Decidable/Undecidable Properties of Conditional Term Rewriting Systems

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Abstract. Decidability/undecidability of properties in subclasses of conditional term rewriting systems (CTRSs, for short) of natural, join, and oriented types are studied. The properties focused are convertibility, reachability, joinability, confluence, and termination. In particular, it is shown that (1) all of the properties listed above are undecidable for left-linear right-ground CTRSs of each types having at most one left-linear right-ground condition for each rules; (2) all of the properties listed above are decidable for left-linear right-ground CTRSs of each types having ground conditions; (3) termination (confluence) is decidable for right-ground (left-linear right-ground, respectively) CTRSs of join and oriented types having at most one left-linear right-ground constructor condition for each rules; and (4) termination is decidable for left-linear right-ground growing CTRSs of oriented type having at most one left-linear right-ground condition for each rules. These results are closely related to known decidability/undecidability results in (conditional) term rewriting.

1 Introduction

Conditional term rewriting has a wide range of applications both in equational reasoning and design of declarative programming languages. We address in this paper decidable and undecidable properties of conditional term rewriting systems (CTRSs, for short) with equations in conditions of rules. In particular, we deal with CTRSs of natural, join, and oriented type, in which conditions are interpreted as convertibility, joinability, and reachability, respectively—these types of CTRSs have been widely investigated in literature.

The paper is to show how known decidability results for subclasses of term rewriting systems (TRSs, for short) are extended in conditional setting. More precisely, we examine for classes $\mathcal{C}$ of TRSs having a decidable property ϕ, in which conditional extensions of $\mathcal{C}$ the property ϕ is undecidable, as well

as in which extensions of $\mathcal{C}$ the property ϕ remains decidable. The properties considered in this paper are: convertibility, joinability, reachability, termination, and confluence—for these properties, decision problems in term rewriting are rather well-established.

Comparing to sufficient criteria for termination, confluence, etc. of CTRSs, only few decidability/undecidability results are known in conditional term rewriting. This situation is contrasting to that for term rewriting where considerable efforts have been dedicated to the study of decidable/undecidable properties of subclasses of TRSs. For example, it has been shown that all of the properties listed above are decidable for left-linear right-ground TRSs. Furthermore, we believe that this approach gives better understanding of expressiveness of CTRSs subjecting to various restrictions.

One of the best-known undecidability results in conditional term rewriting is that one-step reduction of CTRSs is no longer decidable[2][12]. We will see this fact is quite severe in a sense that even in a limited conditional extension of a well-behaved class of TRSs the situation is the same. Several criteria for decidability of one-step reduction, which also imply termination, are known: e.g. decreasing CTRSs[7], deterministic quasi-reductive CTRSs[8]. These criteria are, however, undecidable in general, and tools to detect these criteria have been investigated. More similar to our approach is a result in [2], which says that whether a given term is normal is decidable for orthogonal normal oriented CTRSs with *subterm property*. They proves in fact decidability of innermost one-step reduction for these CTRSs—this property is, however, out of scope of this paper.

The paper is organized as follows. Section 2 is to fix terminology and notations used in this paper. In section 3, the undecidability result in [12] is sharpen to a class of CTRSs whose underlying TRSs are left-linear and right-ground. In section 4, we consider in which conditional extensions the decidable properties of (left-linear) right-ground TRSs remain decidable. In section 5, we examine a conditional extension of the decidability result on reachability of linear growing TRSs in [11]. In section 6, we study decidability of termination for right-ground CTRSs based on the method in [10]; this yields a characterization of the right-groundness in conditional term rewriting. Our main results are summarized in section 7.

2 Preliminaries

We assume familiarity with the basic concepts and notations in term rewriting (which can be found in e.g. [1], [13]).

Rules of the form $l \to r \Leftarrow u_1 = v_1, \ldots, u_n = v_n$ are called conditional rewrite rules; the rule is *right-ground* if r is ground (i.e. has no variables), *left-linear* if l is linear (i.e. every variable occurs at most once); the rule has *right-ground conditions* if $u_1, \ldots, u_n$ are ground, has *left-linear conditions* if $v_1, \ldots, v_k$

are linear, has *ground conditions* if $u_1, \ldots, u_n, v_1, \ldots, v_n$ are ground, and has *at most one condition* if $n \leq 1$. We say $\mathcal{R}$ is right-ground (left-linear) if all rules involved are right-ground (left-linear, respectively); has right-ground (left-linear, ground) conditions if every rule involved has right-ground (left-linear, ground, respectively) conditions. *We assume the number of rules of CTRSs are finite*; thus a CTRS is written as $\mathcal{R} = \{l_i \to r_i \Leftarrow c_i \mid i \in I\}$ for a finite index set I where c_i stands for the conditions of the rule. We denote by $\mathcal{R}_u$ the underlying TRS $\{l \to r \mid l \to r \Leftarrow c \in \mathcal{R}\}$ of $\mathcal{R}$. *We assume in this paper that CTRSs have no extra-variables,* i.e. every rule $l \to r \Leftarrow c$ in CTRSs satisfies $\mathrm{Var}(r) \cup \mathrm{Var}(c) \subseteq \mathrm{Var}(l)$. Here and hereafter, $\mathrm{Var}(\alpha)$ denotes the set of variables appearing in the expression α. We will denote by $\equiv$ the syntactical equality of entities.

For each CTRS $\mathcal{R} = \{l_i \to r_i \Leftarrow c_i \mid i \in I\}$, we define TRSs $\mathcal{R}_k$ for $k \in \omega$ inductively as follows:

$$\begin{array}{rcl} \mathcal{R}_0 & = & \emptyset \\ \mathcal{R}_{k+1} & = & \bigcup_{i \in I} \{l_i\theta \to r_i\theta \mid \mathcal{R}_k \vdash u\theta = v\theta \text{ for all } u = v \text{ in } c_i\}. \end{array}$$

According to the interpretation of the predicate $\mathcal{R}_k \vdash s = t$, we consider here several types of CTRSs, which are summarized in Table 1. Here, $\overset{*}{\to}_{\mathcal{R}_k}$ ($\overset{*}{\leftrightarrow}_{\mathcal{R}_k}$) is the transitive reflexive closure of $\to_{\mathcal{R}_k}$ ($\leftrightarrow_{\mathcal{R}_k}$, respectively). For each types

Table 1:

type	interpretation of $\mathcal{R}_k \vdash s = t$
natural	$s \overset{*}{\leftrightarrow}_{\mathcal{R}_k} t$
join	$s \overset{*}{\to}_{\mathcal{R}_k} u \overset{*}{\leftarrow}_{\mathcal{R}_k} t$ for some u
oriented	$s \overset{*}{\to}_{\mathcal{R}_k} t$
normal oriented	$(s \overset{*}{\to}_{\mathcal{R}_k} t)$ and t is $\mathcal{R}_u$-normal

of CTRSs $\mathcal{R}$, its rewrite relation $\to_{\mathcal{R}}$ is defined as $\bigcup_{k \in \omega} \to_{\mathcal{R}_k}$. We note that the class of normal oriented CTRSs is a subclass of oriented CTRSs.

