Extending persistency of confluence with ordered sorts

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Abstract

A property $\mathcal{P}$ of TRSs is said to be persistent if for any TRS $\langle \mathcal{F}, \mathcal{R} \rangle$ and any sort attachment τ consistent with $\mathcal{R}$, $\langle \mathcal{F}^\tau, \mathcal{R}^\tau \rangle$ has property $\mathcal{P}$ iff $\langle \mathcal{F}, \mathcal{R} \rangle$ has property $\mathcal{P}$. Here $\langle \mathcal{F}^\tau, \mathcal{R}^\tau \rangle$ is the τ-sorted TRS induced from $\langle \mathcal{F}, \mathcal{R} \rangle$. In this paper, usual many-sorted TRS is extended with ordered sorts. It will be shown here that the persistency of confluence is preserved for this extension. Moreover, a new result on the modularity of confluence will be presented making use of this.

1 Introduction

Confluence is a property that is widely studied in the theory of term rewriting [4], [7],[10]. One of the approaches to guarantee confluence of a given TRS is to infer it from those of its subsystems. It is shown that confluence of the disjoint union of TRSs can be inferred from those of its components [9]. Following standard terminology, we say "a property $\mathcal{P}$ is *modular*" if $\mathcal{P}$ is inferred from those of its subsystems. However, unlike other important properties in the theory of term rewriting, only few results which relax the disjoint limitation of modularity of confluence are known [2], [5], [8].

A property $\mathcal{P}$ of TRSs is said to be persistent if for any many-sorted TRS $\langle \mathcal{F}, \mathcal{R} \rangle$, $\langle \mathcal{F}, \mathcal{R} \rangle$ has the property $\mathcal{P}$ iff its underlying unsorted TRS $\langle \Theta(\mathcal{F}), \Theta(\mathcal{R}) \rangle$ has the property $\mathcal{P}$. Here Θ denotes the sort elimination function. The notion of persistency is formulated by H. Zantema [11]. He showed some of persistent properties including confluence are also modular w.r.t. disjoint union. Thus the notion of persistency is thought to be an extension of the notion of modularity. Also he showed that termination is persistent for TRSs in which not both duplicating rules and collapsing rules occur, generalizing a similar result for modularity.

In [1], the present authors showed the persistency of confluence for TRSs, generalizing a similar result for modularity. The persistency of confluence is useful to infer confluence of a given TRS from those of its subsystems. In this paper, usual many-sorted TRS is extended with ordered sorts. It will be shown here that the persistency of confluence is preserved for this extension. Moreover, a new result on the modularity of confluence will be presented making use of this.

In section 2, many-sorted TRS is formulated on ordered sorts. A notion of sort attachment is also introduced on ordered sorts. Then the persistency of confluence on ordered sorts is shown in section 3 and 4. The proof is again a straightforward modification of a simplified proof of modularity of confluence [6]. In section 5, a modularity application of persistency of confluence on ordered sorts is presented. It is observed that the persistency of confluence on ordered sorts has wider application than that on unordered sorts. Also *a combination of hereditary systems with a common constructor part* is introduced and the modularity of confluence for this combination is shown by making use of the persistency of confluence on ordered sorts. We end with some reviews on the modularity of confluence in section 6.

2 Preliminaries

2.1 Sorted term rewriting systems

In this subsection, we introduce some basic notions of sorted term rewriting on ordered sorts and fix the notations that will be used in this paper.

Let $\mathcal{S}$ be a set of sorts and $\mathcal{V}$ be the set of countably infinite sorted variables. We assume that $\mathcal{S}$ is equipped with a well-founded strict partial order $\succ$. We write $\alpha \succeq \beta$ iff $\alpha \succ \beta$ or $\alpha = \beta$. Without loss of generality we assume there is the set $\mathcal{V}^\alpha$ of countably infinite variables of sort α for each sort $\alpha \in \mathcal{S}$. Let $\mathcal{F}$ be a set of sorted function symbols. We assume each sorted function symbol $f \in \mathcal{F}$ is given with the sorts of its arguments and the sort of its value, all of which are included in $\mathcal{S}$. We write suggestively $f : \alpha_1 \times \cdots \times \alpha_n \to \beta$ iff f takes n arguments of sorts $\alpha_1, \ldots, \alpha_n$ respectively to a value of sort β. Function symbols of 0 arguments are called *constants*. The set $\mathcal{T}(\mathcal{F}, \mathcal{V}) = \bigcup_{\alpha \in \mathcal{S}} \mathcal{T}(\mathcal{F}, \mathcal{V})^\alpha$ of all sorted terms built from $\mathcal{F}$ and $\mathcal{V}$ is defined inductively as follows: (1) $\mathcal{V}^\alpha \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V})^\alpha$; (2) if $f : \beta_1 \times$

$\cdots \times \beta_n \to \beta \in \mathcal{F}$, $t_i \in \mathcal{T}(\mathcal{F},\mathcal{V})^{\alpha_i}$ and $\alpha_i \preceq \beta_i$ for $i = 1,\ldots,n$ then $f(t_1,\ldots,t_n) \in \mathcal{T}(\mathcal{F},\mathcal{V})^{\beta}$. Here $\mathcal{T}(\mathcal{F},\mathcal{V})^{\alpha}$ denotes the set of all terms of sort α. We obtain the set of all *strict* sorted terms if (2) is replaced by (2') if $f : \beta_1 \times \cdots \times \beta_n \to \beta \in \mathcal{F}$, $t_i \in \mathcal{T}(\mathcal{F},\mathcal{V})^{\alpha_i}$, $\alpha_i \preceq \beta_i$ for $i = 1,\ldots,n$ and $\alpha_j = \beta_j$ whenever $t_j \in \mathcal{V}$, then $f(t_1,\ldots,t_n) \in \mathcal{T}(\mathcal{F},\mathcal{V})^{\beta}$. Identity of terms is denoted by $\equiv$. We write $t : \alpha$ to denote t is of sort α. We use $\mathcal{F}(t)$ and $\mathcal{V}(t)$ to denote the set of function symbols and variables that appear in t, respectively. If $\mathcal{F}$ is obvious from the context, $\mathcal{T}(\mathcal{F},\mathcal{V})^{\alpha}$ and $\mathcal{T}(\mathcal{F},\mathcal{V})$ is abbreviated as $\mathcal{T}^{\alpha}$ and $\mathcal{T}$, respectively.

A set of *sorted rewrite rules* on $\mathcal{T}$ is $\mathcal{R} \subseteq \mathcal{T} \times \mathcal{T}$ subject to constraints: for any $\langle l, r\rangle \in \mathcal{R}$, (1) $l \notin \mathcal{V}$; (2) $\mathcal{V}(r) \subseteq \mathcal{V}(l)$; (3) l and r are strict; (4) if $l : \alpha$ and $r : \beta$ then $\alpha \succeq \beta$. We conventionally write $l \to r \in \mathcal{R}$ instead of $\langle l, r\rangle \in \mathcal{R}$. If $r \in \mathcal{V}$ then the rewrite rule $l \to r$ is said to be *collapsing*. For $\mathcal{F}' \subseteq \mathcal{F}$, the set $\{l \to r \in \mathcal{R} \mid \mathcal{F}(l) \cup \mathcal{F}(r) \subseteq \mathcal{F}'\}$ is written as $\mathcal{R} \mid \mathcal{F}'$. A *sorted term rewriting system* (*STRS* for short) is a pair $\langle \mathcal{F}, \mathcal{R}\rangle$ where $\mathcal{F}$ is a set of sorted function symbols and $\mathcal{R}$ is a set of sorted rewrite rules on $\mathcal{T}(\mathcal{F},\mathcal{V})$. $\langle \mathcal{F}, \mathcal{R}\rangle$ is often abbreviated as $\mathcal{R}$, and in that case $\mathcal{F}$ is defined to be the set of function symbols that appear in $\mathcal{R}$.

