^{-1}-redex by LEMMA.5.17, $\mathcal{D} \equiv \mathcal{D}'$, hence $\mathcal{D} \equiv \mathcal{P}^\circ$.

So, any β-normal proof of A in $\mathrm{NJ}_{\mathcal{M}}$ is same as $\mathcal{P}^\circ$. Therefore β-normal proof of A in $\mathrm{NJ}_{\mathcal{M}}$ is unique, hence in NJ. □

6 Other related works

We now turn to show that both BCK-provable formulas and depth ≤ 2 formulas are provable in NJ without a non-prime contraction. Since BCK-provable formula is provable in NJ without a contraction, also provable in NJ without a non-prime contraction. In the following, we show that any intuitionistically provable formula is also provable in NJ without a non-prime contraction if its depth is ≤ 2.

DEFINITION. 6.1 The ***depth*** $|A|$ of a formula A is defined by induction on the structure of A as follows:

1. If $A \equiv P$, then $|A| = 0$;
2. If $A \equiv B \to C$, then $|A| = max(|B| + 1, |C|)$.

PROPOSITION. 6.2 Let A be a provable formula in NJ such that $|A| \leq 2$. In any β-normal proof of it, any implication introduction is below all implication elimination.

Proof.

If some implication introduction is upper that implication elimination, The lowest implication introduction must be over the minor premise of implication elimination by β-normality. So, the major premise of that implication elimination is depth ≥ 2 and hence the uppermost formula of its track is depth ≥ 2. Therefore the conclusion of the proof is depth > 2. But this contradicts with the assumption that $|A| \leq 2$. □

LEMMA. 6.3 Any formula A provable in NJ is also provable in NJ without a non-prime contraction if $|A| \leq 2$.

Proof.

Let $\mathcal{D}$ be a $\beta\eta^{-1}$-normal proof of formula A such that $|A| \leq 2$. Let P be the right-most propositional variable that occurs in A. By $\beta\eta^{-1}$-normality, $\mathcal{D}$ is form of

$$\cfrac{\cfrac{\begin{matrix}\mathcal{D}' \\ P\end{matrix}}{\text{(implication introductions)}}}{A,}$$

where no implication introductions occur in $\mathcal{D}'$ by the previous proposition. Let $B_1, B_2, \ldots, B_n$ be non-prime assumptions that are contracted. By $\beta\eta^{-1}$-normality, for each B_i, there exists a subproof $\mathcal{D}_i$

$$\dfrac{\dfrac{[B_i] \qquad \vdots}{\text{(implication eliminations)}} \quad \vdots}{Q_i,}$$

where Q_i is the propositional variable that occur in the right-most of B_i. Let $\mathcal{D}_{k1}, \mathcal{D}_{k2}, \ldots, \mathcal{D}_{kn}$ be a permutation of $\mathcal{D}_1, \ldots, \mathcal{D}_n$ such that if $i < j$ then $\mathcal{D}_{ki}$ is not a subproof of $\mathcal{D}_{kj}$ $(j = 1, \ldots, n)$. Then construct a proof

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{D}''\\ P\end{array}}{Q_{k1} \to P} \quad \begin{array}{c}\mathcal{D}_{k1}'\\ Q_{k1}\end{array}}{P}}{Q_{k2} \to P} \qquad \begin{array}{c}\mathcal{D}_{k2}'\\ Q_{k2}\end{array}}{P} \\ \vdots \\ Q_{kn} \to P\end{array} \qquad \begin{array}{c}\mathcal{D}_{kn}'\\ Q_{kn}\end{array}}{P}}{\text{(implication introductions)}}}{A,}$$

where $\mathcal{D}''$ is a proof obtained from $\mathcal{D}'$ by replacing all

$$\dfrac{\dfrac{[B_i] \qquad \vdots}{\text{(implication eliminations)}} \quad \vdots}{Q_i}$$

with

$$[Q_i]$$

for $i = 1, \ldots, n$, and $\mathcal{D}_{ki}'$ is a proof obtained from $\mathcal{D}_{ki}$ by replacing all

$$\dfrac{\dfrac{[B_{k_j}] \qquad \vdots}{\text{(implication eliminations)}} \quad \vdots}{Q_{k_j}}$$

that appears in $\mathcal{D}_{ki}$ with

$$[Q_{k_j}]$$

(, hence $j > i$ by construction). This proof is a proof of exactly same formula that does not use non-prime contractions. $\square$

Theorem. 6.4 The set of J-minimal formulas which are provable without a non-prime contraction properly include the set of J-minimal depth ≤ 2 formulas and J-minimal BCK-provable formulas.

Proof.

We have already mentioned that J-minimal BCK-provable formulas are provable without a non-prime contraction. By LEMMA.6.3, J-minimal depth ≤ 2 formula is also provable without a non-prime contraction. Properness is shown by EXAMPLE.5.3 since $(P \to Q \to R) \to ((P \to R) \to S) \to ((P \to Q) \to S)$ is easily shown to be unprovable in the BCK (, e.g. using decidability of BCK) and it's depth is 3. $\square$

COROLLARY. 6.5 Any J-minimal formula is negatively non-duplicated if it is depth ≤ 2 or BCK-provable.