We say a CTRS $\mathcal{R}$ has decidable one-step reduction if for any terms s,t, whether $s \to_{\mathcal{R}} t$ holds is decidable. Terms s and t are *convertible* if $s \overset{*}{\leftrightarrow}_{\mathcal{R}} t$; terms s and t are *joinable* if $s \overset{*}{\to}_{\mathcal{R}} u \overset{*}{\leftarrow}_{\mathcal{R}} t$ for some term u; a term t is *reachable from* s if $s \overset{*}{\to}_{\mathcal{R}} t$. The convertibility (joinability, reachability) is decidable for a CTRS $\mathcal{R}$ if for any terms s and t whether s and t are convertible (resp. whether s and t are joinable, whether t is reachable from s) is decidable.

The following proposition is an immediate consequence of these definitions.

Proposition 2.1 *Let $\mathcal{R}$ be a natural (join, oriented) CTRS such that its convertibility (resp. joinability, reachability) is decidable. Then one-step reduction of $\mathcal{R}$ is decidable.*

Proof. Let $\mathcal{R}$ be a natural (join, oriented) CTRS satisfying the condition. Whether $s \to_{\mathcal{R}} t$ can be checked as follows. Collect all position $p \in \mathrm{Pos}(s)$ and substitution σ and $l \to r \Leftarrow u_1 = v_1, \ldots, u_n = v_n$ such that $s \equiv C[l\sigma]_p$, $t \equiv C[r\sigma]_p$ and $\mathrm{dom}(\sigma) = \mathrm{Var}(l)$. Here and hereafter, $\mathrm{Pos}(s)$ denotes the set of positions in s and $\mathrm{dom}(\sigma) = \{x \in \mathcal{V} \mid \sigma(x) \not\equiv x\}$. If $n = 0$ then clearly $s \to_{\mathcal{R}} t$. Otherwise it suffices to check whether $u_j\sigma$ and $v_j\sigma$ are convertible (resp. joinable, reachable) for all $j = 1, \ldots, n$. This is possible due to our assumption. ∎

An instance of *Post's Correspondence Problem* (PCP, for short) is like this:

> Let $a_1, \ldots, a_m$ be an alphabet. Given a finite set P of pairs of non-empty words over this alphabet, namely $P = \{\langle p_i, q_i\rangle \mid 1 \le i \le n\}$ where $p_i, q_i \in \{a_1, \ldots, a_m\}^+$, is there a sequence $\delta_1, \ldots, \delta_k \in \{1, \ldots, n\}$ such that $p_{\delta_1} \cdots p_{\delta_k} = q_{\delta_1} \cdots q_{\delta_k}$?

We identify the problem and the set P; and say P is solvable if there exists such a sequence. It is well-known that solvability of P is undecidable even when $m = 2$[17]. Thus, if we find for each P over a fixed alphabet a CTRS $\mathcal{R}$ in the class $\mathcal{C}$ so that $\mathcal{R}$ has the property ϕ if and only if P has a solution (or so that $\mathcal{R}$ has the property ϕ if and only if P has no solutions) then it follows that whether ϕ holds for a CTRS in $\mathcal{C}$ is undecidable. In the succeeding proofs we will use the undecidability of PCP on the fixed alphabets $\{0, 1\}$ to derive our undecidability results. We also provide below unary function symbols $\{0, 1\}$ in our signature (on CTRSs) and a convention that a term $a_1(\cdots(a_l(x))\cdots)$ is denoted by $\overline{w}(x)$ for each word $w = a_1 \cdots a_l \in \{0, 1\}^+$.

3 Undecidability results for right-ground CTRSs

The following fact is well-known.

Theorem 3.1 ([2][12]) *One-step reduction is undecidable for CTRSs.*

It trivially follows from this theorem that convertibility, joinability, reachability, termination, and confluence are all undecidable for CTRSs. These facts, however, are not surprising since these properties are undecidable also for TRSs.

On the other hand, for the class of (left-linear) right-ground TRSs a number of decidability results has been obtained:

Theorem 3.2 ([5]) *Convertibility is decidable for left-linear right-ground TRSs.*

Theorem 3.3 ([16]) *Joinability and reachability are decidable for right-ground TRSs.*

Theorem 3.4 ([10]) *Termination is decidable for right-ground TRSs.*

Theorem 3.5 ([5][14]) *Confluence is decidable for left-linear right-ground TRSs.*

We will present in this section sharper undecidability results in conditional term rewriting on convertibility, joinability, reachability, termination, and confluence by clarifying these properties are undecidable for left-linear right-ground CTRSs.

First we show that for whichever type of left-linear right-ground CTRSs one-step reduction, convertibility, joinability, reachability, and termination are undecidable. In fact, we even show a stronger result:

Theorem 3.6 *One-step reduction, convertibility, joinability, reachability, and termination are undecidable for left-linear right-ground CTRSs of natural, join, and normal oriented types having at most one left-linear right-ground condition for each rules.*

Proof. Let P be a PCP $\{\langle p_i, q_i\rangle \mid 1 \le i \le n\}$ with $p_i, q_i \in \{0,1\}^+$. We consider the following CTRS:

$$\mathcal{R}\left\{\begin{array}{lll}
f(x,y) \to C & \Leftarrow & eq(x,y) = true \\
f(x,y) \to D & \Leftarrow & f(\overline{p_1}(x), \overline{q_1}(y)) = A \\
\quad \cdots\cdots & & \\
f(x,y) \to D & \Leftarrow & f(\overline{p_n}(x), \overline{q_n}(y)) = A \\
C \to A & & \\
D \to A & & \\
D \to B & & \\
E \to F & \Leftarrow & f(\epsilon,\epsilon) = B \\
F \to E & & \\
F \to G & & \\
eq(\epsilon,\epsilon) \to true & & \\
eq(0(x),0(y)) \to true & \Leftarrow & eq(x,y) = true \\
eq(1(x),1(y)) \to true & \Leftarrow & eq(x,y) = true.
\end{array}\right.$$

In CTRSs $\mathcal{R}$ of each types, the one-step reduction $E \to_{\mathcal{R}} F$ occurs if and only if P admits a solution. Also, the one-step reduction $E \to_{\mathcal{R}} F$ occurs if and only if F is convertible with (reachable from) E; because of the rule $F \to G$, the one-step reduction $E \to_{\mathcal{R}} F$ coincides with joinability of F and G; and, because of the rule $F \to E$, the one-step reduction $E \to_{\mathcal{R}} F$ coincides with termination of $\mathcal{R}$. ∎

Remark 1 *We note that neither proof in [2] nor that in [12] works for right-ground CTRSs.*

Next we address confluence of left-linear right-ground CTRSs subjecting to the same restriction.

Theorem 3.7 *Confluence is undecidable for left-linear right-ground CTRSs of natural, join, and normal oriented types having at most one left-linear right-ground condition for each rules.*

Proof. Let P be a PCP $\{\langle p_i, q_i\rangle \mid 1 \le i \le n\}$ with $p_i, q_i \in \{0,1\}^+$. Let

$$\mathcal{R}\left\{\begin{array}{lll}
f(\overline{p_1}(\epsilon), \overline{q_1}(\epsilon)) \to A & & \\
\quad \cdots\cdots & & \\
f(\overline{p_n}(\epsilon), \overline{q_n}(\epsilon)) \to A & & \\
f(\overline{p_1}(x), \overline{q_1}(y)) \to A & \Leftarrow & f(x,y) = A \\
\quad \cdots\cdots & & \\
f(\overline{p_n}(x), \overline{q_n}(y)) \to A & \Leftarrow & f(x,y) = A \\
f(\epsilon, \epsilon) \to true & & \\
f(0(x), 0(y)) \to true & \Leftarrow & f(x,y) = true \\
f(1(x), 1(y)) \to true & \Leftarrow & f(x,y) = true.
\end{array}\right.$$

For CTRSs $\mathcal{R}$ of each types, P has no solutions if and only if $true \leftarrow_{\mathcal{R}} t \to_{\mathcal{R}} A$ for no term t if and only if $\mathcal{R}$ is confluent. ∎

4 Decidability results for right-ground CTRSs

In CTRSs presented in the previous section, we admit at most one left-linear right-ground condition per rule. Naturally, one may wonder what happens if conditions are at all ground. We first give an answer to this.

