On the Finite Model Property of Intuitionistic Modal Logics over MIPC*

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Abstract. It is shown that every normal intuitionistic modal logic L over MIPC has the finite model property if there exists a universal first-order sentence Φ such that (1) L is characterized by the class of Kripke frames satisfying Φ and (2) every Kripke frame that validates L satisfies Φ. Here, MIPC is a well-known intuitionistic modal logic introduced by Prior (1957).

1 Introduction

In this paper, we study the finite model property of (propositional) modal logics based on intuitionistic logic—we call them *intuitionistic modal logics*—with a necessity operator $\Box$ and a possibility operator $\Diamond$. In particular, we focus on those of logics over **MIPC** in the lattice of $\Box\Diamond$-normal intuitionistic modal logics. We show that every $\Box\Diamond$-normal intuitionistic modal logic L which includes **MIPC** has the finite model property if there exists a universal first-order sentence Φ such that (1) L is characterized by the class of Kripke frames satisfying Φ and (2) every Kripke frame that validates L satisfies Φ.

MIPC is an intuitionistic modal logic introduced by Prior [12], and has been studied by several authors [1][2][3][6][7][8][10][11]. **MIPC** is one of the **S5**-like intuitionistic modal logics; that is, if we add $p \vee \neg p$ to **MIPC** then we get the well-known classical modal logic **S5**. Another feature of **MIPC** is that one can obtain an embedding of **MIPC** into intuitionistic predicate logic by interpreting $\Box$ as $\forall$ and $\Diamond$ as $\exists$. Thus, one-variable fragments (i.e. the sets of formulas with

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one fixed variable) of intuitionistic predicate logic can be interpreted in **MIPC** (see [1], [11]). We also note that **MIPC** is an extension of **FS** (Fisher Servi's logic) introduced in [8]. For recent surveys on intuitionistic modal logics we refer the reader to [13], [15].

The finite model property of a logic L ensures the decidability of L when L is finitely axiomatizable, and hence has been one of the central focuses in the study of logics. The finite model property of **MIPC** was shown in [7], [10] and that of **FS** in [5]. Several general results on the finite model property of extensions of **MIPC** are also obtained in [1], [2].

The rest of the paper is organized as follows. In Section 2, we introduce basic notions and fix some notation used in this paper. In Section 3, we present a construction of a finite counter **MIPC**-model for ϕ from a given counter **MIPC**-model for ϕ based on a descriptive **MIPC**-frame. Our main theorem is given at the end of Section 3. We also give sufficient conditions of extensions of **MIPC** to have the finite model property in Section 4. We conclude our result in Section 5.

2 Preliminaries

Familiarity with the basic notions of modal logic (as expounded in e.g. [4]) will be helpful in what follows.

Let $\mathcal{L}$ be the propositional language consisting of infinitely many propositional variables $p, q, r, \ldots$ and the usual primitive connectives $\rightarrow, \wedge, \vee, \bot$. (We put $\top \stackrel{\text{def}}{=} \bot \rightarrow \bot$.) The set of $\mathcal{L}$-formulas is denoted by $\mathbf{For}\,\mathcal{L}$. A set $L \subseteq \mathbf{For}\,\mathcal{L}$ is called a *superintuitionistic logic* (abbrev. *si-logic*) if L includes intuitionistic logic **Int** and is closed under substitution and modus ponens. For a set $\Gamma \subseteq \mathbf{For}\,\mathcal{L}$, the smallest si-logic that includes Γ is denoted by $\mathbf{Int} + \Gamma$; when $\Gamma = \{\phi\}$, $\mathbf{Int} + \Gamma$ will be denoted by $\mathbf{Int} + \phi$.

An *intuitionistic (general) frame* (**Int**-*frame*, for short) is a structure $\langle W, R, P\rangle$ where R is a partial order on W and P is a set of R-cones in W that contains $\emptyset$ and is closed under $\cap, \cup$ and the operation:

$$\alpha \supset \beta \stackrel{\text{def}}{=} \{x \in W \mid \forall z\ ((xRz \text{ and } z \in \alpha) \text{ imply } z \in \beta)\}.$$

Here, $\alpha \subseteq W$ is an R-cone if $\forall x, y\ (x \in \alpha \text{ and } xRy \text{ imply } y \in \alpha)$. It is well-known (see e.g. [4]) that every **Int**-frame $\mathfrak{F} = \langle W, R, P\rangle$ gives rise to its dual Heyting algebra $\mathfrak{F}^+ = \langle P, \cap, \cup, \supset, \emptyset\rangle$; and conversely, every Heyting algebra $\mathfrak{A} = \langle A, \wedge, \vee, \rightarrow, \bot\rangle$ to its dual **Int**-frame $\mathfrak{A}_+ = \langle W_A, \subseteq, P_A\rangle$ where W_A is the set of prime filters in $\mathfrak{A}$ and $P_A = \{|a| \mid a \in A\}$ with $|a| = \{X \in W_A \mid a \in X\}$; and that $\mathfrak{A} \simeq (\mathfrak{A}_+)^+$. An **Int**-frame $\mathfrak{F}$ is *descriptive* if $\mathfrak{F} \simeq (\mathfrak{F}^+)_+$. A *full* **Int**-*frame* (or a *Kripke* **Int**-*frame*) is an **Int**-frame of the form $\langle W, R, \mathrm{Up}\, W\rangle$. Here, $\mathrm{Up}\, W$ is the set of all R-cones in W. For an **Int**-frame $\mathfrak{F} = \langle W, R, P\rangle$, its *underlying full* **Int**-*frame* $\langle W, R, \mathrm{Up}\, W\rangle$ is denoted by $\kappa\mathfrak{F}$. An **Int**-frame $\mathfrak{F}$ is a

frame for a si-logic L when $\mathfrak{F} \models L$. A si-logic L is $\mathcal{D}$*-persistent* if $\kappa\mathfrak{F} \models L$ holds for any descriptive frame $\mathfrak{F}$ for L.

Let $\mathcal{L}_{\Box\Diamond}$ be the propositional bimodal language, i.e. the language $\mathcal{L}$ with new unary connectives $\Box$ and $\Diamond$. The sets of ($\mathcal{L}_{\Box\Diamond}$-)variables and ($\mathcal{L}_{\Box\Diamond}$-)formulas are denoted by $\mathbf{Var}\,\mathcal{L}_{\Box\Diamond}$ and $\mathbf{For}\,\mathcal{L}_{\Box\Diamond}$, respectively. For a formula ϕ, $\mathbf{Sub}(\phi)$ is the set of subformulas of ϕ. A set $L \subseteq \mathbf{For}\,\mathcal{L}_{\Box\Diamond}$ is called a $\Box\Diamond$*-normal intuitionistic modal logic* (a $\Box\Diamond$*-IM-logic*, for short) if it includes **Int**, contains axioms $\Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q$, $\Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$, $\Box\top$ and $\neg\Diamond\bot$, and is closed under substitution, modus ponens and the monotonicity rules: $\phi \to \psi / \Box\phi \to \Box\psi$ and $\phi \to \psi / \Diamond\phi \to \Diamond\psi$. We denote by $\mathbf{IntK}_{\Box\Diamond}$ the smallest $\Box\Diamond$-IM-logic. For a set $\Gamma \subseteq \mathbf{For}\,\mathcal{L}_{\Box\Diamond}$ and a $\Box\Diamond$-IM-logic L, the smallest $\Box\Diamond$-IM-logic that includes L and Γ is denoted by $L \oplus \Gamma$; when $\Gamma = \{\phi\}$, $L \oplus \Gamma$ will be denoted by $L \oplus \phi$. We set $\mathbf{MIPC} \stackrel{\text{def}}{=} \mathbf{IntK}_{\Box\Diamond} \oplus \{\Box p \to p, p \to \Diamond p, \Diamond\Box p \to \Box p, \Diamond p \to \Box\Diamond p, \Diamond(p \to q) \to (\Box p \to \Diamond q)\}$. A $\Box\Diamond$-IM-logic that includes **MIPC** is called an **MIPC**-logic.

