A Proof of the Conjecture of Zantema on a Persistent Property of Term Rewriting Systems

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Abstract. A property $\mathcal{P}$ of term rewriting system is persistent if for any many-sorted term rewriting system $\mathcal{R}$, $\mathcal{R}$ has the property $\mathcal{P}$ iff its underlying term rewriting system $\Theta(\mathcal{R})$, which results from $\mathcal{R}$ by omitting its sort information, has the property $\mathcal{P}$. It is shown that termination is a persistent property of many-sorted term rewriting systems that contain only variables of the same sorts. This is the positive solution to a problem of Zantema, which has been appeared as Rewriting Open Problem 60 in literature.

1 Introduction

Sort introduction technique in term rewriting has caught attention recently [1][4]. We prove in this paper a conjecture which opens up the possibility of new applications of this technique. The conjecture reads: for any terminating many-sorted term rewriting system $\mathcal{R}$, if $\mathcal{R}$ contains only variables of the same sorts then $\Theta(\mathcal{R})$ is also terminating. Here, $\Theta(\mathcal{R})$—called the underlying term rewriting system of $\mathcal{R}$—is the term rewriting system obtained from $\mathcal{R}$ by omitting its sort information. The conjecture was raised by Zantema and adapted in [3] as Rewriting Open Problem 60.

A property $\mathcal{P}$ of TRSs is said to be *persistent* if $\mathcal{R}$ has the property $\mathcal{P}$ iff $\Theta(\mathcal{R})$ has the property $\mathcal{P}$; this notion is due to Zantema [5]. As an obvious extension, we say $\mathcal{P}$ *is persistent for a class* $\mathcal{C}$ *of many-sorted TRSs* if for any $\mathcal{R} \in \mathcal{C}$, $\mathcal{R}$ has the property $\mathcal{P}$ iff $\Theta(\mathcal{R})$ has the property $\mathcal{P}$. Then we can restate the above conjecture as follows: termination is persistent for the class of many-sorted TRSs that contain only variables of the same sorts. It is known that confluence is persistent for the class of many-sorted TRSs [2] and that termination is persistent for the class of many-sorted TRSs that do not contain both duplicating and collapsing rules [5]. Persistent properties in equational rewriting have been studied in [4].

2 Preliminaries

Let $\mathcal{S}$ be a set of sorts (denoted by $\alpha, \beta, \gamma, \ldots$), $\mathcal{F}$ a set of $\mathcal{S}$-sorted function symbols (denoted by $f, g, h, \ldots$), $\mathcal{V}$ a set of $\mathcal{S}$-sorted variables (denoted by $x, y, z, \ldots$). We write $f : \alpha_1 \times \cdots \times \alpha_n \to \beta$ when $f \in \mathcal{F}$ has sort $\alpha_1 \times \cdots \times \alpha_n \to \beta$. We assume that there are countably infinite variables of sort α for each sort $\alpha \in \mathcal{S}$.

We denote by $\mathcal{T}$ (and $\mathcal{T}^\alpha$) the set of terms (of sort α, respectively). We also write "$t : \alpha$" to indicate that $t \in \mathcal{T}^\alpha$. We denote by $\mathcal{V}$ the set of variables, and $\mathcal{V}(t)$ is the set of variables that appear in a term t. Syntactical equality is denoted by $\equiv$.

For each sort α, let $\Box^\alpha$ be a special constant—called a *hole*—of sort α. A *context* is a term possibly containing holes. The set of contexts is denoted by $\mathcal{C}$. We write $C : \alpha_1 \times \cdots \times \alpha_n \to \beta$ when $C \in \mathcal{C}$ has sort β (as a term) and has n holes $\Box^{\alpha_1}, \ldots, \Box^{\alpha_n}$ from left to right in it. If $C : \alpha_1 \times \cdots \times \alpha_n \to \beta$ and $t_1 : \alpha_1, \ldots, t_n : \alpha_n$ then $C[t_1, \ldots, t_n]$ is the term obtained from C by replacing holes with $t_1, \ldots, t_n$ from left to right. A context C is written as $C[\,]$ when C contains precisely one hole. A term t is said to be a *subterm* of s or equivalently s *contains* t ($t \trianglelefteq s$, in symbol) if $s \equiv C[t]$ for some context $C[\,]$; t is *proper subterm* of s or equivalently s *properly contains* t when $s \not\equiv t$ in addition. A context is said to be *empty* when $C \equiv \Box^\alpha$ for some $\alpha \in \mathcal{S}$.

We assume familiarity with position formalism. The empty (or root) position is denoted by Λ. We denote by $\mathrm{Pos}(t)$ the set of *positions* of a term t; by t/p the subterm of t at a position $p \in \mathrm{Pos}(t)$. We write $u \leq v$ when $v = u.w$ for some w; $u \mid v$ when $v \not\leq u$ and $u \not\leq v$. For a context $C[\,]$, we write $C[\,]_p$ when $C[\,]/p$ is a hole. When $s \equiv C[u]_p$, we denote by $s[p \leftarrow v]$ the term $C[v]$. When $p \notin \mathrm{Pos}(s)$, we define $s[p \leftarrow v] \equiv s$. We abbreviate $(\cdots(s[p_1 \leftarrow v_1])\cdots[p_n \leftarrow v_n])$ to $s[p_i \leftarrow v_i \mid 1 \leq i \leq n]$.

A *substitution* σ is a mapping from $\mathcal{V}$ to $\mathcal{T}$ such that x and $\sigma(x)$ have the same sorts. A substitution is extended to a homomorphism from $\mathcal{T}$ to $\mathcal{T}$ in the obvious way; $t\sigma$ stands for $\sigma(t)$ for a substitution σ and a term t.

A *(many-sorted) rewrite rule* is a pair $\langle l, r\rangle$ of terms such that (1) l and r have the same sorts, (2) $l \notin \mathcal{V}$, (3) $\mathcal{V}(r) \subseteq \mathcal{V}(l)$. We conventionally write $l \to r$ instead of $\langle l, r\rangle$. A rewrite rule $l \to r$ is *collapsing* if $r \in \mathcal{V}$. A *many-sorted term rewriting system* (*STRS*, for short) is a set of rewrite rules.

Given an STRS $\mathcal{R}$, a term s reduces to a term t ($s \to_\mathcal{R} t$, in symbol) when $s \equiv C[l\sigma]$ and $t \equiv C[r\sigma]$ for some $C[\,] \in \mathcal{C}$, $l \to r \in \mathcal{R}$ and substitution σ. We call $s \to_\mathcal{R} t$ a *rewrite step* (or a *reduction*). The *redex* of this rewrite step is $l\sigma$. The term t is called a *reduct* of the term s. It is readily checked that s and t have the same sorts whenever $s \to_\mathcal{R} t$. The transitive-reflexive closure and the transitive closure of $\to_\mathcal{R}$ are denoted by $\overset{*}{\to}_\mathcal{R}$ and $\overset{+}{\to}_\mathcal{R}$, respectively. Henceforth, the subscript $_\mathcal{R}$ will be omitted when $\mathcal{R}$ is obvious from the context.