Theorem 4.1 *1. Convertibility and confluence are decidable for left-linear right-ground CTRSs of natural, join, and oriented types having ground conditions.*

2. Joinability, reachability, and termination are decidable for right-ground CTRSs of join and oriented types having ground conditions.

3. Joinability, reachability, and termination are decidable for left-linear right-ground CTRSs of natural type having ground conditions.

Proof. Suppose that $\mathcal{R}$ is a left-linear right-ground CTRS of natural type having ground conditions. We define a sequence of TRSs $\mathcal{S}_0, \mathcal{S}_1, \ldots$ by induction as follows:

$$\begin{array}{lcll}
\mathcal{S}_0 & = & \emptyset, & \\
\mathcal{S}_{k+1} & = & \{\ l \to r \ \mid & l \to r \Leftarrow u_1 = v_1, \ldots, u_n = v_n \in \mathcal{R}, \\
 & & & u_i \overset{*}{\leftrightarrow}_{\mathcal{S}_k} v_i \ \text{ for all } i = 1, \ldots n\ \}.
\end{array}$$

It is readily checked that $\mathcal{S}_k$ is left-linear right-ground TRS for every k; hence $u_i \overset{*}{\leftrightarrow}_{\mathcal{S}_k} v_i$ is decidable by Theorem 3.2, and $\mathcal{S}_0, \mathcal{S}_1, \ldots$ are effectively defined by

induction on k. Moreover, by definition, $\mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \cdots \subseteq \mathcal{R}_u$ holds. Suppose $\mathcal{R}$ has l rules. Then, there are at most $l+1$ different TRSs in this sequence. It should be clear also from definition that $\mathcal{R}_i = \mathcal{R}_{i+1}$ implies $\mathcal{R}_i = \mathcal{R}_j$ for all $j \geq i$. Hence, it follows that $\mathcal{R}_l = \mathcal{R}_{l+1} = \cdots$.

Now we claim that $\rightarrow_{\mathcal{R}_k}$ equals $\rightarrow_{\mathcal{S}_k}$ for each k by induction on k. Base step is trivial. For the induction step, suppose $\rightarrow_{\mathcal{R}_k} = \rightarrow_{\mathcal{S}_k}$. We have to show the rewrite relation of

$$\mathcal{R}_{k+1} \quad = \quad \{\ l\sigma \rightarrow r\sigma \ \mid \quad l \rightarrow r \Leftarrow u_1 = v_1, \ldots, u_n = v_n \in \mathcal{R}, \\ u_i\sigma \stackrel{*}{\leftrightarrow}_{\mathcal{R}_k} v_i\sigma \text{ for all } i = 1, \ldots, n\ \}$$

coincides with that of $\mathcal{S}_{k+1}$. ($\subseteq$) Suppose $s \rightarrow_{\mathcal{R}_{k+1}} t$. Then $s \equiv C[l\sigma\theta]$ and $t \equiv C[r\sigma\theta]$ for some C, θ, $l \rightarrow r \Leftarrow u_1 = v_1, \ldots, u_n = v_n \in \mathcal{R}$ and σ such that $u_i\sigma \stackrel{*}{\leftrightarrow}_{\mathcal{S}_k} v_i\sigma$ for all $i = 1, \ldots, n$. By induction hypothesis, we know $u_i\sigma \stackrel{*}{\leftrightarrow}_{\mathcal{R}_k} v_i\sigma$. This implies $u_i \stackrel{*}{\leftrightarrow}_{\mathcal{R}_k} v_i$ because all terms appearing in the conditionals of the rules are ground. Thus, $l \rightarrow r \in \mathcal{S}_{k+1}$ and hence $s \rightarrow_{\mathcal{S}_{k+1}} t$. ($\supseteq$) Suppose $s \rightarrow_{\mathcal{S}_{k+1}} t$. Then $s \equiv C[l\theta]$ and $t \equiv C[r\theta]$ for some C, θ, $l \rightarrow r \Leftarrow u_1 = v_1, \ldots, u_n = v_n \in \mathcal{R}$ such that $u_i \stackrel{*}{\leftrightarrow}_{\mathcal{S}_k} v_i$ for all $i = 1, \ldots, n$. By induction hypothesis, we have $u_i \stackrel{*}{\leftrightarrow}_{\mathcal{R}_k} v_i$. By taking the identity substitution as σ, we know that $l \rightarrow r \in \mathcal{R}_{k+1}$, and hence $s \rightarrow_{\mathcal{R}_{k+1}} t$.

Thus, we conclude that $\rightarrow_{\mathcal{R}} = \bigcup_{k \in \omega} \rightarrow_{\mathcal{R}_k} = \rightarrow_{\mathcal{R}_l}$. Since $\mathcal{R}_l$ is a left-linear right-ground TRS, for any terms s, t whether $s \stackrel{*}{\leftrightarrow}_{\mathcal{R}_l} t$ holds is decidable by Theorem 3.2. Also, confluence of $\mathcal{R}_l$ is decidable by Theorem 3.5.

The other statements are proved similarly using Theorems 3.2–3.5. ∎

It is very likely that we allowed a defined symbol f to occur in conditions of rules is a key for the undecidability results shown in the previous section. We next study decidable/undecidable properties of (left-linear) right-ground CTRSs containing no defined symbols in conditions of the rules.

The set D of defined symbols is defined like this: $D = \{\mathrm{root}(l) \mid l \rightarrow r \Leftarrow c \in \mathcal{R}\}$. Here $\mathrm{root}(l)$ is the function symbol occurring at the root position of l. A rule $l \rightarrow r \Leftarrow u_1 = v_1, \ldots, u_n = v_n$ is said to have *constructor conditions* if $u_1, \ldots, u_n, v_1, \ldots, v_n$ involve no defined symbols.

We need the following definition.

Definition 4.2 *We say a CTRS $\mathcal{R}$ is variable-separated if for any $l \rightarrow r \Leftarrow u_1 = v_1, \ldots, u_n = v_n \in \mathcal{R}$ the sets $\mathrm{Var}(r), \mathrm{Var}(u_1), \ldots \mathrm{Var}(u_n)$ of variables are pairwise disjoint*[1]*.*

Theorem 4.3 *1. Convertibility and confluence are decidable for left-linear right-ground variable-separated CTRSs of join and oriented types having left-linear right-ground constructor conditions.*

[1] For the discussion in the rest of this section, it suffices to impose disjointness of $\mathrm{Var}(u_1), \ldots, \mathrm{Var}(u_n)$ (because we will assume right-groundness of CTRSs); however, the requirement will be used rigorously in the next section.