An **MIPC**-*algebra* is a structure $\mathfrak{A} = \langle A, \wedge, \vee, \to, \bot, \Box, \Diamond\rangle$ where $\langle A, \wedge, \vee, \to, \bot\rangle$ is a Heyting algebra and $\Box, \Diamond$ are unary operations on A such that $\Box(a \wedge b) = \Box a \wedge \Box b$, $\Box\top = \top$, $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$, $\neg\Diamond\bot = \top$, $\Box a \le a$, $a \le \Diamond a$, $\Diamond\Box a \le \Box a$, $\Diamond a \le \Box\Diamond a$ and $\Diamond a \wedge \Diamond b \le \Diamond(\Diamond a \wedge b)$ for any $a, b \in A$. It is straightforward to see that for any **MIPC**-logic L, there exists a class C of **MIPC**-algebras such that $\phi \in L$ if and only if $\forall\mathfrak{A} \in C\ (\mathfrak{A} \models \phi)$; such C is called *a class of* **MIPC**-*algebras that characterizes* L.

An (general) **MIPC**-*frame* is a structure $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$ where (1) $\langle W, R, P\rangle$ is an **Int**-frame; (2) $R_\Box, R_\Diamond$ are binary relations on W such that $R_\Box$ is reflexive and transitive, $R_\Diamond = (R_\Box)^{-1}$ and $R_\Box = R \circ (R_\Box \cap R_\Diamond)$; and (3) P is closed under the operations:

$$\begin{aligned} \Box\alpha &= \{x \in W \mid \forall y\ (xR_\Box y \text{ implies } y \in \alpha)\}, \\ \Diamond\alpha &= \{x \in W \mid \exists y \in \alpha\ (xR_\Diamond y)\} \end{aligned}$$

(see [10], [11]). Here and hereafter, $\circ$ denotes the composition of relations.

The following lemma is well-known (see e.g. [4]).

Lemma 2.1 *Let $\mathfrak{A} = \langle A, \wedge, \vee, \to, \bot\rangle$ be a Heyting algebra. Suppose $a \in A$ and α is a filter in $\mathfrak{A}$ such that $a \notin \alpha$. Then there exists a prime filter ∇ in $\mathfrak{A}$ which includes α and does not contain a such that ∇ is maximal in the set $\{\beta \mid \beta$ is a filter in $\mathfrak{A}$ such that $\alpha \subseteq \beta$ and $a \notin \beta\}$.*

Using this lemma, it is not hard to show that, for any **MIPC**-frame $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$, $\mathfrak{F}^+ \stackrel{\text{def}}{=} \langle P, \cap, \cup, \supset, \emptyset, \Box, \Diamond\rangle$ is an **MIPC**-algebra; that, for any **MIPC**-algebra $\mathfrak{A} = \langle A, \wedge, \vee, \to, \bot, \Box, \Diamond\rangle$, $\mathfrak{A}_+ \stackrel{\text{def}}{=} \langle W_A, \subseteq, \hat{R}_\Box, \hat{R}_\Diamond, P_A\rangle$ is an **MIPC**-frame, where (1) W_A is the set of prime filters in $\mathfrak{A}$, (2) $\alpha\hat{R}_\Box\beta$ if and only if $\Box a \in \alpha$ implies $a \in \beta$ for any $a \in A$, (3) $\alpha\hat{R}_\Diamond\beta$ if and only if $a \in \beta$ implies

$\Diamond a \in \alpha$ for any $a \in A$, and (4) $P_A = \{|a| \mid a \in A\}$ with $|a| = \{X \in W_A \mid a \in X\}$; and that $\mathfrak{A} \simeq (\mathfrak{A}_+)^+$. An **MIPC**-frame $\mathfrak{F}$ is *descriptive* if $\mathfrak{F} \simeq (\mathfrak{F}^+)_+$. Thus, for any **MIPC**-logic L, there exists a class C of descriptive **MIPC**-frames such that $\phi \in L$ iff $\forall \mathfrak{F} \in C\ (\mathfrak{F} \models \phi)$; such C is called *a class of descriptive* **MIPC***-frames that characterizes* L.

A *full* **MIPC***-frame* (or a *Kripke* **MIPC***-frame*) is an **MIPC**-frame of the form $\langle W, R, R_\Box, R_\Diamond, \mathrm{Up}\, W\rangle$. For an **MIPC**-frame $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$, we denote by $\kappa\mathfrak{F}$ its *underlying full* **MIPC***-frame* $\langle W, R, R_\Box, R_\Diamond, \mathrm{Up}\, W\rangle$. An **MIPC**-frame $\mathfrak{F}$ is a *frame for an* **MIPC***-logic* L when $\mathfrak{F} \models L$. An **MIPC**-logic L is $\mathcal{D}$*-persistent* if $\kappa\mathfrak{F} \models L$ holds for any descriptive **MIPC**-frame $\mathfrak{F}$ for L. It is straightforward to show that **MIPC** is $\mathcal{D}$-persistent (see [14]).

3 Constructing a finite counter MIPC-model

First, we review some basic notions on **MIPC**-models, which will be used in this section.

A *valuation* on an **MIPC**-frame $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$ is a mapping from $\mathbf{Var}\,\mathcal{L}_{\Box\Diamond}$ to P. An **MIPC***-model* is a pair $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ where $\mathfrak{F}$ is an **MIPC**-frame and V is a valuation on it. $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ is said to be an **MIPC**-model *based on* $\mathfrak{F}$.

Let $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ be an **MIPC**-model based on $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$. For $x \in W$ and $\phi \in \mathbf{For}\,\mathcal{L}_{\Box\Diamond}$, we define a relation $\langle \mathfrak{M}, x\rangle \models \phi$ by induction on the construction of ϕ:

$\langle \mathfrak{M}, x\rangle \models p$ iff $x \in V(p)$;
$\langle \mathfrak{M}, x\rangle \models \psi \wedge \xi$ iff $\langle \mathfrak{M}, x\rangle \models \psi$ and $\langle \mathfrak{M}, x\rangle \models \xi$;
$\langle \mathfrak{M}, x\rangle \models \psi \vee \xi$ iff $\langle \mathfrak{M}, x\rangle \models \psi$ or $\langle \mathfrak{M}, x\rangle \models \xi$;
$\langle \mathfrak{M}, x\rangle \models \psi \to \xi$ iff $\forall y \in W\ (xRy$ and $\langle \mathfrak{M}, y\rangle \models \psi$ imply $\langle \mathfrak{M}, x\rangle \models \xi)$;
$\langle \mathfrak{M}, x\rangle \models \Box\psi$ iff $\forall y \in W\ (xR_\Box y$ implies $\langle \mathfrak{M}, y\rangle \models \psi)$;
$\langle \mathfrak{M}, x\rangle \models \Diamond\psi$ iff $\exists y \in W\ (xR_\Diamond y$ and $\langle \mathfrak{M}, y\rangle \models \psi)$.

When no confusion occurs, $\langle \mathfrak{M}, x\rangle \models \phi$ will be abbreviated as $x \models \phi$.