When $\mathcal{S} = \{*\}$, an STRS is called a TRS. Given an arbitrary STRS $\mathcal{R}$, by identifying each sort with $*$, we obtain the TRS $\Theta(\mathcal{R})$—called the underlying

TRS of $\mathcal{R}$.

Note that according to the definition of rewrite relation, $\Theta(\mathcal{R})$ acts on the set of terms possibly non-well-sorted with respect to the sorts of function symbols in $\mathcal{R}$. With a fixed $\mathcal{R}$, we henceforth denote $\Theta(\mathcal{T})$ and $\Theta(\mathcal{C})$ the sets of terms and contexts possibly non-well-sorted with respect to the sorts of function symbols in $\mathcal{R}$ respectively. On the other hand, we preserve notation $t : \alpha$, $C : \alpha_1 \times \cdots \times \alpha_n \to \beta$ only for well-sorted terms and well-sorted contexts; more precisely, we will write $t : \alpha$ and $C : \alpha_1 \times \cdots \times \alpha_n \to \beta$ only when t and C are well-sorted with respect to the sorts of function symbols in $\mathcal{R}$ respectively.

3 A Proof of the Conjecture

We address in this paper the following theorem conjectured by Zantema.

Theorem 3.1 *Let $\mathcal{R}$ be an STRS that contains only variables of the same sorts. If $\mathcal{R}$ is terminating then so is $\Theta(\mathcal{R})$.*

For this purpose, we assume in the sequel that there exists a special sort $0 \in \mathcal{S}$, and fix an STRS $\mathcal{R}$ such that every variable occurring in rules of $\mathcal{R}$ has sort 0. Further, we assume that l and r have sort 0 for any $l \to r \in \mathcal{R}$—this assumption does not weaken our theorem, because of the following lemma:

Lemma 3.2 *Let $\mathcal{R}$ be an STRS such that all variables appearing in $\mathcal{R}$ have sort 0. Let $\hat{\mathcal{R}}$ be an STRS $\{C[l] \to C[r] \mid l \to r \in \mathcal{R}^\alpha,\ C : \alpha \to 0,\ \alpha \in \mathcal{S}\}$. Here $\mathcal{R}^\alpha$ is the set of rewrite rules in $\mathcal{R}$ of sort α. Then, $\mathcal{R}$ is terminating if and only if $\hat{\mathcal{R}}$ is terminating.*

Proof. ($\Leftarrow$) Suppose $\mathcal{R}$ is not terminating. Then there exists an infinite $\mathcal{R}$-reduction sequence $s_0 \to_{\mathcal{R}} s_1 \to_{\mathcal{R}} s_2 \to_{\mathcal{R}} \cdots$; in particular, that with $s_i : 0$ for all i. Let $s_i \equiv C[l\sigma]$ and $s_{i+1} \equiv C[r\sigma]$. Then we can rename variables in l, r so that they are distinct from any of those appearing in C. Thus $s_i \equiv C[l]\hat{\sigma}$ and $s_{i+1} \equiv C[r]\hat{\sigma}$ for some $\hat{\sigma}$; hence $s_0 \to_{\hat{\mathcal{R}}} s_1 \to_{\hat{\mathcal{R}}} s_2 \to_{\hat{\mathcal{R}}} \cdots$. Thus, $\hat{\mathcal{R}}$ is also not terminating. ($\Rightarrow$) Suppose $\hat{\mathcal{R}}$ is not terminating. Let $s_0 \to_{\hat{\mathcal{R}}} s_1 \to_{\hat{\mathcal{R}}} s_2 \to_{\hat{\mathcal{R}}} \cdots$. By definition, $s_i \equiv C'[C[l]\sigma] \to_{\hat{\mathcal{R}}} C'[C[r]\sigma] \equiv s_{i+1}$; this means $s_i \equiv C'[(C\sigma)[l\sigma]]$ and $s_{i+1} \equiv C'[(C\sigma)[r\sigma]]$, and hence $s_i \to_{\mathcal{R}} s_{i+1}$ for each i. Thus, we have $s_0 \to_{\mathcal{R}} s_1 \to_{\mathcal{R}} s_2 \to_{\mathcal{R}} \cdots$, i.e. $\mathcal{R}$ is not terminating. ∎

3.1 Sort introduction basics

Definition 3.3 *The sort of a term $t \in \Theta(\mathcal{T})$ is defined by*

$$\mathrm{sort}(t) = \begin{cases} \alpha & \text{if } t \equiv x^\alpha, \\ \beta & \text{if } t \equiv f(t_1, \ldots, t_n) \text{ with } f : \alpha_1 \times \cdots \times \alpha_n \to \beta. \end{cases}$$

We extends the definition of sort for contexts in the obvious way by putting $\mathrm{sort}(\Box^\alpha) = \alpha$.

Definition 3.4 *Let $t \equiv C[t_1, \ldots, t_n] \in \Theta(\mathcal{T})$ $(n \geq 0)$ be a term with non-empty C. We write $t \equiv C[\![t_1, \ldots, t_n]\!]$ if (1) $C : \alpha_1 \times \cdots \times \alpha_n \to \beta$ and (2) $\mathrm{sort}(t_i) \neq \alpha_i$ for $i = 1, \ldots, n$. We call the terms $t_1, \ldots, t_n$ principal subterms of t. We denote by $\mathrm{cap}(t)$ the well-sorted context $C[\Box^{\alpha_1}, \ldots, \Box^{\alpha_n}]$.*

Clearly, every term t is uniquely written as $C[\![t_1, \ldots, t_n]\!]$ by some $C \in \mathcal{C}$ and terms $t_1, \ldots, t_n$.

Definition 3.5 *1. The rank of a term t is defined by:*

$$\mathrm{rank}(t) = \begin{cases} 1 & \text{if } t \in \mathcal{T}, \\ \max\{\mathrm{rank}(t_i) \mid 1 \leq i \leq n\} & \text{if } t \equiv C[\![t_1, \ldots, t_n]\!] \text{ with } n > 0. \end{cases}$$

2. A subterm u of t is said to be special (in t) if either $u \equiv t$ or u is a principal subterm of a special subterm of t.

Convention 3.6 *We denote $C[u]$ by $C[!u]$ if this occurrence of u is a proper special subterm of $C[u]$. We indicate $C[u_1, \ldots, u_n]$ as $C[u_1, \ldots, (u_i)_\gamma, \ldots, u_n]$ when $\Box^\gamma$ is not special in $C[u_1, \ldots, \Box^\gamma, \ldots, u_n]$. In the sequel, we assume every hole in non-empty $C \in \Theta(\mathcal{C})$ is not a special subterm of C; in this spirit, we often omit the superscript α of $\Box^\alpha$.*

Definition 3.7 *For a non-empty position p in a term s, the sort of the position p in s is defined as: $\mathrm{sort}(s, p) = \gamma$ when $s \equiv C[(u)_\gamma]_p$ for some C and u.*

Definition 3.8 *1. For a term $t \in \Theta(\mathcal{T})$, we define the set of disconnections in t as follows:*

$$\mathrm{Dcn}(t) = \{p \in \mathrm{Pos}(t) \mid t/p \text{ is a special subterm of } t\}$$

For a context C, $\mathrm{Dcn}(C)$ is defined similarly. It is clear that for every non-empty position p, we have $p \in \mathrm{Dcn}(t)$ iff $\mathrm{sort}(t, p) \neq \mathrm{sort}(t/p)$.