7 Concluding remarks

We said an implication introduction in NJ as a non-prime contraction when cancelled assumption appear more than one place and cancelled assumption is not a propositional variable. And we showed provability in lack of non-prime contractions is a sufficient condition of β-UP as for minimal formula in the implicational intuitionistic logic. This result gives an answer to the Y. Komori's problem in [7]. Also we showed that it includes both BCK-provability and depth ≤ 2 conditions which are previously known conditions of β-UP proved by S. Hirokawa and M. Tatsuta, respectively. As a corollary, we have shown any minimal formula in the implicational intuitionistic logic which satisfies at least one of these conditions is negatively non-duplicated. Here negatively non-duplicated formula is a formula in which any propositional variable has at most one negative occurrence in it, and for which $\beta\eta$-UP was shown to hold in [2]. These results are surveyed as below.

$$\begin{array}{ccc}
A \text{ is J-minimal and BCK-provable/depth} \leq 2 & & \\
\Downarrow & & \\
A \text{ is J-minimal and provable} & & \\
\text{without a non-prime contraction} & \Longrightarrow & A \text{ has } \beta\text{-UP} \\
\Downarrow & & \Downarrow \\
A \text{ is negatively non-duplicated} & \Longrightarrow & A \text{ has } \beta\eta\text{-UP}
\end{array}$$

In the figure 3, we also survey inclusion relations on various conditions of unique normal proof property obtained so far. Note that conditions for $\beta\eta$-UP is illustrated with rectangles and those for β-UP is illustrated with ellipses.

Note that provability in lack of non-prime contractions does not always mean that resulting unique β-normal proof does not use a non-prime contraction. For example,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[Q \to Q \to S]^2 \quad \dfrac{[P \to Q]^3 \quad [P]^1}{Q}}{Q \to S} \qquad \dfrac{[P \to Q]^3 \quad [P]^1}{Q}}{S}}{P \to S}\,1}{(Q \to Q \to S) \to P \to S}\,2}{(P \to Q) \to (Q \to Q \to S) \to P \to S}\,3$$

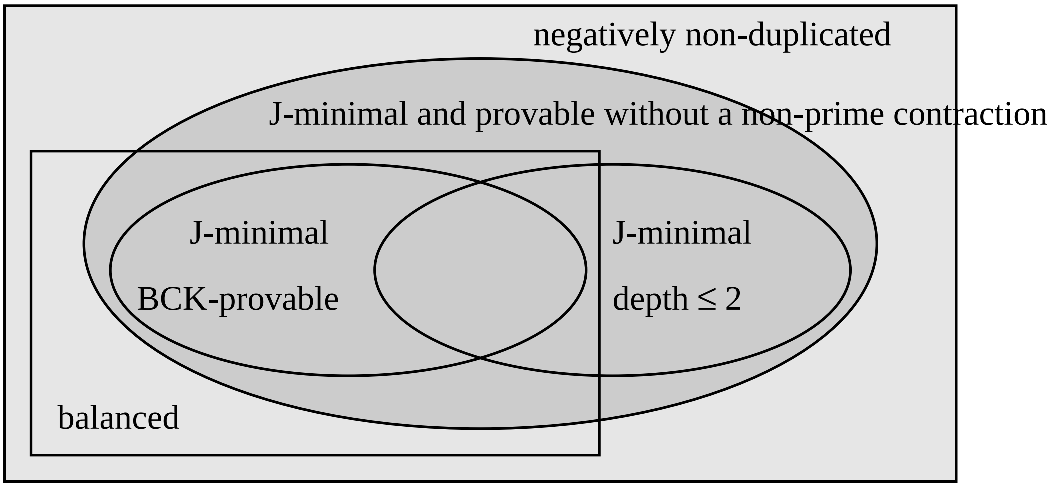

Figure 3: UP in the implicational intuitionistic logic

use a non-prime contraction. But it is easily seen that there exists a (non-β-normal) proof of this formula which does not use any non-prime contractions. Hence provable without a non-prime contraction, and therefore β-UP do hold also for this formula.

If we pay attention to the depth of contracted formula, yet a weaker condition of β-UP for implicational formulas that is minimal in the intuitionistic logic might exist.

Question. 7.1 Does any implicational formula A that is minimal in the intuitionistic logic have a unique β-normal proof in NJ, if it is provable in NJ without a contraction whose formula is depth > 1?

Readers will immediately agree that counterexamples in [10] and [12] does not refuse this question nor our result does not affirm it.

Let us call a contraction whose cancelled assumption is formula of depth i a depth i contraction. Thus prime contractions corresponds to depth 0 contractions. The counterexample by G. Mints ([10]) use a depth 4 contraction and that by M. Tatsuta ([12]) use a depth 3 contraction. However, any depth i ($i \geq 2$) contraction can be reduced to depth 2 contraction by depth-reducing transformation, and there exists no difference between depth i ($i \geq 2$) contraction and depth 2 contraction in the point of view from unique normal proof property. Therefore admissibility of depth 1 contractions would arise as a remaining question.

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