A formula ϕ is *valid* in an **MIPC**-model $\mathfrak{M}$ (notation: $\mathfrak{M} \models \phi$) if $\forall x \in W\ (\langle \mathfrak{M}, x\rangle \models \phi)$; otherwise ϕ is *refuted* in $\mathfrak{M}$ (or $\mathfrak{M}$ is a *counter model for* ϕ). For a set L of formulas, $\mathfrak{M} \models L$ stands for $\forall \psi \in L\ (\mathfrak{M} \models \psi)$. For an **MIPC**-model $\mathfrak{M}$ and an **MIPC**-logic L, $\mathfrak{M}$ is an **MIPC***-model of* L when $\mathfrak{M} \models L$. An **MIPC**-frame $\mathfrak{F}$ is an **MIPC***-frame for* **MIPC***-logic* L (notation: $\mathfrak{F} \models L$) if $\mathfrak{M} \models L$ for any **MIPC**-model based on $\mathfrak{F}$.

If $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$ then $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ is said to be *finite* if the cardinality of W (notation: $|W|$) is finite. An **MIPC**-logic L has *the finite model property* if, for any $\phi \notin L$, there exists a finite **MIPC**-model of L that refutes ϕ.

We begin our study with showing basic properties of an **MIPC**-model. We write $x\vec{R}y$ if xRy and not yRx.

Definition 3.1 *Let $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ be an* **MIPC***-model based on an* **MIPC***-frame $\langle W, R, R_\Box, R_\Diamond, P \rangle$ and $\phi \in \mathbf{For}\mathcal{L}_{\Box\Diamond}$. Then $y \in W$ is said to be R-maximal relative to ϕ if and only if $y \not\models \phi$ and $\forall z \in W$ ($y\vec{R}z$ implies $z \models \phi$). The set of all R-successors of x that are R-maximal relative to ϕ is written as R-max(x, ϕ).*

Lemma 3.2 *Let $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ be an* **MIPC***-model based on a descriptive* **MIPC***-frame $\langle W, R, R_\Box, R_\Diamond, P \rangle$. For any $\phi \in \mathbf{For}\mathcal{L}_{\Box\Diamond}$ and $x \in W$ such that $x \not\models \phi$, R-max(x, ϕ) is non-empty.*

Proof. Since $\mathfrak{F}$ is descriptive, without loss of generality, we can assume $\mathfrak{F} = \mathfrak{A}_+$ for some **MIPC**-algebra $\mathfrak{A}$. Then there exists $a \in \mathfrak{A}$ such that $V(\phi) = |a|$. Note that every $y \in W$ is a prime filter in $\mathfrak{A}$ and it holds that $y \models \phi$ iff $a \in y$.

Suppose that $x \not\models \phi$, i.e. $a \notin x$. Put $\alpha := x$. Then by Lemma 2.1, we obtain a prime filter ∇ satisfying $a \notin \nabla$ and $x \subseteq \nabla$ such that, for any filter y, $\nabla \subsetneq y$ implies $a \in y$. Since $R = \subseteq$ in $\mathfrak{A}_+$, $xR\nabla$ and ∇ is R-maximal relative to ϕ. ∎

It is easy to see that $R_\Box \cap R_\Diamond$ is an equivalence relation. In the sequel, we abbreviate $R_\Box \cap R_\Diamond$ as $\sim$. One easily checks that $\langle \mathfrak{M}, x \rangle \models \Diamond\psi$ if and only if $\exists y \in W$ ($x \sim y$ and $\langle \mathfrak{M}, y \rangle \models \psi$). For $x \in W$, the $\sim$-equivalence class that contains x is written as $[x]$. For $\alpha \subseteq W$, α is said to be a *precluster* if $\alpha \subseteq [x]$ for some $x \in W$; and when this is the case, we write $[\alpha]$ for $[x]$.

Lemma 3.3 *Let $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ be an* **MIPC***-model based on an* **MIPC***-frame $\langle W, R, R_\Box, R_\Diamond, P \rangle$. Let ϕ be a formula of the form $\Box\xi$ or $\Diamond\xi$ for some $\xi \in \mathbf{For}\mathcal{L}_{\Box\Diamond}$. Then for any $x, y \in W$ with $x \sim y$, we have $x \models \phi$ iff $y \models \phi$.*

Proof. Easy. ∎

Thus for any precluster α and $x, y \in \alpha$, we have that $x \models \Box\xi$ iff $y \models \Box\xi$, and that $x \models \Diamond\xi$ iff $y \models \Diamond\xi$ for any $\xi \in \mathbf{For}\mathcal{L}_{\Box\Diamond}$. We write $\alpha \models \Box\xi$ and $\alpha \models \Diamond\xi$ when $x \models \Box\xi$ and $x \models \Diamond\xi$ for some (and hence any) $x \in \alpha$, respectively.

Lemma 3.4 *Let $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ be an* **MIPC***-model based on a descriptive* **MIPC***-frame $\langle W, R, R_\Box, R_\Diamond, P \rangle$. Suppose that $x \in W$ is R-maximal relative to $\Box\phi$. Then for any $y \in [x]$, we have R-max$(y, \Box\phi) \subseteq [x]$.*

Proof. Suppose that $y \sim x \not\sim z$ and $z \in R$-max$(y, \Box\phi)$. Since $x \sim yRz$, there exists $w \in W$ such that $xRw \sim z$. Since $w \sim z \not\sim x$, $w \notin [x]$. Hence $x\vec{R}w$. Then by R-maximality of x, we have $w \models \Box\phi$. On the other hand, $z \not\models \Box\phi$ by R-maximality of z. But this contradicts Lemma 3.3. ∎

Thus for any $\sim$-equivalence class α that contains an element that is R-maximal relative to $\Box\phi$, we have R-max$(x, \Box\phi) \subseteq \alpha$ for any $x \in \alpha$; when this is the case, α is also said to be R-maximal relative to $\Box\phi$.

Lemma 3.5 *Let $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ be an* **MIPC***-model based on a descriptive* **MIPC***-frame $\langle W, R, R_\Box, R_\Diamond, P \rangle$. For any $\sim$-equivalence class α that is R-maximal relative to $\Box\xi$, there exists $y \in \alpha$ such that $y \not\models \xi$.*

Proof. By assumption, there exists $x \in \alpha$ that is R-maximal relative to $\Box\xi$. Since $x \not\models \Box\xi$, there exists $y, z \in W$ such that $xRz \sim y$ and $y \not\models \xi$. Suppose $y \notin \alpha$. Then $x \not\sim z$, and so $x\vec{R}z$. By R-maximality of x, we have $z \models \Box\xi$. Together with $z \sim y$, this implies $y \models \xi$, which is not the case. ∎

Define a relation $\leq$ on the set of $\sim$-equivalence classes by $\alpha \leq \beta$ iff $\exists x \in \alpha\ \exists y \in \beta\ (xRy)$. Then we obtain the following.

Lemma 3.6 *The relation $\leq$ on the set of $\sim$-equivalence classes defined as above is a partial order, i.e. it is a reflexive, transitive and antisymmetric relation.*

Proof. Easy. ∎

Note that $\alpha \leq \beta$ if and only if $\forall x \in \alpha\ \forall y \in \beta\ (x(R \circ \sim)y)$. Moreover, if $\alpha < \beta$ then $\forall x \in \alpha\ \exists y \in \beta\ (x\vec{R}y)$.