2. Let $s \equiv C[!u]_q$. The peak of u is the position p defined by

$$p = \max\{o \leq q \mid \mathrm{sort}(s, o) = 0 \text{ or } o \in \mathrm{Dcn}(s) \backslash \{q\}\}.$$

Clearly, C/p is again a context; we call it the leaf of C and denote it by $\mathrm{leaf}(C)$.

Definition 3.9 *A rewrite step $s \to t$ is said to be inner (written as $s \to^i t$) if*

$$s \equiv C[\![s_1, \ldots, C'[l\sigma], \ldots, s_n]\!] \to C[s_1, \ldots, C'[r\sigma], \ldots, s_n] \equiv t$$

for some terms $s_1, \ldots, s_n$, a substitution σ, $l \to r \in \mathcal{R}$, and $C' \in \Theta(\mathcal{C})$; otherwise it is outer (written as $s \to^o t$). The redex of an inner reduction is a inner redex; that of an outer reduction a outer redex.

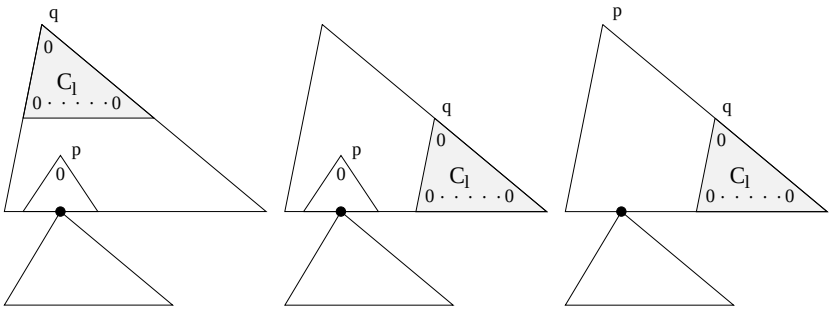

Figure 1: The peak of a principal subterm and an outer redex

Lemma 3.10 *Let $l \to r \in \mathcal{R}$ and $l \equiv C_l[x_1, \ldots, x_n]_{q_1,\ldots,q_n}$ with $C_l \in \mathcal{C}(\mathcal{F}, \emptyset)$. Suppose $s \equiv C[l\sigma]_q$ and $l\sigma$ is an outer redex in s. Then,*

1. *$\mathrm{sort}(C_l) = 0$ and $C[C_l[(x_1\sigma)_0, \ldots, (x_n\sigma)_0]]$;*
2. *for the peak p of each principal subterm of s, we have either $q.q_i \leq p$ for some i, $p < q$ or $p \mid q$.*

Proof. By our assumption on $\mathcal{R}$, we have $C_l : 0 \times \cdots \times 0 \to 0$, and so 1 holds. 2 follows easily from 1 and the definition of peak. See Figure 1. ∎

Definition 3.11 *A rewrite step $s \to^o t$ is said to be destructive at level 1 if $\mathrm{sort}(s) \neq \mathrm{sort}(t)$; $s \to^i t$ is destructive at level $k+1$*

$$s \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!] \to^i C[s_1, \ldots, t_j, \ldots, s_n] \equiv t$$

with $s_j \to t_j$ destructive at level k.

Lemma 3.12 *A rewrite step $s \to^o t$ is destructive if and only if $t \equiv \sigma(x)$ and $s \equiv C[\![s_1, \ldots, \sigma(x), \ldots, s_n]\!]$ for some terms $s_1, \ldots, s_n$, substitution σ, and $C \in \mathcal{C}$ such that $C[s_1, \ldots, \Box, \ldots, s_n] \equiv C'\sigma$ for some $C'[x] \to x \in \mathcal{R}$.*

Proof. ($\Leftarrow$) Suppose $\mathrm{sort}(s) = \mathrm{sort}(t)$. Since $C' \not\equiv \Box$ by the definition of rewrite rules, $\mathrm{sort}(C'[x]) = \mathrm{sort}(s)$; and since $\mathrm{sort}(l) = \mathrm{sort}(r)$ for any $l \to r \in \mathcal{R}$, we know $\mathrm{sort}(x) = \mathrm{sort}(C'[x])$; thus $\mathrm{sort}(x) = \mathrm{sort}(t)$. But then, t can not be principal, since $C[s_1, \ldots, \Box, \ldots, s_n] \equiv C'\sigma : \mathrm{sort}(x) \to \mathrm{sort}(s)$. ($\Rightarrow$) Suppose $\mathrm{sort}(s) \neq \mathrm{sort}(t)$. As $\mathrm{sort}(l) = \mathrm{sort}(r)$ for any $l \to r \in \mathcal{R}$, the rewrite step $s \to_{\mathcal{R}} t$ is an application of a collapsing rule, and the redex of this rewrite step is s. Let the rule be $C'[x] \to x$, and suppose $s \equiv C'[x]\sigma$ and $t \equiv \sigma(x)$. Since $C[s_1, \ldots, \Box, \ldots, s_n] : \mathrm{sort}(x) \to \mathrm{sort}(s)$, it suffices to show $\mathrm{sort}(x) \neq \mathrm{sort}(t)$. We have $\mathrm{sort}(x) = \mathrm{sort}(C'[x]) = \mathrm{sort}(C'[x]\sigma) = \mathrm{sort}(s)$ exactly as above, and so $\mathrm{sort}(x) \neq \mathrm{sort}(t)$ by our assumption. ∎

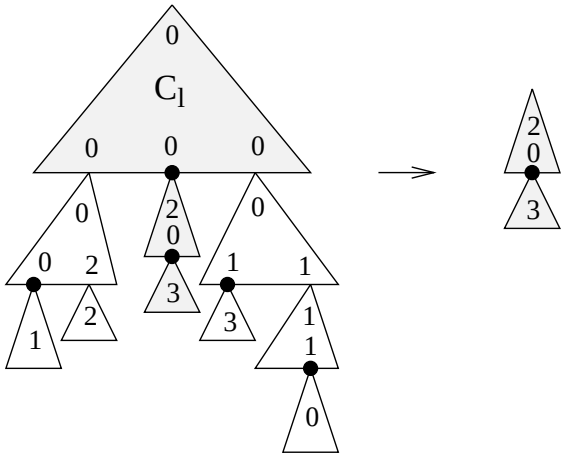

Figure 2: A destructive rewrite step

Lemma 3.13 *Let $s \equiv C[\![s_1, \ldots, s_n]\!] \to s_i$ be a destructive rewrite step of level 1. Let p_j be the peak of s_j in s for each $j = 1, \ldots, n$. Then,*

1. $\mathrm{sort}(s) = 0$ *and* $C[\![s_1, \ldots, (s_i)_0, \ldots, s_n]\!]$;
2. $s/p_i \equiv s_i$ *and* $\forall j \neq i\ (p_j \not\leq p_i)$.