2. *Joinability, reachability, and termination are decidable for right-ground variable-separated CTRSs of join and oriented types having left-linear right-ground constructor conditions.*

Proof. Suppose that $\mathcal{R} = \{l_i \to r_i \Leftarrow c_i \mid i \in I\}$ is a left-linear right-ground variable-separated CTRSs of oriented type having left-linear right-ground constructor conditions.

Let $l_i \to r_i \Leftarrow c_i \in \mathcal{R}$ and suppose that c_i equals $u_1 = v_1, \ldots, u_n = v_n$. By our assumption, every variable occurs in $u_1, \ldots, u_n$ at most once. Thus, since $u_1, \ldots, u_n$ are constructor term, one can assume without loss of generality that there exists a unique ground substitution σ_i such that $\mathrm{dom}(\sigma_i) = \mathrm{Var}(c_i)$ and $u_j\sigma_i \equiv v_j$ for all $j = 1, \ldots, n$. Let $\mathcal{S}$ be a TRS $\{l_i\sigma_i \to r_i \mid i \in I\}$.

We now show that the reduction relation of $\mathcal{S}$ and $\mathcal{R}$ coincide, i.e. $\xrightarrow{*}_{\mathcal{S}} = \xrightarrow{*}_{\mathcal{R}}$. ($\subseteq$) Clear from the fact that $\mathcal{S} \subseteq \mathcal{R}_1$. ($\supseteq$) We show $\xrightarrow{*}_{\mathcal{S}} \supseteq \xrightarrow{*}_{\mathcal{R}_k}$ by induction on k. Base step is trivial. For induction step, suppose $\xrightarrow{*}_{\mathcal{S}} \supseteq \xrightarrow{*}_{\mathcal{R}_k}$. We show $\xrightarrow{*}_{\mathcal{S}} \supseteq \to_{\mathcal{R}_{k+1}}$, then the statement easily follows by induction on the length of $\xrightarrow{*}_{\mathcal{R}_{k+1}}$. Let $s \equiv C[l_i\theta] \to_{\mathcal{R}_{k+1}} C[r_i] \equiv t$. If $l_i \to r_i \in \mathcal{R}$ then $l_i \to r_i \in \mathcal{S}$ and therefore $s \to_{\mathcal{S}} t$ trivially follows. Suppose otherwise, i.e. $l_i \to r_i \Leftarrow c_i \in \mathcal{R}$. Then, for all $u = v \in c_i$, we know $u\theta \xrightarrow{*}_{\mathcal{R}_k} v$ and thus by induction hypothesis, $u\theta \xrightarrow{*}_{\mathcal{S}} v \equiv u\sigma_i$. Since left-hand sides of conditions are constructor terms, we have $\theta(x) \xrightarrow{*}_{\mathcal{S}} \sigma_i(x)$ for each $x \in \mathrm{dom}(\sigma_i)$. Now define a substutution η_i as follows: $\eta_i(x) = \sigma_i(x)$ if $x \in \mathrm{dom}(\sigma_i)$; $\eta_i(x) = \theta(x)$ otherwise. Then $s \equiv C[l_i\theta] \xrightarrow{*}_{\mathcal{S}} C[l_i\eta_i] \equiv C[(l_i\sigma_i)\theta]$ (since σ_i is a ground substitution) $\to_{\mathcal{S}} C[r_i\theta] \equiv C[r_i]$ (since r_i is ground) $\equiv t$.

Since $\mathcal{S}$ is a left-linear right-ground TRS, for any terms s, t whether $s \xleftrightarrow{*}_{\mathcal{S}} t$ holds is decidable by Theorem 3.2. Also, confluence of $\mathcal{S}$ is decidable by Theorem 3.5. For join type, it suffices to note that right-hand sides of conditions are $\mathcal{R}_u$-normal form.

The other statements are proved similarly using Theorems 3.2–3.5. ∎

Remark 2 *A careful inspection of the proof above reveals that for CTRSs of oriented type one can drop the condition that no defined symbols occur on the right-hand sides of conditions. Further, the constructor requirement can be weaken in such a way that for any $l \to r \Leftarrow u_1 = v_1, \ldots, u_n = v_n \in \mathcal{R}$ and $j \in \{1, \ldots, n\}$, every subterm of u_j is unifiable with no $t \in R_l$. Here, $R_l = \{l \mid l \to r \Leftarrow c \in \mathcal{R}\}$.*

Some of the possibilities to extend the decidability results above are rejected by the following propositions. First, since convertibility is undecidable for right-ground TRSs[15], we have

Proposition 4.4 *Convertibility is undecidable for right-ground CTRSs of natural, join, and normal oriented types having at most one ground constructor condition for each rules.*

Proposition 4.5 *Joinability, reachability, and termination are undecidable for right-ground CTRSs of natural type having at most one ground constructor condition for each rules.*

Proof. We use a right-ground TRS E presented in Section 3 of [15]. In [15] it is shown that $\bar{a} \overset{*}{\leftrightarrow}_E \bar{b}$ is undecidable for constructor constants $\bar{a}$ and $\bar{b}$. Hence, by considering a CTRS $\mathcal{R} = E \cup \{A \to C \Leftarrow \bar{a} = \bar{b},\quad B \to C \Leftarrow \bar{a} = \bar{b}\}$ (here A, B, C are new constants), we know joinability (of A and B) is undecidable for right-ground natural CTRSs with ground conditions. Similarly, the undecidability of reachability is shown by a CTRS $\mathcal{R} = E \cup \{A \to B \Leftarrow \bar{a} = \bar{b}\}$ and that of termination by $\mathcal{R} = E \cup \{A \to A \Leftarrow \bar{a} = \bar{b}\}$. ∎

We finally note that, using the proof above, that confluence is decidable for right-ground CTRSs of join and oriented types having ground conditions follows from the decidability of confluence of right-ground TRSs, which, however, is a long-standing open question (#63 in [6]).

A corollary of Theorem 4.3 is

Corollary 4.6 *1. Convertibility and confluence are decidable for left-linear right-ground CTRSs of join and oriented types having at most one left-linear right-ground constructor condition for each rules.*

2. Joinability, reachability, and termination are decidable for right-ground CTRSs of join and oriented types having at most one left-linear right-ground constructor condition for each rules.

This contrasts with $\mathcal{R}$ in the proof of Theorem 3.6 and with that in the proof of Theorem 3.7 which breaks the additional requirement that conditions are constructor terms.

We now present another corollary of Theorem 4.3. For this, we recall a framework of order-sorted TRSs introduced in [3], whose computational interpretation is given by membership conditional CTRSs whose conditions are of the form $x \in \alpha$ for some regular tree language α. In the discussion below, we assume familiarity with order-sorted signature and tree automata (see e.g. [4]).

Let $\langle S, \leq, \Sigma \rangle$ be an order-sorted signature, where S is a set of sorts equipped with a partial-order $\leq$ on it and Σ is a (not necessary disjoint) family $\bigcup_{\alpha_0,\ldots,\alpha_n \in S} \Sigma_{\alpha_1 \times \cdots \times \alpha_n \to \alpha_0}$, where $\Sigma_{\alpha_1 \times \cdots \times \alpha_n \to \alpha_0}$ is the set of function symbols with sort-declaration $\alpha_1 \times \cdots \times \alpha_n \to \alpha_0$. We suppose below that the order-sorted signature (i.e. S and Σ) is finite. Rewrite rules are of the form $l \to r$ such that each $x \in \mathrm{Var}(l)$ is equipped with its sort. The rewrite relation of our order-sorted TRSs is specified as: $s \to_{\mathcal{R}} t$ if $\sigma(x_1){:}\alpha_1, \ldots, \sigma(x_n){:}\alpha_n$, $s \equiv C[l\sigma]$, and $t \equiv C[r\sigma]$ for some $l \to r \in \mathcal{R}$, C, σ such that $l \equiv C_l[x_1{:}\alpha_1, \ldots, x_n{:}\alpha_n]$ with all variables displayed.