We are now going to construct a finite counter **MIPC**-model $\mathfrak{M}_\phi$ for ϕ from a given **MIPC**-model $\mathfrak{M}$ that refutes $\phi \in \mathbf{For}\,\mathcal{L}_{\Box\Diamond}$. To this end, we fix $\phi \in \mathbf{For}\,\mathcal{L}_{\Box\Diamond}$ and **MIPC**-model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ based on a descriptive **MIPC**-frame $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P \rangle$ such that $\mathfrak{M} \not\models \phi$.

We will find a suitable subset W_ϕ of W, based on which the counter model $\mathfrak{M}_\phi$ will be constructed. For this, we define some procedures which act on the subsets X, Y and Z of W satisfying the constraints:

$$\text{all } X, Y \text{ and } Z \text{ are finite;} \tag{1}$$

$$X \text{ is a precluster;} \tag{2}$$

$$[z] \not\leq [y] \text{ for any } y \in Y \text{ and } z \in Z; \tag{3}$$

$$[X] \not\leq [y] \text{ for any } y \in Y; \tag{4}$$

$$[z] \not\leq [X] \text{ for any } z \in Z. \tag{5}$$

Roughly speaking, our construction is carried out by "completing" preclusters one by one in an ascending order. During the construction, X is the precluster that we are focusing now, Y is the set of already selected points consisting of completed preclusters, and Z is a set of points that we should think of later (see Figure 1). Note that it follows from the above constraints that $[X], \{[y] \mid y \in Y\}$ and $\{[z] \mid z \in Z\}$ are pairwise disjoint.

Definition 3.7 *$COPY(X, Y)$ is a procedure that updates X as follows: Let*

$$Y' = \{y \in Y \mid \begin{array}{l} y \text{ is maximal w.r.t. } R \text{ in } Y, \\ \exists x \in X\ (y(\sim \circ R)x), \text{ and not } \exists x \in X\ (yRx)\}. \end{array}$$

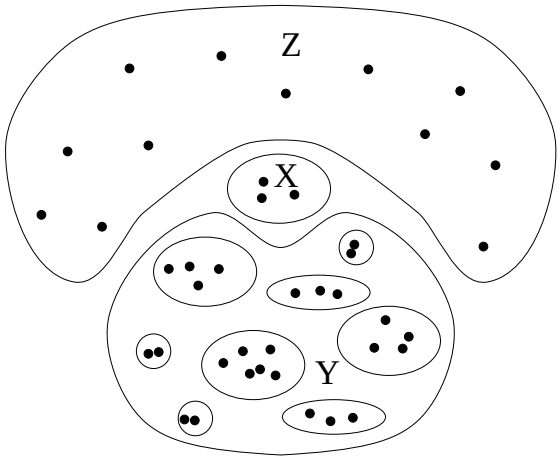

Figure 1: X, Y, Z during the construction

Then, by the definition of an **MIPC***-frame, for each $y \in Y'$, we can take $x_y \in [X]$ such that yRx_y. We put $X := X \cup \{x_y \mid y \in Y'\}$.*

Definition 3.8 *$SAT^{\Box}_{\phi}(X, Z)$ is a procedure that updates X and Z as follows: Let*

$$\Gamma = \{\Box\xi \in \mathbf{Sub}\,\phi \mid [X] \text{ is } R\text{-maximal relative to } \Box\xi\}.$$

Then by Lemma 3.5, for each $\Box\xi \in \Gamma$, we can take $x_\xi \in [X]$ such that $x_\xi \not\models \xi$. We put $X := X \cup \{x_\xi \mid \Box\xi \in \Gamma\}$. Let

$$\Sigma = \{\Box\xi \in \mathbf{Sub}\,\phi \mid \begin{array}{l}[X] \not\models \Box\xi \text{ and} \\ [X] \text{ is not } R\text{-maximal relative to } \Box\xi\}.\end{array}$$

Fix an arbitrary $x \in X$. By Lemma 3.2, for each $\Box\xi \in \Sigma$, we can take $x_\xi \in R\text{-}\max(x, \Box\xi)$. Note that $x_\xi \notin [X]$ by Lemma 3.5. We put $Z := Z \cup \{x_\xi \mid \Box\xi \in \Sigma\}$.

Definition 3.9 *$SAT^{\Diamond}_{\phi}(X, Y)$ is a procedure that updates X as follows: Let*

$$\Gamma = \{\Diamond\xi \in \mathbf{Sub}\,\phi \mid [X] \models \Diamond\xi \text{ and } \forall x \in X\ (x \not\models \xi)\}.$$

Then by definition, for each $\Diamond\xi \in \Gamma$, we can take $x_\xi \in [X]$ such that $x_\xi \models \xi$. We put $X := X \cup \{x_\xi \mid \Diamond\xi \in \Gamma\}$.

Definition 3.10 *$SAT^{\rightarrow}_{\phi}(X, Z)$ is a procedure that updates X and Z as follows: First, we define sequences $X_0, X_1, \ldots$ and $Z_0, Z_1, \ldots$ by induction. Let $X_1 = X$*

and $X_0 = Z_0 = Z_1 = \emptyset$. *Suppose* X_k *and* Z_k $(k \leq n+1)$ *are already defined. We define* X_{n+2} *and* Z_{n+2}. *For each* $x \in X_{n+1} \backslash X_n$, *let*

$$\Gamma_x = \{\chi \to \xi \in \mathbf{Sub}\,\phi \mid \begin{array}{l} x \not\models \chi \to \xi \text{ and} \\ x \text{ is not } R\text{-maximal relative to } \chi \to \xi\}. \end{array}$$

Then by Lemma 3.2, for each $\chi \to \xi \in \Gamma_x$, *we can take* $x_{\chi\to\xi} \in R\text{-max}(x, \chi \to \xi)$. *We let*

$$X_{n+2} = (\{x_{\chi\to\xi} \mid x \in X_{n+1} \backslash X_n \text{ and } \chi \to \xi \in \Gamma_x\} \cap [X]) \cup X_{n+1}$$
$$Z_{n+2} = \{x_{\chi\to\xi} \mid x \in X_{n+1} \backslash X_n \text{ and } \chi \to \xi \in \Gamma_x\} \backslash [X].$$

Finally, put $X := \bigcup_{n\in\omega} X_n$ *and* $Z := Z \cup \bigcup_{n\in\omega} Z_n$.

Lemma 3.11 *Updated* X, Y *and* Z *by the above procedures again satisfy the constraints* (1) $\sim$ (5).

Proof. Let us denote the updated X, Y and Z by $\hat{X}$, $\hat{Y}$ and $\hat{Z}$, respectively.

1. $COPY(X,Y)$. Since $\{x_y \mid y \in Y'\} \subseteq [X]$, $\hat{X}$ is again a precluster and $[X] = [\hat{X}]$. Accordingly (2) $\sim$ (5) hold. By the assumption that Y is finite, Y' is finite. Together with finiteness of X, it follows that $\hat{X}$ is finite. Thus (1) is also satisfied.

2. $SAT^{\Box}_{\phi}(X,Z)$. First, we show that the first part of the procedure preserves the constraints. Since $\{x_\xi \mid \Box\xi \in \Gamma\} \subseteq [X]$, $\hat{X}$ is again a precluster and $[X] = [\hat{X}]$. Accordingly (2) $\sim$ (5) hold. Since $\mathbf{Sub}\,\phi$ is finite, Γ is finite. Together with finiteness of X, it follows that $\hat{X}$ is finite. Thus (1) is also satisfied.