Let $\Box^{\alpha}$ be a special constant (called a *hole*) of sort α. Elements of $\mathcal{T}(\mathcal{F} \cup \{\Box^{\alpha} \mid \alpha \in \mathcal{S}\}, \mathcal{V})$ is called *contexts* over $\mathcal{T}(\mathcal{F},\mathcal{V})$. If $\mathcal{F}$ is obvious from the context, $\mathcal{T}(\mathcal{F} \cup \{\Box^{\alpha} \mid \alpha \in \mathcal{S}\}, \mathcal{V})$ is written just as $\mathcal{C}$. We write suggestively $C : \alpha_1 \times \cdots \times \alpha_n \to \beta$ iff the sort of context C is β and it has n holes $\Box^{\alpha_1},\ldots,\Box^{\alpha_n}$ from left to right in it. If $C : \beta_1 \times \cdots \times \beta_n \to \beta$ and $t_1 : \alpha_1, \ldots, t_n : \alpha_n$ with $\alpha_1 \preceq \beta_1, \ldots, \alpha_n \preceq \beta_n$ then $C[t_1,\ldots,t_n]$ denotes the term obtained from C by replacing holes with $t_1,\ldots,t_n$ from left to right. The context that contains precisely one hole is denoted by $C[\;]$. A term t is said to be a *subterm (occurrence)* of s (written as $s \supseteq t$) iff $s \equiv C[t]$ for some $C \in \mathcal{C}$.

A *substitution* σ is a mapping from $\mathcal{V}$ to $\mathcal{T}$ subjects to a constraint: $x \in \mathcal{V}^{\alpha}$ implies $\sigma(x) \in \mathcal{T}^{\alpha}$. Substitutions are extended to homomorphisms from $\mathcal{T}$ to $\mathcal{T}$. For a substitution σ and a term t, $\sigma(t)$ is conventionally written as $t\sigma$.

Given a STRS $\mathcal{R}$, we write $s \to_{\mathcal{R}} t$ iff $s \equiv C[l\sigma]$ and $t \equiv C[r\sigma]$ for some $C \in \mathcal{C}$, $l \to r \in \mathcal{R}$ and substitution σ. We call $s \to_{\mathcal{R}} t$ a *rewrite step (from s to t) of* $\mathcal{R}$. The *redex* of this rewrite step is defined to be $l\sigma$. The relation $\to_{\mathcal{R}}$ on $\mathcal{T}$ is called *reduction of* $\mathcal{R}$. Then the following proposition holds.

PROPOSITION 2.1. If $s \to_{\mathcal{R}} t$ with $s : \alpha$ and $t : \beta$ then $\alpha \succeq \beta$.

Proof. Easy. $\Box$

The converse relation, the symmetric closure and the reflexive closure of $\to_{\mathcal{R}}$ are denoted by $\leftarrow_{\mathcal{R}}$, $\leftrightarrow_{\mathcal{R}}$ and $\xrightarrow{\equiv}_{\mathcal{R}}$, respectively. The transitive-reflexive closures of $\to_{\mathcal{R}}$, $\leftarrow_{\mathcal{R}}$ and $\leftrightarrow_{\mathcal{R}}$ are denoted by $\xrightarrow{*}_{\mathcal{R}}$, $\xleftarrow{*}_{\mathcal{R}}$ and $\xleftrightarrow{*}_{\mathcal{R}}$, respectively. We also say "s reduces to t" or "t is a reduct of s" if $s \xrightarrow{*}_{\mathcal{R}} t$. Terms t_1 and t_2 are said to be *joinable* if there exists some term t' such that $t_1 \xrightarrow{*}_{\mathcal{R}} t' \xleftarrow{*}_{\mathcal{R}} t_2$. A term t is said to be *confluent* iff any terms t_1 and t_2 are joinable whenever $t_1 \xleftarrow{*}_{\mathcal{R}} t \xrightarrow{*}_{\mathcal{R}} t_2$. A STRS $\mathcal{R}$ is said to be confluent iff any term is confluent with respect to its reduction $\rightarrow_{\mathcal{R}}$. If $\mathcal{R}$ is obvious from the context, the subscript $_{\mathcal{R}}$ is often abbreviated.

Usual TRSs are easily seen as special cases of STRSs, e.g. STRSs on $\mathcal{S} = \{*\}$. In this case $f \in \mathcal{F}_n$ is regarded as an abbreviation for $f : \overbrace{* \times \cdots \times *}^{n\text{-times}} \rightarrow *$. Moreover for each STRS $\langle \mathcal{F}, \mathcal{R} \rangle$, the sort elimination function Θ yields a TRS $\langle \Theta(\mathcal{F}), \Theta(\mathcal{R}) \rangle$ in which every sorts are identified with $*$.

2.2 Sorting of term rewriting systems

In this subsection, we introduce the notion of *sort attachment* to usual TRSs.

Let $\mathcal{S}$ be a set of sorts with well-founded strict partial order $\succ$ on it. $\mathcal{F}$ a set of arity-fixed function symbols. A *sort attachment* τ of $\mathcal{F}$ on $\mathcal{S}$ is a family of mapping $\bigcup_n (\mathcal{F}_n \rightarrow \mathcal{S}^{n+1}) \cup (\mathcal{V} \rightarrow \mathcal{S})$. Without loss of generality we can assume that there are countably infinite variables x with $\tau(x) = \alpha$ for each $\alpha \in \mathcal{S}$. If τ is obvious from its context, $f \in \mathcal{F}$ with $\tau(f) = \langle \alpha_1, \ldots, \alpha_n, \beta \rangle$ is written as $f : \alpha_1 \times \cdots \times \alpha_n \rightarrow \beta$. The set of τ-sorted function symbols from $\mathcal{F}$ and that of τ-sorted variables from $\mathcal{V}$ are denoted by $\mathcal{F}^{\tau}$ and $\mathcal{V}^{\tau}$, respectively. A term t is said to be *well-sorted under* τ *with sort* α (written as $\tau(t) = \alpha$) iff $t \in \mathcal{T}(\mathcal{F}^{\tau}, \mathcal{V}^{\tau})^{\alpha}$. Also, t is said to be *strictly well-sorted under* τ *with sort* α iff $t \in \mathcal{T}(\mathcal{F}^{\tau}, \mathcal{V}^{\tau})^{\alpha}$ is strict. Well-sorted contexts are defined similarly by introducing special constants $\square^{\alpha_1}, \square^{\alpha_2}, \ldots$. Well-sorted terms and contexts are often treated as sorted terms and contexts, respectively.

Let $\langle \mathcal{F}, \mathcal{R} \rangle$ be a TRS. It may be assumed that the variables have been renamed appropriately so that every rewrite rules in $\mathcal{R}$ share no variables. A sort attachment τ of $\mathcal{F}$ on $\mathcal{S}$ is said to be *consistent with* $\mathcal{R}$ iff for any rewrite rule $l \rightarrow r \in \mathcal{R}$, l and r are strictly well-sorted under τ and $\tau(l) \succeq \tau(r)$ holds. The set of τ-sorted rewrite rules of $\mathcal{R}$ is denoted by $\mathcal{R}^{\tau}$. It is obvious that for given $\langle \mathcal{F}, \mathcal{R} \rangle$, a sort attachment τ of $\mathcal{F}$ on $\mathcal{S}$ consistent with $\mathcal{R}$ uniquely induces a STRS $\langle \mathcal{F}^{\tau}, \mathcal{R}^{\tau} \rangle$ called *the* τ*-sorted STRS of* $\langle \mathcal{F}, \mathcal{R} \rangle$. Then the following proposition holds.

PROPOSITION 2.2. Let $\mathcal{R}$ be a TRS and τ a sort attachment that is consistent with $\mathcal{R}$. For any term s which is well-sorted under τ, if $s \rightarrow_{\mathcal{R}} t$ then t is also well-sorted under τ and $s \rightarrow_{\mathcal{R}^{\tau}} t$.

Proof. Easy. $\square$

Now persistency can be formulated as follows.