Observe that for any $\alpha \in S$ and a term t, the condition $t{:}\alpha$ can be specified by $t \xrightarrow{*}_{\mathcal{S}} \alpha$ using the oriented CTRS $\mathcal{S} = \bigcup_{\alpha,\alpha' \in S}\{\alpha \to \alpha' \mid \alpha \leq \alpha'\} \cup \bigcup_{\alpha_0,\ldots,\alpha_n \in S}\{f(x_1,\ldots,x_n) \to \alpha_0 \Leftarrow x_1 = \alpha_1, \ldots, x_n = \alpha_n \mid f \in \Sigma_{\alpha_1 \times \cdots \times \alpha_n \to \alpha_0}\}$. Thus the rewrite relation of an order-sorted TRS $\mathcal{R}$ coincides with that on $\mathcal{T}(\Sigma,\mathcal{V})$ of the oriented CTRS $\mathcal{R}' \cup \mathcal{S}$ where $\mathcal{R}' = \{C[x_1,\ldots,x_n] \to r \Leftarrow x_1 = \alpha_1, \ldots, x_n = \alpha_n \mid C[x_1{:}\alpha_1,\ldots,x_n{:}\alpha_n] \to r \in \mathcal{R}\}$. Note $\mathcal{R}' \cup \mathcal{S}$ is based on the signature $\Sigma \cup S$. However, since for any $\alpha \in S$, α is a constant and $\alpha \to_{\mathcal{R}' \cup \mathcal{S}} \alpha'$ implies $\alpha' \in S$, reachability of $\mathcal{R}$ coincides with that of $\mathcal{R}' \cup \mathcal{S}$ on $\mathcal{T}(\Sigma,\mathcal{V})$; and termination of $\mathcal{R}$ coincides with that of $\mathcal{R}' \cup \mathcal{S}$, provided $\mathcal{S}$ is finite.

Finally, observe that $\mathcal{R}' \cup \mathcal{S}$ is variable-separated and has left-linear right-ground conditions. Thus, by using the first observation in Remark 2, we have

Corollary 4.7 *In the framework of order-sorted TRSs introduced in [3], reachability and termination are decidable for right-ground order-sorted TRSs (provided that the order-sorted signature is finite).*

5 Reachability of growing CTRSs

A TRS $\mathcal{R}$ is growing if for any $l \to r \in \mathcal{R}$ and for any position p, $l/p \in \mathrm{Var}(r)$ implies $|p| \leq 1$. The following theorem is known.

Theorem 5.1 ([11]) *Reachability is decidable for linear growing TRSs.*

In this section, we present an extension of this result for conditional term rewriting and show its limitation as well.

Definition 5.2 *For a conditional rewrite rule $l \to r \Leftarrow u_1 = v_1, \ldots, u_n = v_n$ and $i \in I$, let*

$$\mathcal{O}_i(l \to r \Leftarrow u_1 = v_1, \ldots, u_n = v_n) = \begin{cases} \{\ l \to \nabla_i(u_1,\ldots,u_n,r),\ \nabla_i(v_1,\ldots,v_n,z) \to z\ \} & \text{if } n \geq 1, \\ \{\ l \to r\ \} & \text{if } n = 0, \end{cases}$$

where z is a new variable and ∇_i is a new function symbol with the appropriate arity. Suppose $\mathcal{R} = \{l_i \to r_i \Leftarrow c_i \mid i \in I\}$. Then we define

$$\mathcal{O}(\mathcal{R}) = \bigcup_{i \in I} \mathcal{O}_i(l_i \to r_i \Leftarrow c_i).$$

Symbols in $\{\nabla_i \mid i \in I\}$ are called ∇-symbols. The set of terms containing no ∇-symbols is denoted by $\mathcal{T}$.

From now on until Theorem 5.7, we assume that $\mathcal{R}$ is a CTRS of oriented type having right-ground conditions such that $\mathcal{O}(\mathcal{R})$ is a left-linear TRS.

Definition 5.3 *Let $s \equiv C[g\sigma]_p \to_{\mathcal{O}(\mathcal{R})} C[h\sigma]_p \equiv t$ with $g \to h \in \mathcal{O}(\mathcal{R})$. Let $q \in \mathrm{Pos}(t)$. The antecedent of q is a position in s defined as follows:*

1. *If $p \not< q$ then $q \in \mathrm{Pos}(s)$, and the antecedent of q is q.*
2. *Otherwise, i.e. $p < q$.*
   (a) *Suppose that there exists $p_x \in \mathrm{Pos}(h)$ such that $j/p_x \equiv x \in \mathcal{V}$ and $p.p_x \leq q$. Then, by linearity of g, there exists a unique position $p'_x \in \mathrm{Pos}(g)$ such that $g/p'_x \equiv x$. The antecedent of q is the position $p.p'_x.(q \backslash p_x)$.*
   (b) *Otherwise, the antecedent of q is undefined.*

Definition 5.4 1. *A subterm of t is called a ∇-subterm of t if $\mathrm{root}(t)$ is a ∇-symbol.*

2. *Let $s \xrightarrow{*}_{\mathcal{O}(\mathcal{R})} t$ with $s \in \mathcal{T}$. We now define $\widetilde{v}$ for each subterm v of t, and an origin of u for each ∇-subterm u of t, by induction on the length of $s \xrightarrow{*}_{\mathcal{O}(\mathcal{R})} t$.*
   (a) *Base step. Put $\widetilde{v} \equiv v$ for each subterm v of s. There is no ∇-subterm of s by our assumption.*
   (b) *Induction step. Suppose $s \xrightarrow{*}_{\mathcal{O}(\mathcal{R})} w \to_{\mathcal{O}(\mathcal{R})} t$, $w \equiv C[g\sigma]_p$, and $t \equiv C[h\sigma]_p$ with $g \to h \in \mathcal{O}(\mathcal{R})$. Let $u \equiv t/q$ be a ∇-subterm of t.*
      i. *If $p \not\leq q$ then the origin of u is that of w/q.*
      ii. *If $p = q$ then the origin of u is $\widetilde{w/p}$.*
      iii. *Otherwise, i.e. $p < q$. Then, by the definition of $\mathcal{O}(\mathcal{R})$, there exists a position $p_x \in \mathrm{Pos}(h)$ such that $h/p_x \in \mathcal{V}$ and $p.p_x \leq q$. Thus, there exists a unique antecedent $q' \in \mathrm{Pos}(w)$ of q. The origin of u is that of w/q'.*

   For each subterm v of t, $\widetilde{v}$ results from v by replacing all its maximal ∇-subterms by their respective origins.