   Next, we show the latter part of the procedure preserves the constraints. Since $\mathbf{Sub}\,\phi$ is finite, Σ is finite. Together with finiteness of Z, it follows that $\hat{Z}$ is finite. Thus (1) is satisfied. Clearly, it remains to show (3) and (5). Since xRx_ξ and $x_\xi \notin [\hat{X}]$, we have $[\hat{X}] < [x_\xi]$, and hence (3) and (5) hold.

3. $SAT^{\Diamond}_{\phi}(X,Y)$. Similar to the first part of $SAT^{\Box}_{\phi}(X,Z)$.

4. $SAT^{\rightarrow}_{\phi}(X,Z)$. First, we show that there exists some $l > 0$ such that $X_{l+1} \backslash X_l = \emptyset$. Suppose contrarily that $X_{n+1} \backslash X_n = \emptyset$ for all $n > 0$. Then for any $k \in \omega$, there exists R-chain $x_1 R x_2 R \cdots x_k$ of length k such that, for any i $(k > i > 0)$, there exists $\chi \to \xi \in \mathbf{Sub}\,\phi$ satisfying

   (a) $x_i \not\models \chi \to \xi$,

   (b) x_i is not R-maximal relative to $\chi \to \xi$, and

   (c) $x_{i+1} \in R\text{-max}(x_i, \chi \to \xi)$.

Let

$$\dot{x} = \{\chi \to \xi \in \mathbf{Sub}\,\phi \mid \quad x \models \chi \to \xi \text{ or } x \text{ is } R\text{-maximal relative to } \chi \to \xi\}.$$

Then it follows that $\dot{x}_i \supsetneq \dot{x}_{i+1}$ for any i $(k > i > 0)$. But since each $|\dot{x}_i|$ is bounded by $|\mathbf{Sub}\,\phi|$, there is no R-chain longer than $|\mathbf{Sub}\,\phi|$, which is not the case. Thus there exists $l > 0$ such that $X_{l+1} \backslash X_l = \emptyset$. Then by construction, it follows that $X_l = X_{l+1} = \cdots$ and $\emptyset = Z_{l+1} = Z_{l+2} = \cdots$. Clearly, each X_n and Z_n is finite, hence so are $\hat{X}$ and $\hat{Z}$. Thus (1) is satisfied.

(2) is also satisfied, since $X_i \subseteq [X]$ for each $i \in \omega$.

Clearly, it remains to show (3) and (5). We show for any $z \in \hat{Z} \backslash Z$, $[\hat{X}] < [z]$. Without loss of generality, we suppose $z \in Z_{n+2}$. Then there exists $x \in X_{n+1} \backslash X_n$ such that xRz. Together with $z \notin [X]$, we have $[\hat{X}] = [x] < [z]$. The constraints (3) and (5) immediately follow from this.

∎

We are now going to present the construction of model $\mathfrak{M}_\phi$.

Definition 3.12 *Let $\phi \in \mathbf{For}\,\mathcal{L}_{\Box\Diamond}$, and suppose $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ is a counter* **MIPC***-model for ϕ based on a descriptive* **MIPC***-frame $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$. We construct a model $\mathfrak{M}_\phi$ as follows:*

First, take any point $x_0 \in W$ such that $x_0 \not\models \phi$ (such x_0 always exists since $\mathfrak{M}$ is a counter **MIPC***-model for ϕ).*

Next, we define sequences $U_0, U_1, \ldots$ and $V_0, V_1, \ldots$ by induction. Put $U_0 = \emptyset$ and $V_0 = \{x_0\}$. Suppose U_k and V_k are already defined.

Step 1. Select an element X in $V_k/\!\sim$ such that $[X]$ is minimal w.r.t. $\geq$ in $\{[v] \mid v \in V_k\}$, and put $Y := U_k$ and $Z := V_k \backslash X$.

Step 2. Update X by $COPY(X, Y)$.

Step 3. Update X and Z $SAT^{\Box}_{\phi}(X, Z)$.

Step 4. Update X by $SAT^{\Diamond}_{\phi}(X, Y)$.

Step 5. Update X and Z by $SAT^{\rightarrow}_{\phi}(X, Z)$.

And then define $U_{k+1} = U_k \cup X$ and $V_{k+1} = Z$.

Finally, define $W_\phi = \bigcup_{n\in\omega} U_n$ and $\mathfrak{M}_\phi = \langle \mathfrak{F}_\phi, V_\phi\rangle$, where

$$\mathfrak{F}_\phi = \langle W_\phi, R \restriction W_\phi, R_\Box \restriction W_\phi, R_\Diamond \restriction W_\phi, \mathrm{Up}\, W_\phi\rangle,$$
$$V_\phi(p) = V(p) \cap W_\phi \text{ for any } p \in \mathbf{Var}\,\mathcal{L}_{\Box\Diamond}.$$

Lemma 3.13 *Above construction is well-defined. Moreover, for any* $X \in W_\phi/\sim$*, there exists some* l *such that* $X = U_{l+1}\backslash U_l$ *(hence* X *is finite).*

Proof. We show by induction on n that

i. U_n and V_n are finite,

ii. $[u] \not\leq [v]$ for any $u \in U_n$ and $v \in V_n$.

Base step is obvious. For the induction step, suppose i and ii are satisfied for $n = k$. Then the precluster X selected in step 1 is finite, since V_k is finite. Also, it is clear that X, Y and Z set in step 1 satisfy the constraints (1) $\sim$ (5) by induction hypothesis. In step 2 – 5, by Lemma 3.11, updated X, Y and Z also satisfy the constraints (1) $\sim$ (5). Finally, by the constraints (1) $\sim$ (5), it is easy to see i and ii are again satisfied by U_{k+1} and V_{k+1}.

Now, by the constraints (3) $\sim$ (5) and ii, every element of $W_\phi/\sim$ is $U_{l+1}\backslash U_l$ for some l; it is finite by (1). ∎

Lemma 3.14 $\mathfrak{M}_\phi$ *is a finite* **MIPC***-model.*

Proof. We will show that W_ϕ is finite. Let $\mathbf{Msub}\phi = \{\psi \in \mathbf{Sub}\,\phi \mid \psi \equiv \Box\xi \text{ or } \psi \equiv \Diamond\xi \text{ for some } \xi \in \mathbf{For}\,\mathcal{L}_{\Box\Diamond}\}$. For each $\Gamma \subseteq \mathbf{Msub}\phi$, let $W_\phi(\Gamma) = \{x \in W_\phi \mid \text{ for any } \psi \in \mathbf{Msub}\phi,\ x \models \psi \text{ iff } \psi \in \Gamma\}$. Note, by Lemma 3.3, for any precluster α in W_ϕ there exists a unique $\Gamma \subseteq \mathbf{Msub}\phi$ such that $\alpha \subseteq W_\phi(\Gamma)$. Thus, it suffices to show $W_\phi(\Gamma)$ is finite for each $\Gamma \subseteq \mathbf{Msub}\phi$, since $\mathcal{P}(\mathbf{Msub}\phi)$ is finite. We show this by induction on $|\Gamma|$.

Suppose for any $\Sigma \subseteq \mathbf{Msub}\phi$, $|\Sigma| < |\Gamma|$ implies that $W_\phi(\Sigma)$ is finite. This means, in particular, for any $\Sigma \subsetneq \Gamma$, $W_\phi(\Sigma)$ is finite. Since every precluster in $W_\phi(\Gamma)$ is finite by Lemma 3.13, it suffices to show that $W_\phi(\Gamma)/\sim$ is finite. Moreover, since $W_\phi(\Gamma)/\sim$ is a poset and there is no infinite descending chain in it by the construction, it suffices to show that there are neither infinite ascending chain nor infinite antichain in $W_\phi(\Gamma)/\sim$.