DEFINITION 2.3. A property $\mathcal{P}$ of TRSs is *persistent* iff for any TRS $\langle \mathcal{F}, \mathcal{R} \rangle$ and any sort attachment τ consistent with $\mathcal{R}$,

$$\langle \mathcal{F}^{\tau}, \mathcal{R}^{\tau} \rangle \text{ has property } \mathcal{P} \Leftrightarrow \langle \mathcal{F}, \mathcal{R} \rangle \text{ has property } \mathcal{P}.$$

In [11], persistency is formulated with a different definition via sort elimination function. But it is clear that two definitions are equivalent.

From now on, we assume that a set $\mathcal{S}$ of sorts with well-founded strict partial order $\succ$ on it and a TRS $\langle \mathcal{F}, \mathcal{R} \rangle$ are given and that a sort attachment τ of $\mathcal{F}$ on $\mathcal{S}$ that is consistent with $\mathcal{R}$ is fixed. Then our goal is to show

$$\langle \mathcal{F}^{\tau}, \mathcal{R}^{\tau} \rangle \text{ is confluent } \Leftrightarrow \langle \mathcal{F}, \mathcal{R} \rangle \text{ is confluent.}$$

The implication from right to left immediately follows from PROPOSITION 2.2. Thus we concentrate on the implication of the other direction. In the sequel, we assume confluence of terms that are well-sorted under τ and prove confluence of unsorted terms. In the following sections terms that are well-sorted under τ are often abbreviated just as well-sorted terms.

3 Characterization of unsorted terms

This section provides a characterization of unsorted terms via ordered sorts. Our goal is to show existence of preserved reducts as that in [6].

DEFINITION 3.1. [top sort] The *top sort (under τ)* of an unsorted term t is defined by

$$top(t) = \begin{cases} \tau(t) & \text{if } t \in \mathcal{V}, \\ \beta & \text{if } t \equiv f(t_1, \ldots, t_n) \text{ with } f : \alpha_1 \times \cdots \times \alpha_n \to \beta. \end{cases}$$

DEFINITION 3.2. [principal subterm] Let $t \equiv C[t_1, \ldots, t_n]$ $(n \geq 0)$ be an unsorted term with $C[, \ldots,] \not\equiv \Box$. We write $t \equiv C[\![t_1, \ldots, t_n]\!]$ iff

1. $C : \alpha_1 \times \cdots \times \alpha_n \to \beta$ is a context that is well-sorted under τ;
2. $top(t_i) \not\preceq \alpha_i$ for $i = 1, \ldots, n$.

The $t_1, \ldots, t_n$ are said to be the *principal* subterms of t.

It is clear that an unsorted term t is uniquely written as $C[\![t_1, \ldots, t_n]\!]$ for some $C, t_1, \ldots, t_n$. In the sequel, unsorted terms are often abbreviated just as terms.

DEFINITION 3.3. [inner and outer rewrite step] A rewrite step $s \to t$ is said to be *inner* (written as $s \to^i t$) iff

$$s \equiv C[\![s_1, \ldots, C'[l\sigma], \ldots, s_n]\!] \to C[s_1, \ldots, C'[r\sigma], \ldots, s_n] \equiv t$$

for some $s_1, \ldots, s_n$, $l \to r$, σ and C', otherwise *outer* (written as $s \to^o t$).

DEFINITION 3.4. [vanishing rewrite step] A rewrite step $s \to^o t$ is said to be *vanishing* iff $s \equiv C[\![s_1, \ldots, \sigma(x), \ldots, s_n]\!] \to^o \sigma(x) \equiv t$ for some $s_1, \ldots, s_n, C$ and σ such that $C'\sigma \equiv C[s_1, \ldots, \Box, \ldots, s_n]$ for some $C'[x] \to x \in \mathcal{R}$.

DEFINITION 3.5. [weakened rewrite step] A rewrite step $s \to^o t$ is said to be *weakened* iff $top(s) \succ top(t)$.

DEFINITION 3.6. [destructive rewrite step] A rewrite step $s \to^o t$ is said to be *destructive at level* 1 iff $s \to^o t$ is either vanishing or weakened. The rewrite step $s \to t$ is said to be *destructive at level* $k+1$ iff $s \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!] \to^i C[s_1, \ldots, t_j, \ldots, s_n] \equiv t$ with $s_j \to t_j$ destructive at level k.

PROPOSITION 3.7.

1. If a rewrite step $s \to^o t$ is not vanishing then $top(s) \succeq top(t)$.

2. If a rewrite step $s \to^o t$ is not destructive at level 1 then $top(s) = top(t)$.

Proof.

1. Suppose $top(s) \not\succeq top(t)$. By PROPOSITION 2.1 this can occur only in the case that this reduction is an application of a collapsing rule and whole term s is the redex of this reduction. Let the rule be $C'[x] \to x$. Then $s \equiv C'[x]\sigma$ and $t \equiv \sigma(x)$. By consistency $C'[x]$ is strict, hence $top(x) = \alpha$ whenever $C' : \alpha \to top(C'[x])$. Suppose $top(x) \succeq top(\sigma(x))$. Then $top(C'[x]\sigma) = top(C'[x]) \succeq top(x) \succeq top(\sigma(x))$. But this contradicts with our assumption. Therefore $\alpha \not\succeq top(\sigma(x))$ and $t \equiv \sigma(x)$ is a principal subterm of s. Hence this rewrite step is vanishing.

2. Suppose $top(s) \neq top(t)$. If $top(s) \succ top(t)$ then $s \to^o t$ is weakened. Otherwise $top(s) \not\succeq top(t)$. Then $s \to^o t$ is vanishing by 1. Hence $s \to^o t$ is destructive at level 1 in both case. $\Box$

DEFINITION 3.8. [special subterm] A term t is said to be a *level* 1 *special subterm* of t. If $t \equiv C[\![t_1, \dots, t_n]\!]$ then level k special subterms of $t_1, \dots, t_n$ are said to be *level* $k+1$ *special subterms* of t. The multiset of all level k special subterms of a term t is denoted by $S_k(t)$. Moreover $\sum_{k \geq 1} S_k(t)$ is denoted by $S(t)$.

It is clear that for any $s \in S(t)$, $S(s) \subseteq S(t)$ from the definition.

We write $t \equiv C\langle\!\langle t_1, \dots, t_n \rangle\!\rangle$ if either $t \equiv C[\![t_1, \dots, t_n]\!]$ or $C \equiv \Box$ and $t \equiv t_1$.

PROPOSITION 3.9. [Case analysis]

1. If $s \to^o t$ then

$$\begin{cases} s \equiv C[\![s_1, \dots, s_n]\!], \\ t \equiv C^*\langle\!\langle s_{i_1}, \dots, s_{i_m} \rangle\!\rangle \qquad (i_1, \dots, i_m \in \{1, \dots, n\}) \end{cases}$$

   for some $C, C^*, s_1, \dots, s_n$ and either

   (a) $n = m = 0$ and s, t is well-sorted;

   (b) $n \neq 0$, $s \to^o t$ is vanishing and $t \equiv s_{i_1}$;

   (c) $n \neq 0$, $s \to^o t$ is not vanishing and $t \equiv C^*[\![s_{i_1}, \dots, s_{i_m}]\!]$.

2. If $s \to^i t$ then

$$\begin{cases} s \equiv C[\![s_1, \dots, s_j, \dots, s_n]\!], \\ t \equiv C[s_1, \dots, t_j, \dots, s_n], \\ s_j \to t_j \end{cases}$$

   for some $C, s_1, \dots, s_n, t_j$ and either

   (a) $t \equiv C[\![s_1, \dots, t_j, \dots, s_n]\!]$, or

   (b) $t_j \equiv C'[\![u_1, \dots, u_l]\!]$ and $t \equiv C''[\![s_1, \dots, u_1, \dots, u_l, \dots, s_n]\!]$ for some C', C'' and level 2 or 3 special subterms $u_1, \dots, u_l$ ($l \geq 0$) of s_j.

Proof. Straightforward. Note that in case 1(c) we need the strictness of right-hand side of rewrite rules. $\Box$

The above proposition will be often used implicitly in the sequel.

PROPOSITION 3.10. If $s \equiv C[\![s_1, \dots, s_n]\!] \to^o C^*[\![t_1, \dots, t_m]\!]$ then for any t_i, there exists s_j such that $t_i \in S(s_j)$.