Proof.

1. By Lemma 3.12, $C_l[x_1, \ldots, x_k] \to x_j \in \mathcal{R}$, $s \equiv C_l[x_1\sigma, \ldots, x_k\sigma]$ and $s_i \equiv x_j\sigma$ with non-empty $C_l \in \mathcal{C}(\mathcal{F}, \emptyset)$. Since $C_l : 0 \times \cdots \times 0 \to 0$ by our assumption on $\mathcal{R}$, we get $\mathrm{sort}(s) = 0$ and $C[\![s_1, \ldots, (s_i)_0, \ldots, s_n]\!]$.
2. That $s/p_i \equiv s_i$ follows immediately from 1. Also, $\forall j \neq i\ (p_j \not\leq p_i)$ is readily checked using Lemma 3.10(2). See Figure 2.

∎

3.2 Active subterms and $\Rightarrow$-reduction

Definition 3.14 *Let $s \equiv C[(!u)_\gamma]_q$ and $u \equiv C'[\![u_1, \ldots, u_n]\!]_{q_1,\ldots,q_n}$. Suppose p is the peak of u in s and p_i is the peak of u_i in s for each $i = 1, \ldots, n$.*

1. *This occurrence of u is said to be active in s if (1)* $\mathrm{sort}(u) = 0$ *and (2) for all i $(1 \leq i \leq n)$ u_i is active in t whenever $q = p_i$.*

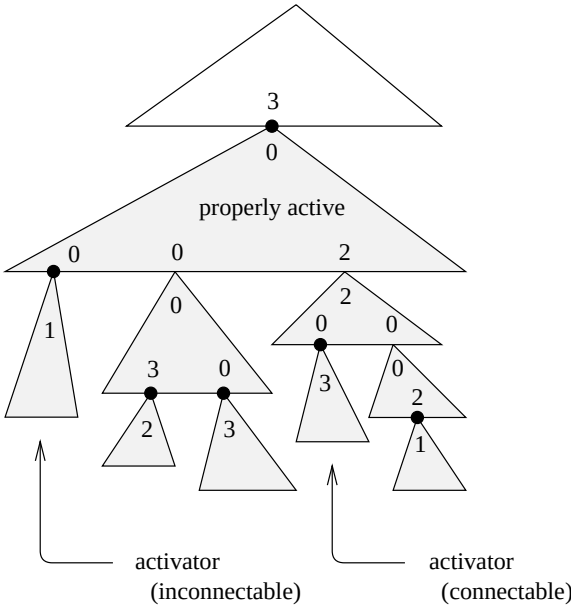

Figure 3: A properly active subterm and activators

2. *This occurrence of u is said to be properly active in s if (1)* $\mathrm{sort}(u) = 0$ *and (2) $q < p_i$ for all i ($1 \leq i \leq n$). When this is the case, u_i ($1 \leq i \leq n$) is called an activator of u in s if (1) p_i is minimal in $\{p_1, \ldots, p_n\}$, (2)* $\mathrm{sort}(u, q_i) = 0$*; if further it satisfies (3)* $\mathrm{sort}(u_i) = \gamma$*, then u_i is called an connectable activator of u in s. Activators that are not connectable are inconnectable activators. (See Figure 3.)*

Clearly, every properly active subterm is active. Also, note that $\mathrm{leaf}(C)$ is non-empty whenever u is an active subterm of $C[!u]$.

The next lemma is an easy consequence of the definition above and Lemma 3.13.

Lemma 3.15 *Let $s \to t$ a rewrite step destructive at level $k > 1$. Then there exists a subterm u of s with $s \equiv C[u]$ a properly active in s and an activator u_i of u in s such that $u \to u_i$ and $t \equiv C[u_i]$.*

Definition 3.16 *Let $t \equiv C[\![t_1, \ldots, t_n]\!]$, and suppose that p_i is the peak of t_i for each $i = 1, \ldots, n$. The term t is said to be root-active if* $\mathrm{sort}(t) = 0$ *and t_i is active whenever $p_i = \Lambda$.*

One can readily check the following lemma.

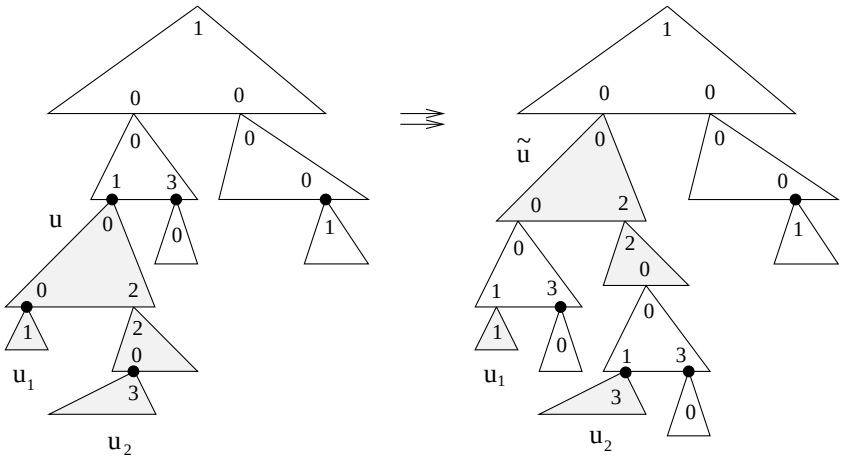

Figure 4: A $\rightrightarrows$-rewrite step

Lemma 3.17 *Let u be a proper special subterm of a term s. Then u is root-active if and only if u is active in s.*

Definition 3.18 *Let $s \equiv C[!u]$ and p the peak of u in s.*

1. *This occurrence of u in s is said to be a $\rightrightarrows$-redex if (1) u is properly active in s and (2) s/p is root-active.*

2. *The $\rightrightarrows$-redex u is a left-most inner-most $\rightrightarrows$-redex if (3) all proper special subterms of s/p with the peak different from p are not $\rightrightarrows$-redex and (4) all principal subterms of $\mathrm{leaf}(C)$ with the peak p placed on the left to the hole are not $\rightrightarrows$-redex.*

3. *Suppose that the $\rightrightarrows$-redex u is a left-most inner-most $\rightrightarrows$-redex. Let $\tilde{u}$ be as follows: Let $u_1, \ldots, u_n$ be activators of u with $n \geq 0$ and $u \equiv C'[u_1, \ldots, u_n]$; then $\tilde{u} \equiv C'[\mathrm{leaf}(C)[u_1], \ldots, \mathrm{leaf}(C)[u_n]]$. Let $t \equiv s[p \leftarrow \tilde{u}]$. Then we say that s $\rightrightarrows$-reduces to t ($s \rightrightarrows t$, in symbol), or t is a $\rightrightarrows$-reduct of s; we call $s \rightrightarrows t$ a $\rightrightarrows$-rewrite step. (See Figure 4.) The term u is called the $\rightrightarrows$-redex of this $\rightrightarrows$-rewrite step and we write $s \rightrightarrows_u t$ when this is the case. A $\rightrightarrows$-normal term is a term s such that $s \rightrightarrows t$ for no term t.*

It is clear that every $\rightrightarrows$-normal term contains no $\rightrightarrows$-redex.