First, we suppose that there is an infinite antichain in $W_\phi(\Gamma)/\sim$ and show that there is also an infinite ascending chain in $W_\phi(\Gamma)/\sim$, so that it will be enough to show that no infinite ascending chain exists in $W_\phi(\Gamma)/\sim$. For this, we introduce a relation $\ll \subseteq <$ on $W_\phi(\Gamma)/\sim$: Let $\alpha \in W_\phi(\Gamma)/\sim$. Then, by Lemma 3.13, $\alpha = U_{l+1}\backslash U_l$ for some l. We let $\beta \ll \alpha$ where $k = \min\{i \in \omega \mid \alpha \cap V_i \neq \emptyset\}$, and $\beta = U_k\backslash U_{k-1} \in W_\phi/\sim$. Clearly, $\beta < \alpha$, and $\beta_1 = \beta_2$ whenever $\beta_1 \ll \alpha$ and $\beta_2 \ll \alpha$. Moreover, it is easy to observe that $\beta \in (\bigcup_{\Sigma\subsetneq\Gamma} W_\phi(\Sigma))/\sim$. Thus, we get an infinite $\ll$-tree rooted at $U_1\backslash U_0$ in $(\bigcup_{\Sigma\subsetneq\Gamma} W_\phi(\Sigma))/\sim$. This tree is finitely branching by the definition of the procedures, and hence there must be an infinite ascending $\ll$-chain (and hence infinite ascending chain) in it by König's Lemma. But by induction hypothesis, there is no such chain in $(\bigcup_{\Sigma\subsetneq\Gamma} W_\phi(\Sigma))/\sim$, and so this infinite ascending chain is contained in $W_\phi(\Gamma)/\sim$.

Suppose that $\alpha_0 < \alpha_1 < \cdots$ is an infinite ascending chain in $W_\phi(\Gamma)/\!\sim$. Let

$$\begin{aligned}
\mathrm{T} &= \bigcup\{\beta \in W_\phi(\Gamma)/\!\sim \mid \beta \text{ is maximal in } W_\phi(\Gamma)/\!\sim\},\\
\mathrm{M} &= \bigcup\{\beta \in W_\phi(\Gamma)/\!\sim \mid \beta \text{ is neither maximal nor minimal in } W_\phi(\Gamma)/\!\sim\},\\
\mathrm{B} &= \bigcup\{\beta \in W_\phi(\Gamma)/\!\sim \mid \beta \text{ is minimal in } W_\phi(\Gamma)/\!\sim\} \text{ and}\\
\mathrm{H} &= \bigcup_{\Sigma \subsetneq \Gamma} W_\phi(\Sigma).
\end{aligned}$$

First note that α_i is contained in $\mathrm{M} \cup \mathrm{B}$ for all i, because otherwise $\alpha_{i+1} \notin W_\phi(\Gamma)/\!\sim$ and this contradicts our assumption.

Let $R' \subseteq \vec{R}$ be a relation on $\mathrm{M} \cup \mathrm{B} \cup \mathrm{H}$ defined as follows: $vR'w$ if either (i) $v = y$ and $z = x_y$ in a stage by $COPY(X,Y)$, (ii) $v = x$ and $z = x_\xi$ in a stage by $SAT^{\rightarrow}_{\phi}(X,Z)$ or (iii) $v = x$ and $z = x_{\chi\to\xi}$ in a stage by the latter half of $SAT^{\Box}_{\phi}(X,Z)$. By definition, R' is finitely branching.

We are now going to claim that $\mathrm{M} \cup \mathrm{B} \cup \mathrm{H}$ has finitely many R'-minimal elements. Let $x \in \mathrm{M}$ and $k = \min\{i \in \omega \mid x \in V_i \cup U_i\}$. Then x is added to W_ϕ at the k-stage either by $COPY(X,Y)$, $SAT^{\rightarrow}_{\phi}(X,Z)$, or the latter half of $SAT^{\Box}_{\phi}(X,Z)$. Indeed, if x is added to W_ϕ by $SAT^{\Diamond}_{\phi}(X,Z)$, then $[x] \cap W_\phi$ is contained in B; and if x is added to W_ϕ by the former half of $SAT^{\Box}_{\phi}(X,Z)$, then $[x] \cap W_\phi$ is contained in T. Consequently, for any element $x \in \mathrm{M}$, there exists $y \in \mathrm{M} \cup \mathrm{B} \cup \mathrm{H}$ such that $yR'x$. By induction hypothesis H is finite, and it can be shown $\mathrm{B}/\!\sim$ is finite exactly as above; hence $\mathrm{B} \cup \mathrm{H}$ is finite by Lemma 3.13. Thus, $\mathrm{M} \cup \mathrm{B} \cup \mathrm{H}$ has finitely many R'-minimal elements.

Next note that there exists infinitely many pairs $x, y \in \mathrm{M} \cup \mathrm{B} \cup \mathrm{H}$ such that $xR'y$ and $x \in R\text{-max}(y, \psi)$ for some $\psi \in \mathbf{Sub}\,\phi$. For, $(\mathrm{M} \cup \mathrm{B} \cup \mathrm{H})/\!\sim$ is infinite, and, for each $\beta \in (\mathrm{M} \cup \mathrm{B} \cup \mathrm{H})/\!\sim$, there exists an element in β which is add to W_ϕ as an element of Z in $SAT^{\rightarrow}_{\phi}(X,Z)$ or the latter half of $SAT^{\Box}_{\phi}(X,Z)$.

Now, let us regard $\mathrm{M} \cup \mathrm{B} \cup \mathrm{H}$ as an R'-DAG. Then, by König's Lemma, there exists an R'-chain $x_0R'x_1R'\cdots$ (and hence R-chain $x_0Rx_1R\cdots$) such that $x_{i+1} \in R\text{-max}(x_i, \psi)$ for some $\psi \in \mathbf{Sub}\,\phi$ for infinitely many $i \in \omega$. But this is impossible, and thus there exists no infinite ascending chain $\alpha_0 < \alpha_1 < \cdots$ in $W_\phi(\Gamma)/\!\sim$.

It remains to show $\mathfrak{M}_\phi$ is an **MIPC**-model, but this easily follows from the construction. ∎

Lemma 3.15 *For any $\psi \in \mathbf{Sub}\,\phi$ and $x \in W_\phi$, $\langle \mathfrak{M}_\phi, x\rangle \models \psi$ if and only if $\langle \mathfrak{M}, x\rangle \models \psi$.*

Proof. Our proof proceeds by induction on the construction of ψ. When $\psi \in \mathbf{Var}\,\mathcal{L}_{\Box\Diamond}$ or $\psi \equiv \bot$, it follows from the definition of V_ϕ. When $\psi \equiv \xi \wedge \chi$ or $\psi \equiv \xi \vee \chi$, it immediately follows from the induction hypothesis. Next we verify the remaining cases. In each case, ($\Leftarrow$) direction is straightforward, and hence we concentrate on the other.