Proof. Straightforward. $\Box$

DEFINITION 3.11. [rank] The *rank* of a term t is defined by

$$rank(t) = \begin{cases} 1 & \text{if } t \text{ is well-sorted,} \\ 1 + max\{rank(t_i) \mid 1 \leq i \leq n\} & \text{if } t \equiv C[\![t_1, \ldots, t_n]\!] \text{ with } n \geq 1. \end{cases}$$

It is clear that for any $s \in \sum_{i \geq 2} S_i(t)$, $rank(t) > rank(s)$ from the definition.

PROPOSITION 3.12.

1. If $s \rightarrow t$ then $rank(s) \geq rank(t)$.
2. If a rewrite step $s \rightarrow t$ is vanishing then $rank(s) > rank(t)$.

Proof.

1. Our proof proceeds by induction on the rank of s. If $rank(s) = 1$, s is well-sorted. Then t is also well-sorted, hence $rank(t) = 1$. Suppose $rank(s) = n (> 1)$.

   (a) Case where $s \rightarrow^o t$. If $s \rightarrow^o t$ is vanishing then it follows from the definition of rank. Otherwise $s \equiv C[\![s_1, \ldots, s_n]\!] \rightarrow^o t \equiv C^*[\![s_{i_1}, \ldots, s_{i_m}]\!]$. Then

   $$\begin{array}{rcl} rank(s) & = & max\{rank(s_1), \ldots, rank(s_n)\} + 1 \\ & \geq & max\{rank(s_{i_1}), \ldots, rank(s_{i_m})\} + 1 \\ & = & rank(t). \end{array}$$

   (b) Case where $s \rightarrow^i t$. Then

   $$\begin{cases} s \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!], \\ t \equiv C[s_1, \ldots, t_j, \ldots, s_n], \\ s_j \rightarrow t_j \end{cases}$$

   for some $C, s_1, \ldots, s_n, t_j$.

   i. Case where $t \equiv C[\![s_1, \ldots, t_j, \ldots, s_n]\!]$. Then

   $$\begin{array}{rcl} rank(s) & = & max\{rank(s_1), \ldots, rank(s_j), \ldots, rank(s_n)\} + 1 \\ & \geq & max\{rank(s_1), \ldots, rank(t_j), \ldots, rank(s_n)\} + 1 \\ & & \quad \text{by induction hypothesis} \\ & = & rank(t) \end{array}$$

ii. Case where $t_j \equiv C[\![u_1, \ldots, u_l]\!]$ and $t \equiv C''[\![s_1, \ldots, u_1, \ldots, u_l, \ldots, s_n]\!]$ for some C', C'' and level 2 or 3 special subterms $u_1, \ldots, u_l$ of s_j. By definition, $rank(t_j) > rank(u_i)$ for all $i = 1, \ldots, n$. Then

$$\begin{aligned}
rank(s) &= max\{rank(s_1), \ldots, rank(s_j), \ldots, rank(s_n)\} + 1 \\
&\geq max\{rank(s_1), \ldots, rank(t_j), \ldots, rank(s_n)\} + 1 \\
&\quad \text{by induction hypothesis} \\
&\geq max\{rank(s_1), \ldots, rank(u_1), \ldots, rank(u_l), \\
&\qquad \ldots, rank(s_n)\} + 1 \\
&= rank(t)
\end{aligned}$$

2. By definition. □

DEFINITION 3.13. [grade] The *grade* ($grade(t) \in \mathbb{N} \times \mathcal{S}$) of a term t is defined by $grade(t) = \langle rank(t), top(t) \rangle$.

Let $>$ be the lexicographic order on $\mathbb{N} \times \mathcal{S}$ induced from from $>$ on $\mathbb{N}$ and $\succ$ on $\mathcal{S}$.

PROPOSITION 3.14.

1. If $s \to t$ then $grade(s) \geq grade(t)$.
2. If a rewrite step $s \to t$ is destructive at level 1 then $grade(s) > grade(t)$.

Proof.

1. By PROPOSITION 3.12, $rank(s) \geq rank(t)$. If $rank(s) > rank(t)$ then it follows from the definition. Otherwise $rank(s) = rank(t)$. Then $s \to t$ can not be vanishing by PROPOSITION 3.12. Then by PROPOSITION 3.7, $top(s) \succeq top(t)$.
2. If $s \to t$ is vanishing then $rank(s) > rank(t)$ by PROPOSITION 3.12. Otherwise $s \to t$ is not vanishing, and $s \to t$ is weakened. Hence $rank(s) \geq rank(t)$ by PROPOSITION 3.12 and $top(s) \succ top(t)$ by definition. □

PROPOSITION 3.15. Let $s \to t$ be a rewrite step that is destructive at level n. Then there exist a level n special subterm s' of s such that $s \equiv C[s']$ and a term t' such that $s' \to t'$ destructive at level 1 and $t \equiv C[t']$.

Proof. Our proof proceeds by induction on the level of destruction. If the level is 1 then it is obvious since $s \in S_1(s)$. Suppose that $s \to t$ is a rewrite step destructive at level $k+1$. Then $s \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!] \to C[s_1, \ldots, t_j, \ldots, s_n] \equiv t$ with $s' \to t'$ destructive at level k. Then by induction hypothesis, there exists a level k special subterm s' of s_j such that $s_j \equiv C'[s']$ and a term t' such that $s' \to t'$ destructive at level 1 and $t_j \equiv C'[t']$. But by definition of the special subterms, s' is a level $k+1$ special subterm of s. Moreover, $s \equiv C[\![s_1, \ldots, C'[s'], \ldots, s_n]\!]$ and $t \equiv C[s_1, \ldots, C'[t'], \ldots, s_n]$. □

PROPOSITION 3.16. Let $s \to t$ be a rewrite step. If $t' \in S(t)$ then there exists $s' \in S(s)$ such that $s' \overset{\equiv}{\to} t'$.

Proof. Let $t' \in S_n(t)$. Then our proof proceeds by induction on n. It is obvious if $n = 1$. Suppose $k \geq 1$ and $t' \in S_{k+1}(t)$.

1. Case where $s \to^o t$. Then t can not be well-sorted, since $n > 1$. Thus we can let $s \equiv C[\![s_1, \ldots, s_n]\!]$, $t \equiv C^*[\![t_1, \ldots, t_m]\!]$ and $t' \in S(t_i)$. Then by PROPOSITION 3.10, $t_i \in S(s_j)$ for some j. Thus the statement follows since $t' \in S(t_i) \subseteq S(s_j) \subseteq S(s)$.

2. Case where $s \to^i t$. Then $s \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!], t \equiv C[s_1, \ldots, t_j, \ldots, s_n]$ and $s_j \to t_j$ for some $C, s_1, \ldots, s_n, t_j$.

   (a) Case where $t \equiv C[\![s_1, \ldots, t_j, \ldots, s_n]\!]$. If $t' \in S_k(s_i)$ $(i \neq j)$ then the statement follows since $S(s_i) \subseteq S(s)$. Otherwise $t' \in S_k(t_j)$. Then since $s_j \to t_j$ and $t' \in S_k(t_j)$, there exists $s' \in S(s_j)$ such that $s' \overset{\equiv}{\to} t'$ by induction hypothesis. Thus the statement follows since $S(s_j) \subseteq S(s)$.

   (b) Case where $t_j \equiv C[\![u_1, \ldots, u_l]\!]$ and $t \equiv C''[\![s_1, \ldots, u_1, \ldots, u_l, \ldots, s_n]\!]$ for some C', C'' and level 2 or 3 special subterms $u_1, \ldots, u_l$ of s_j. Then the statement clearly follows since $t' \in S(s)$ for any $t' \in S(t)$. □

PROPOSITION 3.17. Let $s \equiv C[s'] \to C[t'] \equiv t$ be a rewrite step with $s' \to t'$ and $s' \in S(s)$. If $s' \to t'$ is not destructive at level 1 then $t' \in S(t)$.