Definition 3.19 *Let $A : s \rightrightarrows_u t$ be a $\rightrightarrows$-rewrite step. Suppose $s \equiv C[u]_q$ and p is the peak of u in s. For $o \in \mathrm{Pos}(s)$, the set $A(o)$ of descendants of*

o by A is defined as follows: Let $u_1, \ldots, u_n$ $(n \geq 0)$ be activators of u with $u \equiv C[u_1, \ldots, u_n]_{q_1,\ldots,q_n}$. Then,

$$A(o) = \begin{cases} \{o\} & \text{if } o < p \text{ or } o \mid p, \\ \{p.q_i.(o\backslash p) \mid 1 \leq i \leq n\} & \text{if } p < o \text{ and } q \not\leq o, \\ \{p.(o\backslash q)\} & \text{if } q < o \text{ and } \forall i\ (q.q_i \not\leq o), \\ \{p.q_j.(p\backslash q).(o\backslash(q.q_j))\} & \text{if } q.q_j < o \text{ for some } j\ (1 \leq j \leq n). \\ \emptyset & \text{otherwise.} \end{cases}$$

For $u \equiv s/o$, the set of descendants of u by A is $\{t/o' \mid o' \in A(o)\}$.

One can check the next lemma easily.

Lemma 3.20 *Let $A : s \rightrightarrows_u t$ be a $\rightrightarrows$-rewrite step. Suppose $s \equiv C[u]_q$ and p is the peak of u in s. Let $u_1, \ldots, u_n$ $(n \geq 0)$ be activators of u with $u \equiv C'[u_1, \ldots, u_n]_{q_1,\ldots,q_n}$. Let $P = \bigcup\{A(o) \mid o \in \mathrm{Dcn}(s)\}$ and $Q = \{p.(q_i\backslash q).(q\backslash p) \mid u_i$ is inconnectable$\}$. Then,*

$$\mathrm{Dcn}(t) = \begin{cases} P \cup Q & \text{if } p \notin \mathrm{Dcn}(s), \\ \{p\} \cup P \cup Q & \text{if } p \in \mathrm{Dcn}(s). \end{cases}$$

Lemma 3.21 *Let $A : s \rightrightarrows_u t$ be a $\rightrightarrows$-rewrite step, and p the peak of u in s. Then,*

1. *s/p is active iff t/p is active;*
2. *every descendant of an active subterm is active;*
3. *every active subterm of t different from t/p is a descendant of an active subterm different from u and s/p.*

Proof. By Lemma 3.20 and definition, it suffices to prove that s/o is active iff t/o is active, where $o = \max\{p' \leq p \mid s/p'$ is a special subterm of $s\}$. We distinguish two cases:

1. $o = p$. The case where $p = \Lambda$ is trivial. Otherwise, suppose that s/p is a proper special subterm of s. We show s/p and t/p are active in s and t respectively. Since s/p is root-active by the definition $\rightrightarrows$-rewrite step, s/p is active in s by Lemma 3.17. We know hence $\mathrm{sort}(s/p) = 0$ and $\mathrm{sort}(s, p) \neq 0$ and every principal subterm of s/p with the peak p is active. If u has no activator, then $t/p \equiv u$, which is active in t. Otherwise suppose u has activators $u_1, \ldots, u_n$ $(n > 0)$ with $u \equiv C'[u_1, \ldots, u_n]$. First, $\mathrm{sort}(t/p) = \mathrm{sort}(u) = 0$. Next, note that principal subterms of t/p consists of those of u which are not activators and those contained in $\mathrm{leaf}(C')[u_i]$ for some i $(1 \leq i \leq n)$. Principal subterms from the former ones do not have the peak p in t, because u is active in s. Principal subterms from the latter ones also do not have the peak p, because $\mathrm{sort}(t, p.q_i) = 0$.

2. $o < p$. Then s/p is not special. Thus, $\mathrm{sort}(s,p) = 0$ and so the principal subterms of s/o contained in s/p can not have the peak o. Also, the principal subterms of s/o not contained in s/p coincides those of t/o not contained in t/p. Therefore, s/o is active in s iff t/o is active in t.

∎

Definition 3.22 *For each special subterm u of t, we define* $\mathrm{depth}(u)$ *inductively as follows: (1)* $\mathrm{depth}(t) = 0$. *(2) Suppose u is a special subterm of* $\mathrm{depth}(u) = k$ *and* $u \equiv C[\![(u_1)_{\gamma_1}, \ldots, (u_n)_{\gamma_n}]\!]$. *Then for* $i = 1, \ldots, n$

$$\mathrm{depth}(u_i) = \begin{cases} k \dot{-} 1 & \text{if } \gamma_i = 0, \\ k + 1 & \text{if } \mathrm{sort}(u_i) = 0, \\ k & \text{otherwise.} \end{cases}$$

Since $\mathrm{sort}(u) = 0$ for any active subterm u, the depth of active subterms are non-zero.

The following lemma is readily checked.

Lemma 3.23 *Let u be an active subterm of a term s and p the peak of u. Suppose that v is the minimal special subterm of s that contains s/p. Then* $\mathrm{depth}(u) = \mathrm{depth}(v) + 1$.

Lemma 3.24 *The relation $\rightrightarrows$ is well-founded and confluent.*

Proof. For each $\rightrightarrows$-redex v in s, let $\mathrm{weight}(v) = \langle \mathrm{depth}(v), \mathrm{rank}(v), M \rangle$ where $M = [\langle \mathrm{depth}(v'), \mathrm{rank}(v') \rangle \mid v'$ is an active subterm of s different from v yet has the same peak as $v\,]$.

Let $\succcurlyeq$ be an order defined by: $\langle i_1, j_1 \rangle \succcurlyeq \langle i_2, j_2 \rangle$ iff either (1) $i_1 > i_2$ or (2) $i_1 = i_2$ and $j_1 > j_2$; and $\gg$ the multiset extension of $\succcurlyeq$. Let $\succ$ be an order defined by: $\langle i_1, i_2, M_1 \rangle \succ \langle j_1, j_2, M_1 \rangle$ iff either (1) $i_1 > i_2$, (2) $i_1 = i_2$ and $j_1 > j_2$ or (3) $i_1 = i_2$, $j_1 = j_2$ and $M_1 \gg M_2$; and $\succ\!\!\succ$ the multiset extension of $\succ$. We are going to verify that $s \rightrightarrows t$ implies $\|s\| \succ\!\!\succ \|t\|$.