1. $\psi \equiv \xi \to \chi$. Suppose x is R-maximal relative to $\xi \to \chi$. By definition, $\langle \mathfrak{M}, x\rangle \not\models \xi \to \chi$. But for any $z \in W$ such that $x\vec{R}z$, we have $\langle \mathfrak{M}, z\rangle \models \xi \to \chi$. Thus, $\langle \mathfrak{M}, x\rangle \models \xi$ and $\langle \mathfrak{M}, x\rangle \not\models \chi$. Then, by induction hypothesis, $\langle \mathfrak{M}_\phi, x\rangle \models \xi$ and $\langle \mathfrak{M}_\phi, x\rangle \not\models \chi$. Hence, $\langle \mathfrak{M}_\phi, x\rangle \not\models \xi \to \chi$. Otherwise suppose x is not R-maximal relative to $\xi \to \chi$. Then by Lemma 3.13, $x \in U_{k+1}\backslash U_k$ for some k and hence there exists $y \in W_\phi$ such that $y \in R\text{-max}(x, \xi \to \chi)$ by $SAT^{\to}_{\phi}(X, Z)$. Then, analogously, we have $\langle \mathfrak{M}_\phi, y\rangle \models \xi$ and $\langle \mathfrak{M}_\phi, y\rangle \not\models \chi$. Hence $\langle \mathfrak{M}_\phi, x\rangle \not\models \xi \to \chi$.

2. $\psi \equiv \Box\xi$. Suppose x is R-maximal relative to $\Box\xi$. Then since $[x]$ is R-maximal relative to $\Box\xi$, there exists $x' \in [x] \cap W_\phi$ such that $\langle \mathfrak{M}, x'\rangle \not\models \xi$ by $SAT^{\Box}_{\phi}(X, Z)$. Then by induction hypothesis, we have $\langle \mathfrak{M}_\phi, x'\rangle \not\models \xi$ and hence $\langle \mathfrak{M}_\phi, x\rangle \not\models \Box\xi$. Otherwise suppose $[x]$ is not R-maximal relative to $\Box\xi$. Then there exists $x' \in [x] \cap W_\phi$ and $y \in W_\phi$ such that $y \in R\text{-max}(x', \Box\xi)$ by $SAT^{\Box}_{\phi}(X, Z)$. Then, analogously, there exists $y' \in [y] \cap W_\phi$ such that $\langle \mathfrak{M}, y'\rangle \not\models \xi$. Then by induction hypothesis, we have $\langle \mathfrak{M}_\phi, y'\rangle \not\models \xi$. But since $x \sim x'Ry \sim y'$, we have $x(R \circ \sim)y'$. Thus $\langle \mathfrak{M}_\phi, x\rangle \not\models \Box\xi$.

3. $\psi \equiv \Diamond\xi$. Suppose $\langle \mathfrak{M}, x\rangle \models \Diamond\xi$. Then there exists $y \in W_\phi$ such that $x \sim y$ and $\langle \mathfrak{M}, y\rangle \models \xi$ by $SAT^{\Diamond}_{\phi}(X, Y)$. Then by induction hypothesis, we have $\langle \mathfrak{M}_\phi, y\rangle \models \xi$. Thus $\langle \mathfrak{M}_\phi, x\rangle \models \Diamond\xi$.

∎

Thus, we arrive at

Theorem 3.16 *Let L be an* **MIPC***-logic such that for any descriptive* **MIPC***-frame $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$,*

$$\mathfrak{F} \models L \text{ implies } \forall V \subseteq W\ (\kappa\mathfrak{F}\restriction V \models \mathbf{MIPC} \text{ implies } \kappa\mathfrak{F}\restriction V \models L),$$

where $\kappa\mathfrak{F}\restriction V = \langle V, R\restriction V, R_\Box\restriction V, R_\Diamond\restriction V, \mathrm{Up}\, V\rangle$. Then L has the finite model property.

Proof. Suppose $\phi \notin L$. Then there exists a descriptive **MIPC**-frame $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$ for L and a model $\mathfrak{M}$ based on it such that $\mathfrak{M} \not\models \phi$. Then by Lemma 3.14 and Lemma 3.15, there exists a finite subset V of W, such that $\kappa\mathfrak{F}\restriction V$ is an **MIPC**-frame and a model $\mathfrak{M}$ based on it such that $\mathfrak{M} \not\models \phi$. By assumption, $\kappa\mathfrak{F}\restriction V \models L$. Thus, $\mathfrak{M}$ is a finite counter **MIPC**-model of L for ϕ. ∎

4 MIPC-logics with the finite model property

In this section, we give some corollaries of Theorem 3.16 in accessible forms and present some examples of **MIPC**-logics with the finite model property. For this, we use the following result given in [14].

Proposition 4.1 *If a $\Box\Diamond$-IM-logic L is characterized by a class of full $\Box\Diamond$-frames which is closed under elementary equivalence (in the first-order language with the predicates $=$, R, $R_\Box$ and $R_\Diamond$), then L is $\mathcal{D}$-persistent.*

From this and Theorem 3.16, we obtain

Corollary 4.2 *Let L be an* **MIPC***-logic. If there exists a universal first-order sentence Φ with the predicates $=$, R, $R_\Box$ and $R_\Diamond$ such that*

1. *L is characterized by the class of full* **MIPC***-frames satisfying Φ and*
2. *every full* **MIPC***-frame that validates L satisfies Φ,*

then L has the finite model property.

Proof. We show that the condition of Theorem 3.16 is satisfied: Let $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$ be a descriptive **MIPC**-frame for L. By condition 1 and Proposition 4.1, L is $\mathcal{D}$-persistent. Consequently, $\kappa\mathfrak{F} \models L$, and so $\kappa\mathfrak{F}$ satisfies Φ by condition 2. Since Φ is a universal sentence, the class of full **MIPC**-frames satisfying Φ is closed under substructure (see e.g. [9]). Hence, for any $V \subseteq W$, $\kappa\mathfrak{F}\restriction V$ satisfies Φ, and thus $\kappa\mathfrak{F}\restriction V \models L$ by condition 1. ∎

Example 4.3 *The following* **MIPC***-logics have the finite model property; hence are decidable.*

1. **MIPC** $\oplus$ $\boldsymbol{sc}$, *where* $\boldsymbol{sc} = \Box(\Box p \to q) \vee \Box(\Box q \to p)$.
2. **MIPC** $\oplus$ $(\Box p \to q) \vee (\Box q \to p)$.
3. **MIPC** $\oplus$ $\Box(p \to q) \vee \Box(q \to p)$.

Proof. 1. It is straightforward to show that every full **MIPC**-frame satisfies

$$\forall x, y, z\ (xR_\Box y \wedge xR_\Box z \to zR_\Box y \vee yR_\Box z) \tag{6}$$

if and only if it validates $\boldsymbol{sc}$. Thus, it remains to show that **MIPC** $\oplus$ $\boldsymbol{sc}$ is characterized by the class of full **MIPC**-frames satisfying (6).

We are now going to show that every descriptive frame $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$ for **MIPC** $\oplus$ $\boldsymbol{sc}$ satisfies condition (6). Let $\mathfrak{M}$ be an **MIPC**-model based on $\mathfrak{F}$. Without loss of generality, we can assume $\mathfrak{F} = \mathfrak{A}_+$ for some **MIPC**-algebra $\mathfrak{A}$. Suppose that $\mathfrak{F}$ does not satisfy condition (6). Then $xR_\Box y, xR_\Box z, \neg(yR_\Box z)$ and $\neg(zR_\Box y)$. From $\neg(yR_\Box z)$, we know that there exist $a \in \mathfrak{A}$ such that $y \in \Box a$ and $z \notin a$. Similarly, from $\neg(zR_\Box y)$, $b \in \mathfrak{A}$ such that $z \in \Box b$ and $y \notin b$. Thus, $y \notin \Box a \to b$ and $z \notin \Box b \to a$. Then since $xR_\Box y$ and $xR_\Box z$, $x \notin \Box(\Box a \to b)$ and $x \notin \Box(\Box b \to a)$. Thus, $x \notin \Box(\Box a \to b) \vee \Box(\Box b \to a)$, i.e. $\mathfrak{F} \not\models \Box(\Box a \to b) \vee \Box(\Box b \to a)$.