Proof. If $s' \to^i t'$ then $top(s') = top(t')$ by definition. Also if $s' \to^o t'$ then $top(s') = top(t')$ by PROPOSITION 3.7. Thus $top(s') = top(t')$ and the statement immediately follows from this. □

DEFINITION 3.18. [preservation] A term s is said to be *preserved* iff there are no reduction sequences starting from s that contain a destructive rewrite step.

Definition 3.19. [collapsing reduction] We write $s \to_c t$ iff there exist a special subterm s' of s such that $s \equiv C[s']$ and a term t' such that $s' \xrightarrow{*} t'$, $t \equiv C[t']$ and $s' \xrightarrow{*} t'$ contains a level 1 destructive rewrite step. The relation $\to_c$ is called *collapsing reduction* and s' is called the *collapsing redex.*

Proposition 3.20.

1. If $s \to_c t$ then $s \xrightarrow{*} t$.

2. A term is preserved iff it contains no collapsing redex.

Proof. 1. By definition, $s \equiv C[s']$, $t \equiv C[t']$ and $s' \xrightarrow{*} t'$ for some s', t', C. Since $s' \xrightarrow{*} t'$, $s \equiv C[s'] \xrightarrow{*} C[t'] \equiv t$ by the definition of reduction.

2. (If)-part. Assume there exists a reduction sequence starting from s that contains a destructive rewrite step, i.e. $s \xrightarrow{*} t \to u$ and $t \to u$ is destructive. Then by Proposition 3.15, there exist a special subterm t' of t such that $t \equiv C'[t']$ and a term u' such that $t' \to u'$ is destructive at level 1 and $u \equiv C'[u']$. Then repeated applications of Proposition 3.16 yield $s' \in S(s)$ with $s' \xrightarrow{*} t' \to u'$. Obviously s' is a collapsing redex in s. (Only if)-part. Suppose $s \to_c t$. Then by definition, there exists a special subterm s' of s such that $s \equiv C[s']$ and a term t' such that $s' \xrightarrow{*} t'$, $t \equiv C[t']$ and $s' \xrightarrow{*} t'$ contains a level 1 destructive rewrite step. Let the first destructive step in $s' \xrightarrow{*} t'$ be $u' \to v'$. Then since $s' \in S(s)$, $u' \in S(C[u'])$ by Proposition 3.17. Therefore $C[u'] \to C[v']$ is destructive. Hence $s \equiv C[s'] \xrightarrow{*} C[u'] \to C[v'] \xrightarrow{*} C[t'] \equiv t$ contains a destructive rewrite step. □

In the following proposition, we use the fact that $\succ$ is well-founded.

Proposition 3.21. Every term has a preserved reduct.

Proof. We show there can not be infinite collapsing reduction sequences. Then the proposition immediately follows by Proposition 3.20.

For any term t, assign multiset $\|t\|$ as: $\|t\| = [grade(s) \mid s \in S(t)]$. Then we show $s \to_c t$ implies $\|s\| \gg \|t\|$. Here $\gg$ is the multiset extension of the ordering $>$.

Suppose $s \to_c t$. Then by definition, there exist a special subterm s' of s such that $s \equiv C[s']$ and a term t' such that $s' \xrightarrow{*} t'$ contains a level 1 destructive rewrite step and $t \equiv C[t']$. Then by Proposition 3.14, $grade(s') > grade(t')$. Hence $\|s'\| \gg \|t'\|$. From this follows $\|s\| \gg \|t\|$, since this s' is a special subterm of s.

It is easily seen that $>$ on $\mathbb{N} \times \mathcal{S}$ is well-founded since both of $>$ on $\mathbb{N}$ and $\succ$ on $\mathcal{S}$ are well-founded. Then the relation $\gg$ is well-founded (e.g. see [3]) and hence there can not be infinite collapsing reduction sequences. □

4 Persistency of confluence

In this section, the persistency of confluence on ordered sorts is proved. The main difference from [1] is PROPOSITION 4.2.

LEMMA 4.1. Let t be strictly well-sorted term. Suppose $t \equiv C[x_1, \ldots, x_n]$ with all variables displayed and $C : \alpha_1 \times \cdots \times \alpha_n \to \beta$. Then for any substitution σ, if $x_i \equiv x_j$ and $\sigma(x_i)$ is a principal subterm of $t\sigma$ then $\sigma(x_j)$ is also a principal subterm of $t\sigma$.

Proof. By strictness of t, $x_i \equiv x_j$ implies $\alpha_i = \alpha_j$. Then since $\sigma(x_i) \equiv \sigma(x_j)$ and $\sigma(x_i)$ is a principal subterm of $t\sigma$, $\sigma(x_j)$ is also a principal subterm of $t\sigma$. □

Let $s_1, \ldots, s_n$ and $t_1, \ldots, t_n$ be terms. We write $\langle s_1, \ldots, s_n\rangle \propto \langle t_1, \ldots, t_n\rangle$ if for any $1 \leq i, j \leq n$, $s_i \equiv s_j$ implies $t_i \equiv t_j$. Moreover, we write $\langle s_1, \ldots, s_n\rangle \infty \langle t_1, \ldots, t_n\rangle$ iff $\langle s_1, \ldots, s_n\rangle \propto \langle t_1, \ldots, t_n\rangle$ and $\langle t_1, \ldots, t_n\rangle \propto \langle s_1, \ldots, s_n\rangle$.

PROPOSITION 4.2. If $C[\![s_1, \ldots, s_n]\!] \to^o C'\langle\!\langle s_{i_1}, \ldots, s_{i_m}\rangle\!\rangle$ and $\langle s_1, \ldots, s_n\rangle \propto \langle t_1, \ldots, t_n\rangle$ then $C[t_1, \ldots, t_n] \to^o C'[t_{i_1}, \ldots, t_{i_m}]$.

Proof. Let the redex of this reduction be $l\sigma$. If $l\sigma$ does not contain any principal terms then it is obvious. Else let $l\sigma \equiv C^*[\![s_{j_1}, \ldots, s_{j_l}]\!]$. It suffices to show $C^*[t_{j_1}, \ldots, t_{j_l}]$ is again a redex. But this straightforwardly follows using LEMMA 4.1 and strictness of l. □

The above proposition does not hold without strictness of l. Suppose $f(x, x) \to c$ is a rewrite rule with $f : 0 \times 1 \to 1$ and let $a : 1$. Then $f(a, a) \equiv C[\![a]\!]$ with $C[\ ] \equiv f(\Box, a)$. Thus $\langle a\rangle \propto \langle c\rangle$ and $f(a, a) \to_{\mathcal{R}} c$ but not $f(a, c) \to_{\mathcal{R}} c$.

PROPOSITION 4.3. If rewrite steps $t_1 \overset{*}{\leftarrow}{}^o\ t \overset{*}{\rightarrow}{}^o\ t_2$ do not contain a destructive rewrite step except the last steps then t_1 and t_2 are joinable.

Proof. We may write $t \equiv C[\![s_1, \ldots, s_n]\!]$, $t_1 \equiv C_1\langle\!\langle s_{i_1}, \ldots, s_{i_m}\rangle\!\rangle$ and $t_2 \equiv C_2\langle\!\langle s_{j_1}, \ldots, s_{j_p}\rangle\!\rangle$ by PROPOSITION 3.9. Let $C : \alpha_1 \times \cdots \times \alpha_n \to \beta$. Then choose fresh variables $x_i \in \mathcal{V}_{\alpha_i}(i = 1, \ldots, n)$ such that $\langle x_1, \ldots, x_n\rangle \infty \langle s_1, \ldots, s_n\rangle$. Let $t' \equiv C[x_1, \ldots, x_n]$, $t'_1 \equiv C_1[x_{i_1}, \ldots, x_{i_m}]$ and $t'_2 \equiv C_2[x_{j_1}, \ldots, x_{j_p}]$. Then obviously t' is well-sorted. Moreover repeated applications of PROPOSITION 2.2 and PROPOSITION 4.2 yield well-sorted rewrite steps $t'_1 \overset{*}{\leftarrow} t' \overset{*}{\rightarrow} t'_2$. Therefore t'_1 and t'_2 are joinable, i.e. $t'_1 \overset{*}{\rightarrow} C^*[x_{k_1}, \ldots, x_{k_q}] \overset{*}{\leftarrow} t'_2$ for some C^*. Instantiating the valley yields $t_1 \overset{*}{\rightarrow}{}^o\ C^*\langle\!\langle s_{k_1}, \ldots, s_{k_q}\rangle\!\rangle \overset{*}{\leftarrow}{}^o\ t_2$. □

Definition 4.4. [representation] Let S be a set of confluent terms. A set $\widehat{S}$ of terms *represents* S iff the following two conditions are satisfied:

1. every term s in S has a unique reduct $\widehat{s}$ in $\widehat{S}$, which will be called the *representative of* s;
2. joinable terms in S have the same representative in $\widehat{S}$.