Lemma 5.5 *Let $s \xrightarrow{*}_{\mathcal{O}(\mathcal{R})} t$ with $s,t \in \mathcal{T}$. Then, $s \xrightarrow{*}_{\mathcal{R}} t$.*

Proof. We prove that $\widetilde{s} \xrightarrow{*}_{\mathcal{R}} \widetilde{t}$ for any $s \in \mathcal{T}$ and t such that $s \xrightarrow{*}_{\mathcal{O}(\mathcal{R})} t$. Then, since $t \equiv \widetilde{t}$ when $t \in \mathcal{T}$, the statement of the lemma follows immediately. The proof proceeds by induction on the length of $s \xrightarrow{*}_{\mathcal{O}(\mathcal{R})} t$. Base case is trivial. For the induction step, suppose $s \xrightarrow{*}_{\mathcal{O}(\mathcal{R})} w \equiv C[g\sigma]_p \to_{\mathcal{O}(\mathcal{R})} C[h\sigma]_p \equiv t$ with $g \to h \in \mathcal{O}(\mathcal{R})$.

Suppose there exists a maximal ∇-subterm w/q of w such that $q < p$. Then, clearly, t/q is a maximal ∇-subterm of t and since the origins of w/q and t/q are same, we have $\widetilde{w} \equiv \widetilde{t}$. Assume otherwise, i.e. that for any maximal ∇-subterm w/q of w, either $p \mid q$ or $p \leq q$ holds. We first show $\widetilde{g\sigma} \xrightarrow{*}_{\mathcal{R}} \widetilde{h\sigma}$.

1. $g \to h$ has form $l_i \to r_i$ for $l_i \to r_i \in \mathcal{R}$. Then if w/q is a maximal ∇-subterm of w then $p.p_x \leq q$ for some p_x with $l_i/p_x \in \mathcal{V}$. Let $\widetilde{\sigma}$ be a substitution defined by $\widetilde{\sigma}(x) \equiv \widetilde{\sigma(x)}$ for all $x \in \mathrm{Var}(g)$. Since g, h do not contain ∇_i and g is linear, we have $\widetilde{g\sigma} \equiv g(\widetilde{\sigma}) \to_{\mathcal{R}} h(\widetilde{\sigma}) \equiv \widetilde{h\sigma}$.

2. $g \to h$ has form $l_i \to \nabla_i(u_1, \ldots, u_n, r_i)$ for $l_i \to r_i \Leftarrow u_1 = v_1, \ldots, u_n = v_n \in \mathcal{R}$. Then, by definition, we have $\widetilde{g\sigma} \equiv g\widetilde{\sigma} \equiv \widetilde{h\sigma}$.

3. $g \to h$ has form $\nabla_i(v_1, \ldots, v_n, z) \to z$ for $l_i \to r_i \Leftarrow u_1 = v_1, \ldots, u_n = v_n \in \mathcal{R}$. Then $g\sigma$ is a maximal ∇-subterm of w by our assumption, and, since $v_1, \ldots, v_n$ are ground, we have $g\sigma \equiv \nabla_i(v_1, \ldots, v_n, z\sigma)$. Suppose $s \xrightarrow{*}_{\mathcal{O}(\mathcal{R})} w$ equals $s \equiv s_0 \to_{\mathcal{O}(\mathcal{R})} s_1 \to_{\mathcal{O}(\mathcal{R})} \cdots \to_{\mathcal{O}(\mathcal{R})} s_m \equiv w$. Then there exist positions $p_l \in \mathrm{Pos}(s_l)$, $p_{l+1} \in \mathrm{Pos}(s_{l+1}), \ldots, p_m = p \in \mathrm{Pos}(s_m)$ $(0 \leq l < m)$ such that (1) p_i is the antecedent of p_{i+1} for all $i = l, \ldots, m-1$; and (2) $s_l \equiv C[l_i\theta]_{p_l} \to_{\mathcal{O}(\mathcal{R})} C[\nabla_i(u_1\theta, \ldots, u_n\theta, r_i\theta)]_{p_{l+1}} \equiv s_{l+1}$. Clearly, we have $s_i/p_i \equiv s_{i+1}/p_{i+1}$ or $s_i/p_i \to_{\mathcal{O}(\mathcal{R})} s_{i+1}/p_{i+1}$ for each $i = l, \ldots, m-1$. Hence, we obtain a reduction sequence $l_i\theta \to_{\mathcal{O}(\mathcal{R})} \nabla_i(u_1\theta, \ldots, u_n\theta, r_i\theta) \xrightarrow{*}_{\mathcal{O}(\mathcal{R})} \nabla_i(v_1, \ldots, v_n, z\sigma)$, which is properly shorter than $s \xrightarrow{*}_{\mathcal{O}(\mathcal{R})} t$. From this it follows $u_j\theta \xrightarrow{*}_{\mathcal{O}(\mathcal{R})} v_j$ for all $j = 1, \ldots, n$ and $r_i\theta \xrightarrow{*}_{\mathcal{O}(\mathcal{R})} z\sigma$. By induction hypothesis, we have $\widetilde{u_j\theta} \equiv u_j\widetilde{\theta} \xrightarrow{*}_{\mathcal{R}} \widetilde{v_j} \equiv v_j$ for all $j = 1, \ldots, n$, and $\widetilde{r_i\theta} \equiv r_i\widetilde{\theta} \xrightarrow{*}_{\mathcal{R}} \widetilde{z\sigma} \equiv \widetilde{h\sigma}$. Hence, we have $\widetilde{g\sigma} \equiv l_i\widetilde{\theta} \to_{\mathcal{R}} r_i\widetilde{\theta} \xrightarrow{*}_{\mathcal{R}} \widetilde{h\sigma}$.

Finally, we have $\widetilde{C[g\sigma]} \equiv \widetilde{C}[\widetilde{g\sigma}] \xrightarrow{*}_{\mathcal{R}} \widetilde{C}[\widetilde{h\sigma}] \equiv \widetilde{C[h\sigma]}$. ∎

Remark 3 *In the lemma above, our assumption that conditions of rewrite rules in $\mathcal{R}$ are right-ground can not be dropped. To see this, let*

$$\mathcal{R} \left\{ \begin{array}{ll} a(c) \to b(d) & \\ A(x) \to B & \Leftarrow \quad a(x) = b(x). \end{array} \right.$$

Then we have $\mathcal{O}(\mathcal{R}) = \{a(c) \to b(d),\ A(x) \to \nabla_2(a(x), B),\ \nabla_2(b(x), z) \to z\}$, and thus $A(c) \to_{\mathcal{O}(\mathcal{R})} \nabla_2(a(c), B) \to_{\mathcal{O}(\mathcal{R})} \nabla_2(b(d), B) \to_{\mathcal{O}(\mathcal{R})} B$ holds. But since there is no term t satisfying $a(t) \xrightarrow{*}_{\mathcal{R}} b(t)$, we have $A(t) \xrightarrow{*}_{\mathcal{R}} B$ for no term t.

Remark 4 *Similar translations and preservation results have been appeared in [2], [9], [18]. But neither of them is effective for our theorem below.*

We define growingness of a CTRS as follows.