Let C be the class of full **MIPC**-frames that satisfy condition (6). If $\mathbf{MIPC} \oplus \boldsymbol{sc} \not\models \phi$, then there exists a descriptive **MIPC**-frame $\mathfrak{F}$ with $\mathfrak{F} \not\models \phi$ and $\mathfrak{F} \models \boldsymbol{sc}$. Then, by the argument above, $\mathfrak{F}$ satisfies condition (6), and hence so does $\kappa\mathfrak{F}$. Thus, $\kappa\mathfrak{F} \in C$, $\kappa\mathfrak{F} \models \mathbf{MIPC} \oplus \boldsymbol{sc}$ and $\kappa\mathfrak{F} \not\models \phi$. On the other hand, if $\phi \in \mathbf{MIPC} \oplus \boldsymbol{sc}$, then $\mathfrak{F} \models \phi$ for all $\mathfrak{F} \in C$. Thus, $\mathbf{MIPC} \oplus \boldsymbol{sc}$ is characterized by the class C. Therefore, by Corollary 4.2, $\mathbf{MIPC} \oplus \boldsymbol{sc}$ has the finite model property.

2 and 3 can be shown similarly by replacing condition (6) with

$$\forall x, y, z \ (xRy \wedge xRz \to zR_{\Box}y \vee yR_{\Box}z) \tag{7}$$

$$\forall x, y, z \ (xR_{\Box}y \wedge xR_{\Box}z \to zRy \vee yRz) \tag{8}$$

respectively. ∎

In [1], it has been proved that if a si-logic $\mathbf{Int} + \Gamma$ is tabular then $\mathbf{MIPC} \oplus \{\phi[\Box p/p]_{p \in \mathrm{Var}(\phi)} \mid \phi \in \Gamma\}$ has the finite model property. Here, $\phi[\Box p/p]_{p \in \mathrm{Var}(\phi)}$ denotes the formula obtained from ϕ by replacing every propositional variable p appearing in ϕ by $\Box p$. Also, the complete list of pre-tabular **MIPC**-logics has been obtained in [1]. However, these results are not directly applicable to the **MIPC**-logics in Example 4.3.

Next, we turn our attention to non-modal extensions of **MIPC**.

Corollary 4.4 *For $\Gamma \subseteq \mathbf{For}\,\mathcal{L}$, If there exists a universal first-order sentence Φ with the predicates $=$, R such that*

1. $\mathbf{Int} + \Gamma$ *is characterized by the class of full* **Int***-frames satisfying Φ and*
2. *every full* **Int***-frame that validates Γ satisfies Φ,*

then $\mathbf{MIPC} \oplus \Gamma$ has the finite model property.

Proof. Suppose $\Gamma \subseteq \mathbf{For}\,\mathcal{L}$ and $\mathbf{Int} + \Gamma$ is characterized by the class of full **Int**-frames satisfying a universal first-order sentence Φ. Then it is well-known that $\mathbf{Int} + \Gamma$ is $\mathcal{D}$-persistent (see Theorem 10.22 in [4]).

We are now going to show $\mathbf{MIPC} \oplus \Gamma$ is characterized by the class of full **MIPC**-frames satisfying Φ. Let $\phi \notin \mathbf{MIPC} \oplus \Gamma$. Then $\mathfrak{F} \not\models \phi$ for some descriptive **MIPC**-frame $\mathfrak{F}$ such that $\mathfrak{F} \models \mathbf{MIPC} \oplus \Gamma$. It is clear that $\kappa\mathfrak{F} \not\models \phi$; thus, it remains to show that $\kappa\mathfrak{F}$ satisfies Φ. By the $\mathcal{D}$-persistency of $\mathbf{Int} + \Gamma$, we have $\kappa\mathfrak{F} \models \Gamma$, since $\kappa\mathfrak{F}$ is also a descriptive **Int**-frame for $\mathbf{Int} + \Gamma$. Hence, by condition 2, $\kappa\mathfrak{F}$ satisfies Φ. On the other hand, if a full **MIPC**-frames satisfies Φ, then $\mathfrak{F} \models \Gamma$ by condition 1, and so $\mathfrak{F} \models \mathbf{MIPC} \oplus \Gamma$.

We next show that for any full **MIPC**-frames $\mathfrak{F}$, $\mathfrak{F}$ satisfies Φ whenever $\mathfrak{F} \models \mathbf{MIPC} \oplus \Gamma$. Indeed, if $\mathfrak{F} \models \mathbf{MIPC} \oplus \Gamma$ then $\mathfrak{F} \models \Gamma$, and so $\mathfrak{F}$ satisfies Φ by assumption 2.

Therefore, by Corollary 4.2, $\mathbf{MIPC} \oplus \Gamma$ has the finite model property. ∎

Example 4.5 *The following* **MIPC***-logics have the finite model property; hence are decidable.*

1. $\mathbf{MIPC} \oplus \boldsymbol{bd}_n$ *($n \geq 1$), where* $\boldsymbol{bd}_n$ *is defined inductively as: (1)* $\boldsymbol{bd}_1 = p_1 \vee \neg p_1$*; (2)* $\boldsymbol{bd}_{n+1} = p_{n+1} \vee (p_{n+1} \to \boldsymbol{bd}_n)$.
2. $\mathbf{MIPC} \oplus \boldsymbol{bw}_n$ *($n \geq 1$), where* $\boldsymbol{bw}_n = \bigvee_{i=0}^{n}(p_i \to \bigvee_{i \neq j} p_j)$.
3. $\mathbf{MIPC} \oplus \boldsymbol{bc}_n$ *($n \geq 1$), where* $\boldsymbol{bc}_n = p_0 \vee (p_0 \to p_1) \vee \cdots \vee (p_0 \wedge \cdots \wedge p_{n-1} \to p_n)$.

Proof. It immediately follows from Corollary 4.4 and the well-known characterizations of $\mathbf{Int} + \boldsymbol{bd}_n$, $\mathbf{Int} + \boldsymbol{bw}_n$ and $\mathbf{Int} + \boldsymbol{bc}_n$ (see e.g. [4]). ∎

It should be noted that in [2] the finite model property of $\mathbf{MIPC} \oplus \boldsymbol{bw}_n$ and that of (every extensions of) $\mathbf{MIPC} \oplus \boldsymbol{bd}_n$ have been already obtained. Also, it is shown in [1] that, for every $\Gamma \subseteq \mathbf{For}\mathcal{L}$, $\mathbf{MIPC} \oplus \Gamma$ has the finite model property if either (1) Γ consists of $\{\wedge, \to, \bot\}$-formulas; or (2) $\mathbf{Int} + \Gamma$ is locally tabular.

5 Conclusion

In this paper we gave a sufficient condition for intuitionistic modal logics over **MIPC** (in the lattice of $\Box\Diamond$-normal intuitionistic modal logics) to have the finite model property. We showed that every **MIPC**-logic has the finite model property if there exists a universal first-order sentence Φ such that (1) L is characterized by the class of full **MIPC**-frames satisfying Φ and (2) every full **MIPC**-frame that validates L satisfies Φ. We also applied our result to show the finite model property of some intuitionistic modal logics over **MIPC** whose finite model property had been previously unknown.

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