Proposition 4.5. Every finite set S of confluent terms can be represented.

Proof. Since S consists of confluent terms, joinability is an equivalence relation on S. Since S is finite, each equivalence class is also finite and has a common reduct. It is obvious that the set of such common reducts represents S. □

Definition 4.6. [inner preservation] A term s is said to be *inner preserved* iff its all principal subterms are preserved.

Note that principal subterms of an inner preserved term are also inner preserved.

Proposition 4.7. Any reduction sequence starting from an inner preserved term has at most one vanishing rewrite step.

Proof. Let $s \equiv C[\![s_1, \ldots, s_n]\!]$ be inner preserved term. If $s \xrightarrow{*} s'$ does not contain a vanishing rewrite step then s' is of form $s' \equiv C'[\![s'_{i_1}, \ldots, s'_{i_m}]\!]$ for respective reduct $s'_{i_1}, \ldots, s'_{i_m}$ of $s_{i_1}, \ldots, s_{i_m}$ by repeated applications of Proposition 3.9. Thus any reduct s'_j of vanishing step is a reduct of some principal subterm s_j of s. Since s_j is preserved, there is no reduction sequences starting from s_j that contains a destructive rewrite step. Hence there are no reduction sequences from s'_j that contain a vanishing rewrite step. □

Lemma 4.8. Inner preserved terms are confluent.

Proof. We show that any inner preserved term t is confluent by induction on the rank of t. If $rank(t) = 1$ then t is a well-sorted term, hence confluent. Suppose that $t \equiv C[\![s_1, \ldots, s_n]\!]$ is an inner preserved term with $rank(t) = n(> 1)$ and $t_1 \xleftarrow{*} t \xrightarrow{*} t_2$.

Firstly note that vanishing rewrite steps occur in $t \xrightarrow{*} t_i \quad (i = 1, 2)$ only at most once by Proposition 4.7.

Define a set $\mathcal{L}(t \xrightarrow{*} t_i)$, say the set of lower parts of $t \xrightarrow{*} t_i$, $(i = 1, 2)$ as follows.

1. Suppose $t \xrightarrow{*} t_i$ does not contain a vanishing rewrite step. Then let T be the set of all terms that appear between $t \xrightarrow{*} t_i$. Then define

$$\mathcal{L}(t \xrightarrow{*} t_i) = \{u \mid u \text{ is a principal subterm of some } s \in T\}.$$

2. Otherwise $t \xrightarrow{*} t_i$ contains a vanishing rewrite step, i.e. there exists some t_i', t_i'' such that $t \xrightarrow{*} t_i' \to t_i'' \xrightarrow{*} t_i$, $t_i' \to t_i''$ is vanishing. Note that $t \xrightarrow{*} t_i'$ and $t_i'' \xrightarrow{*} t_i$ do not contain a vanishing rewrite step. Let T be the set of all terms that appear between $t \xrightarrow{*} t_i'$. Then define

$$\begin{array}{rcll}\mathcal{L}(t \xrightarrow{*} t_i) & = & & \{u \mid u \text{ is a principal subterm of some } s \in T\} \\ & & \cup & \{v \mid v \text{ appears between } t_i'' \xrightarrow{*} t_i\}.\end{array}$$

Let S be $\mathcal{L}(t \xrightarrow{*} t_1) \cup \mathcal{L}(t \xrightarrow{*} t_2)$. Then for any elements $s \in S$,

1. s is also inner preserved, because s is preserved; and
2. s have less rank than n by definition of rank and Proposition 3.12.

Therefore we can apply induction hypothesis to each element of S, hence S is a set of confluent terms. Then by Proposition 4.5, S has a representation $\widehat{S}$.

Next we define $\tilde{u}$ for each term u that appears in $t_1 \xleftarrow{*} t \xrightarrow{*} t_2$ as follows. If u is a term that appears before the vanishing rewrite step occurs then define $\tilde{u}$ to be a term obtained from u by replacing all its principal subterms with its representative. Otherwise define $\tilde{u}$ to be the representative of u. Note that $u \xrightarrow{*} \tilde{u}$ in each case by definition of representation.

Then we can show $\tilde{u_1} \xrightarrow{\equiv}{}^{o} \tilde{u_2}$ for any $u_1 \to u_2$ not preceded by the vanishing rewrite step and $\tilde{u_1} \equiv \tilde{u_2}$ for any $u_1 \to u_2$ that is preceded by the vanishing rewrite step as follows.

1. Let $u_1 \to u_2$ be a rewrite step in $t_1 \xleftarrow{*} t \xrightarrow{*} t_2$ not preceded by vanishing rewrite step.
   (a) Case where $u_1 \to^o u_2$. Then $u_1 \equiv C[\![s_1, \ldots, s_n]\!]$ and $u_2 \equiv C'\langle\!\langle s_{i_1}, \ldots, s_{i_m}\rangle\!\rangle$. It is easily seen $\tilde{u_1} \equiv C[\![\widehat{s_1}, \ldots, \widehat{s_n}]\!]$ and $\tilde{u_2} \equiv C'\langle\!\langle \widehat{s_{i_1}}, \ldots, \widehat{s_{i_m}}\rangle\!\rangle$. Since $\langle s_1, \ldots, s_n\rangle \propto \langle \widehat{s_1}, \ldots, \widehat{s_n}\rangle$ by definition of representation, $\tilde{u_1} \to^o \tilde{u_2}$ by Proposition 4.2.
   (b) Case where $u_1 \to^i u_2$. Then since t is inner preserved, $u_1 \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!]$, $u_2 \equiv C[\![s_1, \ldots, s_j', \ldots, s_n]\!]$ and $s_j \to s_j'$ by Proposition 3.9. Then since $\widehat{s_j} \equiv \widehat{s_j'}$, $\tilde{u_1} \equiv \tilde{u_2}$.

2. Let $u_1 \to u_2$ be a rewrite step that is preceded by the vanishing rewrite step. Then $u_1, u_2 \in S$ and $\widehat{u_1} \equiv \widehat{u_2}$ by the definition of representation, hence $\tilde{u_1} \equiv \tilde{u_2}$.

From this, we conclude that $\tilde{t_1} \overset{*}{\leftarrow}{}^{o} \tilde{t} \overset{*}{\rightarrow}{}^{o} \tilde{t_2}$ do not contain a vanishing rewrite step except the last steps. Therefore $\tilde{t_1}$ and $\tilde{t_2}$ are joinable by PROPOSITION 4.3, hence so are t_1 and t_2. See also figure 1. $\square$

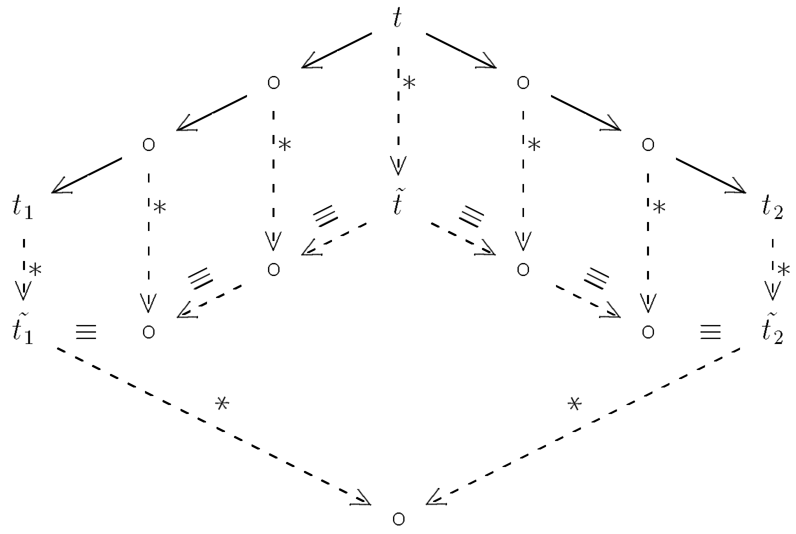

Figure 1: LEMMA 4.8

DEFINITION 4.9. [witness] Let $s \equiv C[\![s_1, \ldots, s_n]\!]$ be a term. Then a *witness* of s is a term $t \equiv C[t_1, \ldots, t_n]$ which satisfies: (1) t is inner preserved; (2) $s_i \overset{*}{\to} t_i$ for $i = 1, \ldots, n$; and (3) $\langle s_1, \ldots, s_n \rangle \propto \langle t_1, \ldots, t_n \rangle$.