Suppose $A : s \rightrightarrows_u t$, and let p be the peak of u in s. By the definition of $\rightrightarrows$-reduction, if v is a $\rightrightarrows$-redex in t then either (1) $v \equiv t/p$, (2) v is a descendant of a $\rightrightarrows$-redex in s different from u and s/p, or (3) v is a descendant of a properly active subterm of s with the peak $q.q_i$ for some i.

If t/p is a $\rightrightarrows$-redex then $\mathrm{weight}(u) \succ \mathrm{weight}(t/p)$ by Lemma 3.23. Also, if v is a descendant of a properly active subterm of s with the peak $q.q_i$ for some i, then we have $\mathrm{weight}(u) \succ \mathrm{weight}(v)$, because $\mathrm{depth}(u) = \mathrm{depth}(v)$ and $\mathrm{rank}(u) > \mathrm{rank}(v)$. Thus, because u has no descendant, it suffices to show that $[\mathrm{weight}(v)] \underline{\succcurlyeq} [\mathrm{weight}(t/o') \mid o' \in A(v)]$ for any $\rightrightarrows$-redex v of s different from u and s/p.

Let v be a $\rightrightarrows$-redex in s and o the position of v in s. If either $o < p$ or $o \mid p$ then $A(o) = \{o\}$ and $\mathrm{weight}(v) = \mathrm{weight}(t/o)$ by Lemma 3.21. Thus, we only

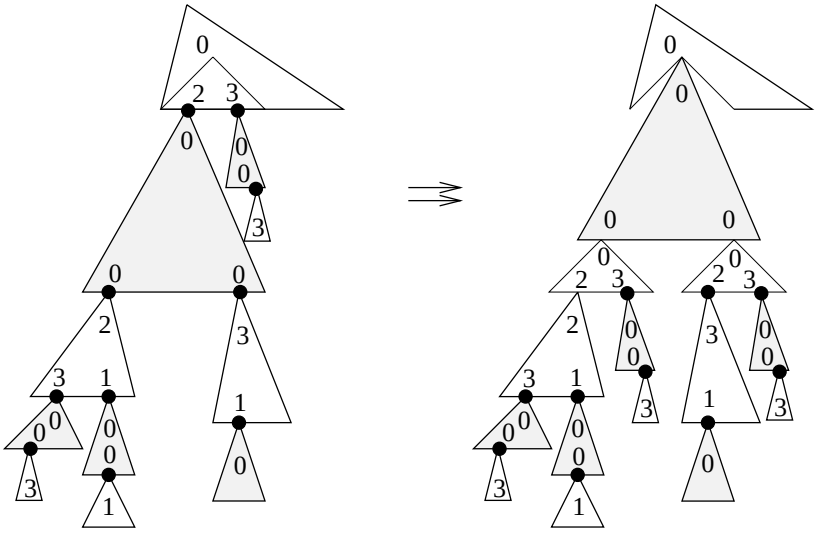

Figure 5: $\rightrightarrows$ is well-founded

need to consider the case where $p < o$. By definition, $\rightrightarrows$-redexes contained in s/p have the peak p. For any descendant v' of v, $\mathrm{depth}(v') = \mathrm{depth}(v)$, $\mathrm{rank}(v') = \mathrm{rank}(v)$ and $\pi_3(\mathrm{weight}(v')) > \pi_3(\mathrm{weight}(v))$; for, $\pi_3(\mathrm{weight}(v'))\backslash\pi_3(\mathrm{weight}(v)) = [\langle \mathrm{depth}(u), \mathrm{rank}(u)\rangle]$ and we have $\mathrm{depth}(u) = \mathrm{depth}(w)$ and $\mathrm{rank}(u) > \mathrm{rank}(w)$ for any $\langle \mathrm{depth}(w), \mathrm{rank}(w)\rangle \in \pi_3(\mathrm{weight}(v'))\backslash\pi_3(\mathrm{weight}(v))$. (See Figure 5.)

Thus $s \rightrightarrows t$ implies $\|s\| \succ\succ \|t\|$. It is readily checked $\succ\succ$ is well-founded; hence so is $\rightrightarrows$.

Next, $\rightrightarrows$ is locally confluent; for, if two left-most inner-most $\rightrightarrows$-redexes occur in a term then their peaks are disjoint. Therefore $\rightrightarrows$ is confluent. ∎

Thus every term s has the unique $\rightrightarrows$-normal form; we denote it by $s{\downarrow\downarrow}$.

3.3 Simulating $\Theta(\mathcal{R})$-reductions in $\rightrightarrows$-normal terms

We write $s \to_{[p,l\to r]} t$ when $s \equiv C[l\sigma]_p$ and $t \equiv C[r\sigma]_p$ for some context C and substitution σ.

Definition 3.25 *Let $A : s \to_{[p,l\to r]} t$ be a rewrite step. For $o \in \mathrm{Pos}(s)$, the set $A(o)$ of descendants of o by A is defined as follows:*

$$A(o) = \begin{cases} \{o\} & \text{if } o < p \text{ or } o \mid p, \\ \{p.p_3.p_2 \mid r/p_3 = l/p_1\} & \text{if } o = p.p_1.p_2 \text{ with } p_1 \in \mathrm{Pos}_{\mathcal{V}}(l), \\ \emptyset & \text{otherwise.} \end{cases}$$

For $u \equiv s/q$, the set of descendants of u by A is the set $\{t/q' \mid q' \in A(q)\}$.

Lemma 3.26 *If $s \to t$ and s is $\rightrightarrows$-normal, then t is also $\rightrightarrows$-normal.*

Proof. It is clear by Lemma 3.10 that the statement holds when $s \to t$ is not destructive. It is also so in case where $s \to t$ is destructive at level 1. Suppose $s \to t$ is destructive at level $k > 0$. Then by Lemma 3.15 we have $s \equiv C[!u]$, $C[u_i] \equiv t$ and $u \to u_i$ for u properly active in s and u_i an activator of u in s. Let p the peak of u in s. Suppose $\gamma \neq \mathrm{sort}(u_i)$. Then the only possible $\rightrightarrows$-redex in t is u_i and the minimal special subterm of t that contains t/p. But since $\mathrm{sort}(u_i) \neq 0$, each can not be a new $\rightrightarrows$-redex. Otherwise, suppose $\gamma = \mathrm{sort}(u_i)$. Then the only possible $\rightrightarrows$-redex in t is a principal subterm of u_i and the minimal special subterm of t that contains t/p. If w is a new $\rightrightarrows$-redex from the former candidate, then its new peak must be p; but since u is properly active and s is $\rightrightarrows$-normal, s/p is not root-active, and so is t/p. Thus w can not be a new $\rightrightarrows$-redex. Also, if w is a new $\rightrightarrows$-redex from the latter candidate, then w must be properly active and the minimal special subterm v of s that contains s/p must not be properly active. This can happen only when $w \equiv t/p$ and $v \equiv s/p$, which is not the case. ∎

Definition 3.27 *Let $t \equiv C[\![t_1, \ldots, t_n]\!]$, and p_i the peak of t_i for each $i = 1, \ldots, n$. The head of t is defined as:* $\mathrm{head}(t) = t[p_i \leftarrow \Box \mid 1 \leq i \leq n]$. *We write $t \equiv C'[\langle\!\langle u_1, \ldots, u_k \rangle\!\rangle]$ when $t \equiv C'[u_1, \ldots, u_k]$ and C' is the head of t.*

Definition 3.28 *A rewrite step $s \to t$ is said to be a body reduction (written as $s \to^{bd} t$) if*

$$s \equiv C[\langle\!\langle s_1, \ldots, C'[l\sigma], \ldots, s_n \rangle\!\rangle] \to C[s_1, \ldots, C'[r\sigma], \ldots, s_n] \equiv t$$

for some terms $s_1, \ldots, s_n$, a substitution σ, $l \to r \in \mathcal{R}$, and $C' \in \mathcal{C}$; otherwise it is a head reduction (written as $s \to^{hd} t$). The redex of a body reduction is a body redex; that of a head reduction a head redex.