Definition 5.6 *A CTRS $\mathcal{R}$ are said to be growing if for any $l \to r \Leftarrow c \in \mathcal{R}$ and for any position $p \in \mathrm{Pos}(l)$, $l/p \in \mathrm{Var}(r) \cup \mathrm{Var}(c)$ implies $|p| \leq 1$.*

Theorem 5.7 *Reachability is decidable for linear growing variable-separated CTRSs of oriented type having left-linear right-ground conditions.*

Proof. Observe that $\mathcal{O}(\mathcal{R})$ is linear and growing; for, $\mathcal{R}$ is variable-separated, $v_1, \ldots, v_k$ are ground, and l is linear and growing. Thus, it only remains to show that $\xrightarrow{*}_{\mathcal{O}(\mathcal{R})}$ over the set $\mathcal{T}$ coincides with $\xrightarrow{*}_{\mathcal{R}}$, i.e. $(\xrightarrow{*}_{\mathcal{O}(\mathcal{R})} \cap \mathcal{T}^2) = \xrightarrow{*}_{\mathcal{R}}$. ($\supseteq$) We show $(\xrightarrow{*}_{\mathcal{O}(\mathcal{R})} \cap \mathcal{T}^2) \supseteq \xrightarrow{*}_{\mathcal{R}_k}$ by induction on k. Base step is trivial. For the induction step, suppose $(\xrightarrow{*}_{\mathcal{O}(\mathcal{R})} \cap \mathcal{T}^2) \supseteq \xrightarrow{*}_{\mathcal{R}_k}$. We show $(\xrightarrow{*}_{\mathcal{O}(\mathcal{R})} \cap \mathcal{T}^2) \supseteq \rightarrow_{\mathcal{R}_{k+1}}$, then the statement easily follows by induction on the length of $\xrightarrow{*}_{\mathcal{R}_{k+1}}$. Suppose $s \rightarrow_{\mathcal{R}_{k+1}} t$ with $C[l_i\sigma] \equiv s$, $C[r_i\sigma] \equiv t$, $l_i \rightarrow r_i \Leftarrow c_i \in \mathcal{R}$, and $u\sigma \xrightarrow{*}_{\mathcal{R}_k} v$ for all $u = v$ in c_i. If c_i is empty, then the statement trivially follows. Suppose otherwise. For any $u = v \in c_i$, we have $u\sigma \xrightarrow{*}_{\mathcal{O}(\mathcal{R})} v$ by induction hypothesis. Therefore, we have $C[l_i\sigma] \rightarrow_{\mathcal{O}(\mathcal{R})} C[\nabla_i(u_1, \ldots, u_n, r_i)\sigma] \equiv C[\nabla_i(u_1\sigma, \ldots, u_n\sigma, r_i\sigma)] \xrightarrow{*}_{\mathcal{O}(\mathcal{R})} C[\nabla_i(v_1, \ldots, v_n, r_i\sigma)] \rightarrow_{\mathcal{O}(\mathcal{R})} C[r_i\sigma]$ with c_i being $u_1 = v_1, \ldots, u_n = v_n$. ($\subseteq$) This follows from Lemma 5.5. ∎

Some of these conditions can not be dropped, as we show now. First, the CTRS $\mathcal{R}$ in the proof of Theorem 3.6 is also effective to show

Proposition 5.8 *Reachability is undecidable for linear variable-separated oriented CTRSs with left-linear right-ground conditions.*

Using a TRS which appears in the proof of Theorem 16 in [11], one easily shows

Proposition 5.9 *Reachability is undecidable for right-linear growing variable-separated CTRSs of oriented type having left-linear right-ground conditions.*

Proposition 5.10 *Reachability is undecidable for linear growing variable-separated CTRSs of natural and join types having left-linear right-ground conditions.*

Proof. Let P be a PCP $\{\langle p_i, q_i\rangle \mid 1 \leq i \leq n\}$ with $p_i, q_i \in \{0,1\}^+$. In the following CTRS $\mathcal{R}$, for each of natural and join types, F is reachable from E if and only if P has a solution.

$$\mathcal{R}\left\{\begin{array}{lll}
S \rightarrow \langle 0(\epsilon), 0(\epsilon)\rangle & & \\
S \rightarrow \langle 1(\epsilon), 1(\epsilon)\rangle & & \\
\langle x, y\rangle \rightarrow \langle 0(x), 0(y)\rangle & & \\
\langle x, y\rangle \rightarrow \langle 1(x), 1(y)\rangle & & \\
f(x, y) \rightarrow A & \Leftarrow & \langle x, y\rangle = S \\
f(x, y) \rightarrow A & \Leftarrow & f(\overline{p_1}(x), \overline{q_1}(y)) = A \\
\quad \cdots\cdots & & \\
f(x, y) \rightarrow A & \Leftarrow & f(\overline{p_n}(x), \overline{q_n}(y)) = A \\
E \rightarrow F & \Leftarrow & f(\epsilon, \epsilon) = A
\end{array}\right.$$

∎

Proposition 5.11 *Reachability is undecidable for linear growing variable-separated oriented CTRSs with left-linear conditions.*

Proof. Let P be a PCP $\{\langle p_i, q_i\rangle \mid 1 \leq i \leq n\}$ with $p_i, q_i \in \{0,1\}^+$. In the following oriented CTRS, F is reachable from E if and only if P has a solution.

$$\mathcal{R}\left\{\begin{array}{lll} f(x,y) \to C & \Leftarrow & x = y \\ f(x,y) \to D & \Leftarrow & f(\overline{p_1}(x), \overline{q_1}(y)) = A \\ \quad\cdots\cdots & & \\ f(x,y) \to D & \Leftarrow & f(\overline{p_n}(x), \overline{q_n}(y)) = A \\ C \to A & & \\ D \to A & & \\ D \to B & & \\ E \to F & \Leftarrow & f(\epsilon,\epsilon) = B \end{array}\right.$$

∎

As the translation $\mathcal{O}$ suggests, whether the other conditions—right-linearity, variable-separatedness, left-linearity of conditions—are necessary is closely related to the problem whether right-linearity is necessary to guarantee the decidability of linear growing TRSs, which, as far as we know, remains unsolved.

A corollary of Theorem 5.7 is

Corollary 5.12 *Reachability is decidable for left-linear right-ground growing CTRSs of oriented type having at most one left-linear right-ground condition for each rules.*

This contrasts with $\mathcal{R}$ in the proof of Theorem 3.6, which breaks growingness condition.

6 Termination of right-ground CTRSs

A proof analogous to that of decidability of termination of (right-)ground TRSs[10] is effective to establish a relation between decidability of termination and that of one-step reduction for right-ground CTRSs.[2]

The next lemma is proved in the same way as the one for TRSs.

Lemma 6.1 *Let $\mathcal{R} = \{l_i \to r_i \Leftarrow c_i \mid i \in I\}$ be a right-ground CTRS. If $\mathcal{R}$ is non-terminating then there exists $i \in I$ such that r_i is non-terminating.*

The proof of the following lemma[3] needs an alternation, for the original way in [10] (which is adopted also in [13] and [1]) to use induction on the number of rewrite rules does not work.

[2] Original lemmas are presented under an additional (auxiliary) assumption that left-hand sides of rewrite rules are also ground.

[3] The original lemma shows $l_i \xrightarrow{+} C[l_i]$ instead of $r_i \xrightarrow{+} C[r_i]$.