PROPOSITION 4.10. Every term has a witness.

Proof. Let $s \equiv C[\![s_1, \ldots, s_n]\!]$. By PROPOSITION 3.21, there exists a preserved reduct t_i of s_i for any $i = 1, \ldots, n$. Taking the same preserved term for identical terms ensures $\langle s_1, \ldots, s_n \rangle \propto \langle t_1, \ldots, t_n \rangle$. Moreover any principal term of $t \equiv C[t_1, \ldots, t_n]$ is a subterm of some t_i. But every t_i is preserved, so are all the principal terms of t. Thus t is a witness of s. $\square$

In the following, an arbitrary witness of a term s is denoted by $\dot{s}$.

LEMMA 4.11. Let s, t be terms such that $s \to t$. If all the principal subterms of s are confluent then $\dot{s}$ and $\dot{t}$ are joinable.

Proof. Let $s \equiv C[\![s_1, \ldots, s_n]\!]$ and $\dot{s} \equiv C[u_1, \ldots, u_n]$ for respective reducts $u_1, \ldots, u_n$ of $s_1, \ldots, s_n$.

1. Case where $s \to^o t$. Then we may write $t \equiv C^*\langle\!\langle s_{i_1}, \ldots, s_{i_m}\rangle\!\rangle$. Suppose $\dot{t} \equiv C^*[t_{i_1}, \ldots, t_{i_m}]$ for respective reducts $t_{i_1}, \ldots, t_{i_m}$ of $s_{i_1}, \ldots, s_{i_m}$. Since $s \equiv C[\![s_1, \ldots, s_n]\!] \to^o C^*\langle\!\langle s_{i_1}, \ldots, s_{i_m}\rangle\!\rangle \equiv t$ and $\langle s_1, \ldots, s_n\rangle \propto \langle u_1, \ldots, u_n\rangle$, $\dot{s} \equiv C[u_1, \ldots, u_n] \to C^*[u_{i_1}, \ldots, u_{i_m}]$ by PROPOSITION 4.2. Now for $j = i_1, \ldots, i_m$, since $u_j \overset{*}{\leftarrow} s_j \overset{*}{\to} t_j$ and s_j is a principal subterm of s, u_j and t_j are joinable by assumption. Therefore $\dot{s}$ and $\dot{t}$ are joinable. See figure 2.

$$\begin{tikzcd}
s \equiv C[\![s_1, \ldots, s_n]\!] \arrow[r] \arrow[d, "*"] & C^*\langle\!\langle s_{i_1}, \ldots, s_{i_m}\rangle\!\rangle \equiv t \arrow[d, "*"] \\
\dot{s} \equiv C[u_1, \ldots, u_n] \arrow[d, dashed] & C^*[t_{i_1}, \ldots, t_{i_m}] \equiv \dot{t} \arrow[d, dashed, "*"] \\
C^*[u_{i_1}, \ldots, u_{i_m}] \arrow[r, dashed, "*"] & {}
\end{tikzcd}$$

Figure 2: LEMMA 4.11 Case 1.

2. Case where $s \to^i t$. Then $t \equiv C[s_1, \ldots, s'_j, \ldots, s_n]$ and $s_j \to s'_j$. It is easily seen that $\dot{t} \equiv C[t_1, \ldots, t_n]$ for some reducts $t_1, \ldots, t_n$ of $s_1, \ldots, s'_j, \ldots, s_n$. Now for $k = 1, \ldots, n$, since $u_k \overset{*}{\leftarrow} s_k \overset{*}{\to} t_k$ and s_k is a principal subterm of s, u_k and t_k are joinable by assumption. Therefore $\dot{s} \equiv C[\![u_1, \ldots, u_n]\!]$ and $\dot{t} \equiv C[t_1, \ldots, t_n]$ are joinable. □

THEOREM 4.12. Confluence is persistent property of TRSs on ordered sorts.

Proof. Our proof proceeds by induction on the rank. If $rank(t) = 1$ then t is a well-sorted term, hence confluent. Suppose $rank(t) = n (> 1)$ and $t_1 \overset{*}{\leftarrow} t \overset{*}{\to} t_2$. Note that any term that appears in $t_1 \overset{*}{\leftarrow} t \overset{*}{\to} t_2$ has the rank less than or equal to n by PROPOSITION 3.12, and therefore its principal subterms have ranks less than n, hence confluent by induction hypothesis. Now, by PROPOSITION 4.10, any term that appears in $t_1 \overset{*}{\leftarrow} t \overset{*}{\to} t_2$ has a witness. Then repeated applications of LEMMA 4.11 yield a conversion between $\dot{t}_1$ and $\dot{t}_2$ by way of witnesses. Since witnesses are inner preserved, we can apply LEMMA 4.8 repeatedly so that $\dot{t}_1$ and $\dot{t}_2$ are joinable. Hence so are t_1 and t_2. See figure 3. □

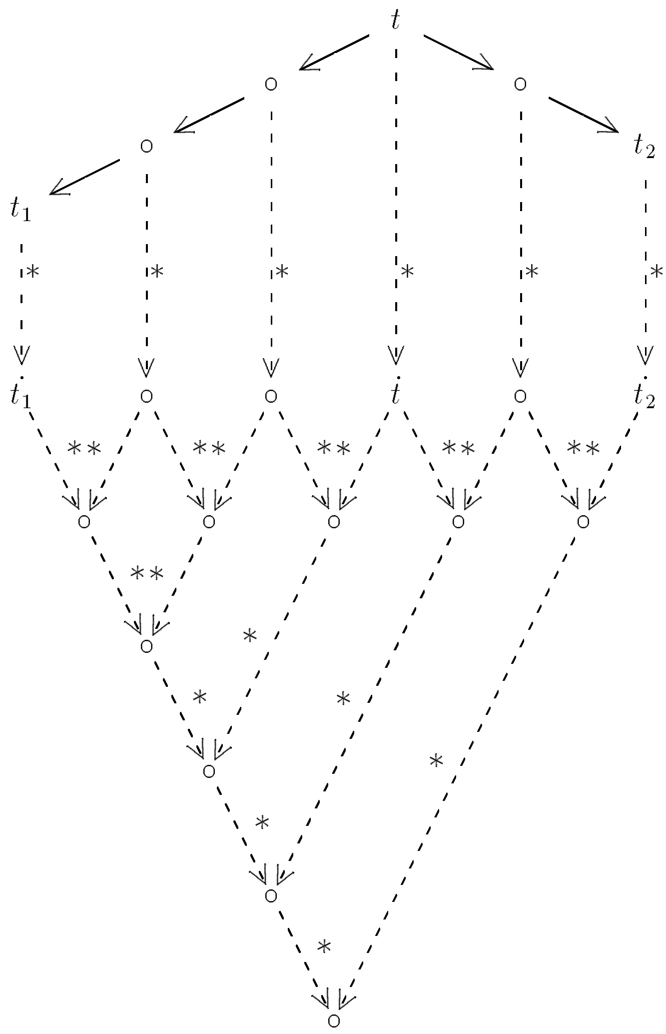

Figure 3: Theorem 4.12

5 Modularity application

The persistency of confluence is useful to infer confluence of a given TRS from those of its subsystems. The persistency of confluence on ordered sorts extend the modularity application of persistency in some extent. Below we will explain this with a simple example.