Lemma 3.29 *If $s \to t$ and $s \rightrightarrows s_1$, then there exists s_2, t_1 such that $s_1 \overset{*}{\rightrightarrows} s_2 \overset{*}{\to} t_1 \overset{*}{\leftleftarrows} t$; further if $s \to^{hd} t$ then $s_2 \overset{+}{\to}{}^{hd} t_1$.*

Proof. Suppose $A : s \equiv \hat{C}[l\sigma]_o \to \hat{C}[r\sigma] \equiv t$ and $B : s \rightrightarrows_u s_1$. Let $s \equiv C[u]_q$, p the peak of u. We distinguish the cases according to the position of the redex of $s \to t$:

1. $o \mid p$. Then, we have $s_1 \equiv s_2 \to_{[o, l \to r]} t_1 \leftleftarrows_{t/q} t$ for some t_1.
2. $o < p$. Then $o.o_x \leq p$ for some $o_x \in \mathrm{Pos}_{\mathcal{V}}(l)$ by Lemma 3.10. Let $\{o_1, \ldots, o_l\} = \{o' \in \mathrm{Pos}_{\mathcal{V}}(l) \mid l/o' = l/o_x, o' \neq o_x\}$ and $\{v_1, \ldots, v_k\} = A(s/o_x) \cup \bigcup_{1 \leq i \leq l} A(s/o_i)$. Then v_i is $\rightrightarrows$-redex for all $i = 1, \ldots, k$ and so is $s/(o.o_i.(q \backslash o_x))$ for all $i = 1, \ldots, l$. Thus, we have $s_1 \rightrightarrows_{o_1} \cdots \rightrightarrows_{o_l} s_2 \to_{[o, l \to r]} t_1 \leftleftarrows_{v_k} \cdots \leftleftarrows_{v_1} t$ for some t_1.

3. $p \leq o$. We distinguish two cases:

   (a) u has activators $u_1, \ldots, u_n$ $(n > 0)$ and $u \equiv C'[u_1, \ldots, u_n]$.

      i. $q \mid o$. Then $C[\,]_q \to_{[o,l\to r]} C^*[\,]_q$ and $\mathrm{leaf}(C) \to_{[o\backslash p,l\to r]} \mathrm{leaf}(C^*)$ for some C^*. Let $B(o) = \{o_1, \ldots, o_n\}$. Then s_1/o_i is a redex for any $i = 1, \ldots, n$, and so we have $s_1 \equiv s_2 \to_{[o_1,l\to r]} \cdots \to_{[o_n,l\to r]} t_1 \leftleftarrows_{t/q} t$.

      ii. $q.q_i \not\leq o$ for all i $(1 \leq i \leq n)$. If $o = q$ and $s \to t$ is destructive, then $r\sigma$ is an activator by Lemma 3.15, and so we have $s_1 \to_{[p,l\to r]} t_1 \equiv t$. (See Figure 6.) Otherwise, for any $i = 1, \ldots, n$, either $o \mid q.q_i$ or $o.o_x \leq q.q_i$ for some $o_x \in \mathrm{Pos}_{\mathcal{V}}(l)$. Hence, t/q' is an activator of t/p for any $q' \in \bigcup_i B(q.q_i)$. Also, $s_1/p.(q\backslash o)$ is a $(\mathcal{R}$-)redex, because $u_i \equiv u_j$ implies $\mathrm{leaf}(C')[u_i] \equiv \mathrm{leaf}(C')[u_j]$. Thus, we have $s_1 \to_{[p.(o\backslash q),l\to r]} t_1 \leftleftarrows_{t/q} t$. (Note that there may be the case that $\bigcup_i B(q.q_i) = \emptyset$; however, by the definition of $\rightrightarrows$-reduction, we have $t_1 \leftleftarrows_{t/q} t$ also in this case.)

      iii. $q.q_i \leq o$ for some i $(1 \leq i \leq n)$. Let $u_j \to u_j'$ with $s \equiv C'[u_j]$ and $t \equiv C'[u_j']$. Since $\mathrm{sort}(u_j) \neq 0$, $u_j \to u_j'$ is not destructive at level 1 by Lemma 3.13. Hence $\mathrm{sort}(u_j) = \mathrm{sort}(u_j')$ and so u_j' is an activator of the descendants of u. Thus, we have $s_1 \to_{[p.q_i.(p\backslash q).(o\backslash(q.q_i)),l\to r]} t_1 \leftleftarrows_{t/q} t$.

   (b) u has no activator.

      i. $q \mid o$. Then the descendant of u by A also has no activator. Thus, we have $s_1 \equiv s_2 \equiv t_1 \equiv t$. (See Figure 7.)

      ii. $q \leq o$. Then $s \to t$ can not be destructive whenever u is the redex of this reduction. By Lemma 3.13, it is clear that no descendant of u by any reduction has a principal subterm of sort 0. Hence the descendant of u by A also has no activator. Thus, we have $s_1 \to_{[p.(o\backslash q),l\to r]} t_1 \leftleftarrows_{t/q} t$.

Finally, observe that $s \to^{hd} t$ only in the cases 1 or 2. ∎

Lemma 3.30 *If $s \to t$, then $s{\downarrow\downarrow} \xrightarrow{*} t{\downarrow\downarrow}$; further, if $s \to^{hd} t$, then $s{\downarrow\downarrow} \xrightarrow{+}{}^{hd} t{\downarrow\downarrow}$.*

Proof. It easily follows from Lemma 3.26 and Lemma 3.29. ∎

Definition 3.31 *Let $s \to t$ be a destructive rewrite step at level $k > 1$. Then by Lemma 3.15 we have $s \equiv C[!u]$, $C[u_i] \equiv t$ and $u \to u_i$ for u properly active in s and u_i an activator of u in s. Let p be the peak of u in s. We say $s \to t$ critically destructive if s/p is root-active.*

Lemma 3.32 *Suppose $\mathcal{R}$ is terminating. Every infinite $\Theta(\mathcal{R})$-reduction sequence containing infinitely many head reductions contains infinitely many rewrite steps that are critically destructive.*