Lemma 6.2 *Let $\mathcal{R} = \{l_i \to r_i \Leftarrow c_i \mid i \in I\}$ be a right-ground CTRS. If $\mathcal{R}$ is non-terminating then there exists $i \in I$ such that $r_i \xrightarrow{+} C[r_i]$.*

Proof. Suppose $\mathcal{R}$ is non-terminating but there is no $i \in I$ such that $r_i \xrightarrow{+} C[r_i]$. By Lemma 6.1, there exists $i \in I$ such that r_i is non-terminating. Let $A : r_i \to u_0 \to u_1 \to \cdots$ be an infinite reduction sequence. If there exist infinitely many rewrite steps in A occurring at the root position, then, since I is finite, we know $r_j \to r_j$ for some j and this contradicts our assumption. So, suppose otherwise. Let $A : r_j \xrightarrow{*} r_k \to \cdots$ and assume that $r_k \to \cdots$ contains no rewrite steps occurring at the root position. Clearly, there is a smallest proper subterm r_k/p_1 of r_k that is non-terminating; and hence, without loss of generality one may assume any rewrite steps in $r_k \to \cdots$ occur at positions $\geq p_1$ and at least one rewrite step occurs at the position p_1. Thus, we know that there exists an infinite reduction sequence $r_j \xrightarrow{*} r_k \xrightarrow{*}_{\geq p_1} C_1[l_{j_1}\sigma_1]_{p_1} \to C_1[r_{j_1}]_{p_1} \xrightarrow{*}_{\geq p_1} \cdots$. Here, $\xrightarrow{*}_{\geq p_i}$ denotes a reduction sequence in which all rewrite steps occur at positions $\geq p_i$. By using our assumption in a similar way, we know that this reduction $C_1[r_{j_1}]_{p_1} \xrightarrow{*}_{\geq p_1} \cdots$ is of the form $C_1[r_{j_1}]_{p_1} \xrightarrow{*}_{\geq p_1} C_1[r_{k_1}]_{p_1} \xrightarrow{*}_{>p_1} \cdots$. Again without loss of generality one can take a smallest proper subterm $C[r_{k_1}]/p_2$ with $p_1 < p_2$ that is non-terminating, and can assume that rewrite steps in $C_1[r_{k_1}]_{p_1} \xrightarrow{*}_{>p_1} \cdots$ occur at positions $\geq p_2$. Repeating the similar arguments, we know that there exists an infinite reduction sequence $r_j \xrightarrow{*} r_k \xrightarrow{*}_{\geq p_1} C_1[l_{j_1}\sigma_1]_{p_1} \to C_1[r_{j_1}]_{p_1} \xrightarrow{*}_{\geq p_1} C_1[r_{k_1}]_{p_1} \xrightarrow{*}_{\geq p_2} C_2[l_{j_2}\sigma_2]_{p_2} \to C_2[r_{j_2}]_{p_2} \xrightarrow{*}_{\geq p_2} C_2[r_{k_2}]_{p_2} \xrightarrow{*}_{\geq p_3} C_3[l_{j_3}\sigma_3]_{p_3} \to C_3[r_{j_3}]_{p_3} \xrightarrow{*}_{\geq p_3} C_3[r_{k_3}]_{p_3} \xrightarrow{*}_{\geq p_4} \cdots$ with $p_1 < p_2 < \cdots$. Now, since I is finite, $r_k, r_{k_1}, r_{k_2}, \ldots$ can not be all distinct. Thus, we conclude $r_i \xrightarrow{+} C[r_i]$ for some $i \in I$. ∎

Theorem 6.3 *Let $\mathcal{R}$ be a right-ground CTRS. If one-step reduction of $\mathcal{R}$ is decidable then termination is a decidable property of $\mathcal{R}$.*

Proof. It is clear that if one-step reduction of $\mathcal{R}$ is decidable then one can compute all $\mathcal{R}$-reducts of a given term. Thus, the proof in [10] works similarly. ∎

Theorem 6.4 *For any right-ground CTRS $\mathcal{R}$ if $\mathcal{R}$ is non-terminating then there exists a natural number k such that $\mathcal{R}_k$ is non-terminating.*

Proof. Let $\mathcal{R} = \{l_i \to r_i \Leftarrow c_i \mid i \in I\}$. Then, by Lemma 6.2, we have $r_i \xrightarrow{+} C[r_i]$ for some $i \in I$. Clearly, it suffices to take a number k such that $r_i \xrightarrow{+}_{\mathcal{R}_k} C[r_i]$.

Remark 5 *The theorem above contrasts to non-right-ground case. Let*

$$\mathcal{R}\left\{\begin{array}{lll} g(x) \to g(f(x)) & \Leftarrow & h(x) = b \\ h(a) \to b & & \\ h(f(x)) \to b & \Leftarrow & h(x) = b. \end{array}\right.$$

Then, for CTRSs $\mathcal{R}$ of each types,

$$\begin{array}{lcl} \mathcal{R}_0 & = & \emptyset \\ \mathcal{R}_1 & = & \{h(a) \to b\} \\ \mathcal{R}_2 & = & \mathcal{R}_1 \cup \{g(a) \to g(f(a)), h(f(a)) \to b\} \\ \mathcal{R}_3 & = & \mathcal{R}_2 \cup \{g(f(a)) \to g(f(f(a))), h(f(f(a))) \to b\} \\ \mathcal{R}_4 & = & \mathcal{R}_3 \cup \{g(f(f(a))) \to g(f(f(f(a)))), h(f(f(f(a)))) \to b\} \\ & & \cdots\cdots \end{array}$$

are all terminating, while $\mathcal{R}$ is non-terminating as:

$$g(a) \to_{\mathcal{R}} g(f(a)) \to_{\mathcal{R}} g(f(f(a))) \to_{\mathcal{R}} \cdots.$$

By combining Proposition 2.1, Corollary 5.12, and Theorem 6.3, we obtain

Corollary 6.5 *Termination is decidable for left-linear right-ground growing CTRSs of oriented type having at most one left-linear right-ground condition for each rules.*

This contrasts with $\mathcal{R}$ in the proof of Theorem 3.6, which breaks growingness condition.

7 Conclusion

In this paper we studied decidability/undecidability of convertibility, joinability, reachability, termination, and confluence for subclasses of CTRSs of natural, join, and oriented types—in particular, those that are related to known decidability/undecidability results for (C)TRSs.

We summarize our main results in the tables below. (Here, n, j, and o stand for natural, join, and oriented, respectively.)

1. Convertibility/Confluence of CTRSs:

type	restriction	decidability	Theorem
any	left-linear right-ground rules		
	& left-linear right-ground conditions		
	& at most one condition per rule	undecidable	3.6
any	left-linear right-ground rules		
	& ground conditions	decidable	4.1
j/o	left-linear right-ground rules		
	& left-linear right-ground conditions		
	& at most one condition per rule		
	& constructor conditions	decidable	4.3

2. Joinability of CTRSs:

type	restriction	decidability	Theorem
any	left-linear right-ground rules & left-linear right-ground conditions & at most one condition per rule	undecidable	3.6
n	left-linear right-ground rules & ground conditions	decidable	4.1
j/o	right-ground rules & ground conditions	decidable	4.1
j/o	right-ground rules & left-linear right-ground conditions & at most one condition per rule & constructor conditions	decidable	4.3

3. Reachability/Termination of CTRSs:

type	restriction	decidability	Theorem
any	left-linear right-ground rules & left-linear right-ground conditions & at most one condition per rule	undecidable	3.6
n	left-linear right-ground rules & ground conditions	decidable	4.1
j/o	right-ground rules & ground conditions	decidable	4.1
j/o	right-ground rules & left-linear right-ground conditions & at most one condition per rule & constructor conditions	decidable	4.3
o	left-linear right-ground rules & left-linear right-ground conditions & at most one condition per rule & growing rules	decidable	(Cor.) 5.12, 6.5

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