Let $\mathcal{R}$ be

$$\begin{cases} f(x,A) \to G(x) & (r1) \\ f(x,f(x,y)) \to y & (r2) \\ G(C) \to C & (r3) \\ F(x) \to F(G(x)) & (r4). \end{cases}$$

To show the confluence of this system directly seems difficult from known results, since this system is neither terminating nor left-linear. It is easily seen that the modularity for disjoint union can not apply. Also we can neither apply results of [2], [5] nor [8].

Let $\mathcal{S}$ be a set $\{0,1,2\}$ of ordered sorts with $1 \succ 0$ and $2 \succ 0$ and no other relations. Think of sort attachment τ:

$$\begin{cases} A : 1 \\ f : 0 \times 1 \to 1 \\ G : 0 \to 0 \\ C : 0 \\ F : 0 \to 2. \end{cases}$$

It is easily checked this is consistent with $\mathcal{R}$.

Now, $\mathcal{R}^\tau$ is confluent iff any well-sorted term in $\mathcal{T}$ is confluent, i.e. any term in $\mathcal{T}^0$, $\mathcal{T}^1$ or $\mathcal{T}^2$ is confluent.

- Since rules applicable to terms in $\mathcal{T}^0$ are $(r3)$, they are confluent.
- Since rules applicable to terms in $\mathcal{T}^1$ are $(r1),(r2),(r3)$, they are confluent since it is terminating and there are no critical pairs.
- Since rules applicable to terms in $\mathcal{T}^2$ are $(r3),(r4)$, they are confluent since it is orthogonal.

Hence $\mathcal{R}^\tau$ is confluent. Then confluence of $\mathcal{R}$ follows by persistency of confluence on ordered sorts. Moreover, readers easily convince that we can not divide it into subsystems by any unordered sort attachments.

5.1 Combination of hereditary systems with a common constructor part

In this subsection, we present a syntactical condition for the modularity of confluence making use of the persistency of confluence on ordered sorts.

DEFINITION 5.1. [parents] The set of *parents* of a term t in a term s is defined as $parents(t,s) = \{f \in \mathcal{F}(s) \mid s \supseteq f(\ldots,t,\ldots)\}$.

DEFINITION 5.2. [hereditary TRS] Let $\langle \mathcal{F}, \mathcal{R} \rangle$ be a TRS. If $\mathcal{F}$ is partitioned into disjoint sets $\mathcal{D}, \mathcal{M}, \mathcal{C}$ such that

1. for any $x \in \mathcal{V}(l)$, either $parents(x,l) \subseteq \mathcal{D}$ or $parents(x,l) \subseteq \mathcal{M} \cup \mathcal{C}$;
2. for any $x \in \mathcal{V}(l)$, if $parents(x,l) \subseteq \mathcal{D}$ then $parents(x,r) \subseteq \mathcal{D}$;
3. if $root(l) \in \mathcal{C}$ then $root(r) \in \mathcal{C} \cup \mathcal{V}$;
4. for any $f(t_1,\ldots,t_n) \subseteq l$ or $f(t_1,\ldots,t_n) \subseteq r$, if $f \in \mathcal{M} \cup \mathcal{C}$ then $root(t_1)\ldots, root(t_n) \in \mathcal{C} \cup \mathcal{V}$;

for any $l \to r \in \mathcal{R}$, then $\langle \mathcal{F}, \mathcal{R} \rangle$ is said to be a *hereditary TRS*. Elements of $\mathcal{D}$ are called defined symbols. Elements of $\mathcal{M}$ are called minor defined symbols. Elements of $\mathcal{C}$ are called constructors.

DEFINITION 5.3. A combination of TRSs $\langle \mathcal{F}_1, \mathcal{R}_1 \rangle$ and $\langle \mathcal{F}_2, \mathcal{R}_2 \rangle$ such that

1. both are hereditary systems with constructors $\mathcal{F}_1 \cap \mathcal{F}_2$; and
2. $\mathcal{R}_1 \mid (\mathcal{F}_1 \cap \mathcal{F}_2) = \mathcal{R}_2 \mid (\mathcal{F}_1 \cap \mathcal{F}_2)$

is said to be a *combination of hereditary systems with a common constructor part*. Then the system $\langle \mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_1 \cup \mathcal{R}_2 \rangle$ is said to be a *combined hereditary system with a common constructor part.*

Then we have the following corollary.

COROLLARY 5.4. Confluence is modular for combinations of hereditary systems with a common constructor part.

Proof. Let $\langle \mathcal{D}_1 \cup \mathcal{M}_1 \cup \mathcal{C}, \mathcal{R}_1 \rangle$ and $\langle \mathcal{D}_2 \cup \mathcal{M}_2 \cup \mathcal{C}, \mathcal{R}_2 \rangle$ be hereditary TRSs with a common constructor part.

Let $\mathcal{S}$ be a set $\{0,1,2\}$ of ordered sorts with $1 \succ 0$ and $2 \succ 0$ and no other relations. Think of sort attachment τ:

- $\tau(f) = i \times \cdots \times i \to i$ for $f \in \mathcal{D}_i$ $(i = 1,2)$;
- $\tau(f) = 0 \times \cdots \times 0 \to i$ for $f \in \mathcal{M}_i$ $(i = 1,2)$;
- $\tau(f) = 0 \times \cdots \times 0 \to 0$ for $f \in \mathcal{C}$.

It is clear that this sort attachment is consistent with the combined system.

Now, $\mathcal{R}^\tau$ is confluent iff any well-sorted term in $\mathcal{T}$ is confluent, i.e. any term in $\mathcal{T}^0$, $\mathcal{T}^1$ or $\mathcal{T}^2$ is confluent. Since rules applicable to terms in $\mathcal{T}^i$ are $\mathcal{R}_i$, they are confluent by assumption ($i = 1, 2$). It remains to show confluence of terms in $\mathcal{T}^0$. But this immediately follows from confluence of $\mathcal{R}_1$ (or $\mathcal{R}_2$), since 0 is an minimal sort in $\mathcal{S}$ and hence closed under reduction. □

Example 5.5. Let $\mathcal{R}$ be

$$\begin{cases} D(x,x) \to x & (r1) \\ D(x,G(x)) \to C & (r2) \\ G(C) \to C & (r3) \\ F(x) \to F(F(C)) & (r4). \end{cases}$$

This is a combination of hereditary systems with a common constructor part by letting $\mathcal{M}_1 = \{D\}, \mathcal{D}_2 = \{F\}, \mathcal{C} = \{G, C\}, \mathcal{D}_1 = \mathcal{M}_2 = \emptyset$. Hence divided into $\{(r1),(r2),(r3)\}$ and $\{(r3),(r4)\}$. But $\{(r1),(r2),(r3)\}$ is terminating and all critical pairs are joinable, hence confluent. Also $\{(r3),(r4)\}$ is orthogonal, hence confluent. Thus the whole system is confluent.

Remark 5.6. The above syntactical condition do not make full use of the persistency of confluence on ordered sorts. For example, we can not use sort attachment of $f : 0 \times 1 \to 1$ as is in the first example in this section.

6 Concluding remarks

We have extended the persistency of confluence [1] to that on ordered sorts. Making use of this, we have provided a new result on the modularity of confluence.

The present authors developed a top-down labelling technique and showed that modularity results for disjoint union of TRSs are lifted to two kinds of combinations of TRSs [2]. The modularity of confluence for those combination includes the modularity of confluence for constructor systems with neither collapsing nor constructor-lifting rules [8] and the modularity application of persistency of confluence on unordered sorts [1]. But it does not include the modularity application of persistency of confluence on ordered sorts. It is a fascinating question whether it is also possible to give a simpler proof of the modularity application of persistency of confluence on ordered sorts by lifting the modularity of confluence for disjoint union of TRSs.

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