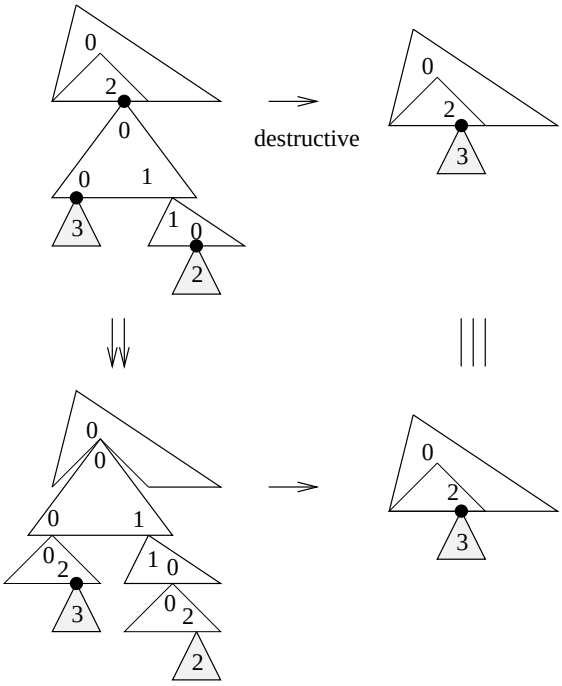

Figure 6: Destructive step from an active subterm

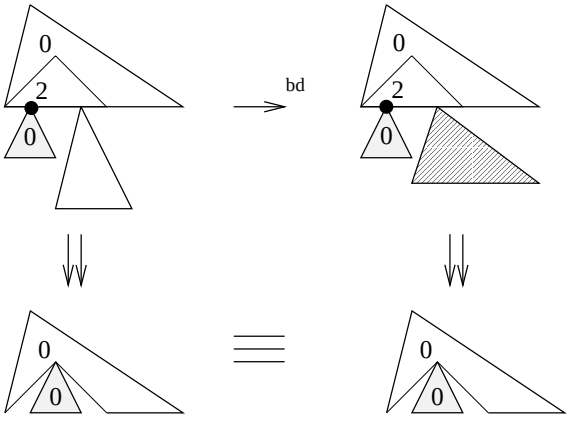

Figure 7: Elimination of a body reduction

Proof. Suppose that $D : s_1 \to s_2 \to s_3 \cdots$ is an infinite $\Theta(\mathcal{R})$-reduction sequence that contains infinitely many head reductions and only finitely many rewrite steps that are critically destructive. Without loss of generality, we may further assume that D contains no critically destructive step. We have $\mathrm{head}(s_i) \to \mathrm{head}(s_{i+1})$ whenever $s_i \to^{hd} s_{i+1}$. Suppose $s_i \to^{bd} s_{i+1}$ and $\mathrm{head}(s_i) \not\equiv \mathrm{head}(s_{i+1})$. Then there exists a principal subterm u of s_i and a principal subterm u_j of u such that $u \to u_j$ is destructive, $s_i \equiv C[u]$ and $s_{i+1} \equiv C[u_j]$, and there is no other principal subterm of u with the same peak as u_j. By Lemma 3.15, u is properly active and u_i is an activator of u, because $s_i \to^{bd} s_{i+1}$ is destructive at level 2. Hence s/p is root-active, where p is the peak of u in s. It thus follows that $s_i \to^{bd} s_{i+1}$ is critically destructive, which is not the case. Hence $s_i \to^{bd} s_{i+1}$ implies $\mathrm{head}(s_i) \equiv \mathrm{head}(s_{i+1})$. Therefore we know $\mathrm{head}(s_1) \xrightarrow{*} \mathrm{head}(s_2) \xrightarrow{*} \mathrm{head}(s_3) \cdots$ is an infinite $\mathcal{R}$-reduction sequence. But this is impossible as $\mathcal{R}$ is terminating. ∎

Lemma 3.33 *If $s \to t$ and s is $\rightrightarrows$-normal then $s \to t$ is not critically destructive.*

Proof. Suppose that $s \to t$ is critically destructive, i.e. $s \equiv C[!u] \to C[u_i] \equiv t$ and $u \equiv C'[\![u_1, \ldots, u_n]\!]$, with properly active subterm u in s and an activator u_i of u in s. Let p be the peak of u in s. Since s/p is root-active by definition, thus u is a $\rightrightarrows$-redex in s. ∎

Lemma 3.34 *If $\mathcal{R}$ is terminating then so is $\Theta(\mathcal{R})$.*

Proof. Suppose $\mathcal{R}$ is terminating. Let $s_0 \to s_1 \to \cdots$ be an infinite $\Theta(\mathcal{R})$-reduction sequence. Without loss of generality, we can assume that there are infinitely many head reductions in it. Then by Lemma 3.30, we have a reduction sequence $s_0{\downarrow\downarrow} \xrightarrow{*} s_1{\downarrow\downarrow} \xrightarrow{*} \cdots$, which contains infinitely many head reductions. By Lemma 3.32, it must contain infinitely many critically destructive rewrite steps. As $s_i{\downarrow\downarrow}$ is $\rightrightarrows$-normal, this contradicts Lemma 3.33. ∎

Now, Theorem 3.1 immediately follows from Lemma 3.2 and Lemma 3.34.

Example 3.35 *Let*

$$\mathcal{R} \left\{ \begin{array}{l} f(g(A), g(B), x) \to f(x, x, x) \\ h(x, y) \to x \\ h(x, y) \to y \end{array} \right. \qquad \left\{ \begin{array}{l} f : 0 \times 0 \times 0 \to 0 \\ g : 1 \to 0 \\ A, B : 1 \\ h : 0 \times 0 \to 0. \end{array} \right.$$

Then we have an infinite $\Theta(\mathcal{R})$-reduction sequence:

$$\begin{array}{ll} & f(g(A), g(B), g(h(A, B))) \\ \to & f(g(h(A, B)), g(h(A, B)), g(h(A, B))) \\ \to & f(g(A), g(h(A, B)), g(h(A, B))) \\ \to & f(g(A), g(B), g(h(A, B))) \\ \to & \cdots. \end{array}$$

Note the subterm $h(A,B)$ is a $\rightrightarrows$-redex in each terms. By simulating this reduction sequence by $\rightrightarrows$-normal terms, we have

$$\begin{array}{ll} & f(g(A),g(B),h(g(A),g(B))) \\ \to & f(h(g(A),g(B)),h(g(A),g(B)),h(g(A),g(B))) \\ \to & f(g(A),h(g(A),g(B)),h(g(A),g(B))) \\ \to & f(g(A),g(B),h(g(A),g(B))) \\ \to & \cdots, \end{array}$$

which is an infinite $\mathcal{R}$-reduction sequence.

4 Concluding Remarks

We have proved that for any terminating many-sorted TRS $\mathcal{R}$, if $\mathcal{R}$ contains only variables of the same sorts then its underlying TRS is also terminating. This is the positive solution to the problem of Zantema that has been appeared as Rewriting Open Problem 60 in [3].

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