Persistency of confluence

Takahito Aoto
aoto@jaist.ac.jp

Yoshihito Toyama
toyama@jaist.ac.jp

School of Information Science
Japan Advanced Institute of Science and Technology,
Hokuriku
Asahidai Tatsunokuchi, Ishikawa 923-12
Japan

Abstract

A property $\mathcal{P}$ of TRSs is said to be persistent if for any many-sorted TRS $\langle \mathcal{F}, \mathcal{R} \rangle$, $\langle \mathcal{F}, \mathcal{R} \rangle$ has the property $\mathcal{P}$ iff its underlying unsorted TRS $\langle \Theta(\mathcal{F}), \Theta(\mathcal{R}) \rangle$ has the property $\mathcal{P}$. Here Θ denotes the sort elimination function. Persistency of confluence will be shown here. This can be obtained by a straightforward modification of the proof of modularity of confluence, whose details are given in this report.

1 Introduction

Confluence is a property that is widely studied in the theory of term rewriting [5],[2],[9]. One of the approaches to guarantee confluence of a given TRS is to infer it from those of its subsystems. Y. Toyama showed that confluence of the disjoint sum of TRSs can be inferred from those of its components [8]. Following standard terminology, we say "a property $\mathcal{P}$ is *modular*" if $\mathcal{P}$ is inferred from those of its components. However, unlike other important properties in the theory of term rewriting, only few results which relax the disjoint limitation for modularity of confluence is known [6], [3].

A property $\mathcal{P}$ of TRSs is said to be persistent if for any many-sorted TRS $\langle \mathcal{F}, \mathcal{R} \rangle$, $\langle \mathcal{F}, \mathcal{R} \rangle$ has the property $\mathcal{P}$ iff its underlying unsorted TRS $\langle \Theta(\mathcal{F}), \Theta(\mathcal{R}) \rangle$ has the property $\mathcal{P}$. Here Θ denotes the sort elimination function. The notion of persistency is formulated by H. Zantema in [10]. He showed some of persistent properties including confluence are also modular w.r.t. disjoint union. Thus the notion of persistency is thought to be an extension of the notion of modularity. Also he showed that termination is

persistent for TRSs in which not both duplicating rules and collapsing rules occur, generalizing a similar result for modularity.

In this report, we will show persistency of confluence for TRSs, generalizing a similar result for modularity. Persistency of confluence is useful to infer confluence of a given TRS from those of its subsystems. It also overcomes the disjoint limitation to some extent. We will exhibit this in the last section.

Although no proofs of persistency of confluence have been appeared in the literature, it was conjectured to be feasible by H. Zantema, Y. Toyama and A. Middeldorp (in a private communication). It is also mentioned in [10] that the straightforward modification of Toyama's proof in [8] seems applicable to show persistency of confluence. In this report, we rather show it by a straightforward modification of a simplified proof of modularity of confluence in [4].

2 Preliminaries

2.1 Sorted term rewriting systems

In this subsection, we review some basic notions of sorted term rewriting and fix the notations that will be used in this report.

Let $\mathcal{S}$ be a set of sorts and $\mathcal{V}$ be the set of countably infinite sorted variables. Without loss of generality we assume there is the set $\mathcal{V}^\alpha$ of countably infinite variables of sort α for each sort $\alpha \in \mathcal{S}$. Let $\mathcal{F}$ be a set of sorted function symbols. We assume each sorted function symbol $f \in \mathcal{F}$ is given with the sorts of its arguments and the sort of its value, all of which are included in $\mathcal{S}$. We write suggestively $f : \alpha_1 \times \cdots \times \alpha_n \to \beta$ iff f takes n arguments of sorts $\alpha_1, \ldots, \alpha_n$ respectively to a value of sort β. Function symbols with no arguments are called *constants*. The set $\mathcal{T}(\mathcal{F}, \mathcal{V})$ of all sorted terms built from $\mathcal{F}$ and $\mathcal{V}$ is defined inductively as follows: (1) $\mathcal{V}^\alpha \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V})^\alpha$; (2) If $f : \alpha_1 \times \cdots \times \alpha_n \to \beta \in \mathcal{F}$ and $t_1 \in \mathcal{T}(\mathcal{F}, \mathcal{V})^{\alpha_1}, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})^{\alpha_n}$ then $f(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{F}, \mathcal{V})^\beta$. Here $\mathcal{T}(\mathcal{F}, \mathcal{V})^\alpha \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V})$ denotes the set of all terms of sort α. We write $t : \alpha$ iff the term t is of sort α. We use $\mathcal{V}(t)$ to denote the set of all variables that appear in t. Identity of terms is denoted by $\equiv$. If $\mathcal{F}$ is obvious from the context, $\mathcal{T}(\mathcal{F}, \mathcal{V})^\alpha$ and $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is abbreviated as $\mathcal{T}^\alpha$ and $\mathcal{T}$, respectively.

A set of rewrite rules on $\mathcal{T}$ is $\mathcal{R} \subseteq \bigcup_\alpha \mathcal{T}^\alpha \times \mathcal{T}^\alpha$ subject to constraints: (1) $l \notin \mathcal{V}$; (2) $\mathcal{V}(r) \subseteq \mathcal{V}(l)$; for any $\langle l, r \rangle \in \mathcal{R}$. We conventionally write $l \to r \in \mathcal{R}$ instead of $\langle l, r \rangle \in \mathcal{R}$. A *sorted term rewriting system* (*STRS* for short) is a pair $\langle \mathcal{F}, \mathcal{R} \rangle$ where $\mathcal{F}$ is a set of sorted function symbols and $\mathcal{R}$ is a set of rewrite rules on $\mathcal{T}(\mathcal{F}, \mathcal{V})$. $\langle \mathcal{F}, \mathcal{R} \rangle$ is often abbreviated as $\mathcal{R}$, and in that case $\mathcal{F}$ is defined to be the set of function symbols that appear in $\mathcal{R}$.

Let $\Box^\alpha$ be a special constant (called a *hole*) of sort α. Elements of $\mathcal{T}(\mathcal{F} \cup \{\Box^\alpha \mid \alpha \in \mathcal{S}\}, \mathcal{V})$ is called *contexts* over $\mathcal{T}(\mathcal{F}, \mathcal{V})$. If $\mathcal{F}$ is obvious from the context, $\mathcal{T}(\mathcal{F} \cup \{\Box^\alpha \mid \alpha \in \mathcal{S}\}, \mathcal{V})$ is written just as $\mathcal{C}$. We write suggestively $C : \alpha_1 \times \cdots \times \alpha_n \to \beta$ iff the sort of C is β and it has n holes $\Box^{\alpha_1}, \ldots, \Box^{\alpha_n}$ from left to right in it. If $C : \alpha_1 \times \cdots \times \alpha_n \to \beta$ and $t_1 : \alpha_1, \ldots, t_n : \alpha_n$ then $C[t_1, \ldots, t_n]$ denotes the term obtained from C by replacing holes with $t_1, \ldots, t_n$ from left to right. The context that contains precisely one hole is denoted by $C[\ ]$. A term t is said to be a *subterm (occurrence)* of s iff $s \equiv C[t]$ for some $C \in \mathcal{C}$.

A *substitution* σ is a mapping from $\mathcal{V}$ to $\mathcal{T}$ subjects to constraints: $x \in \mathcal{V}^\alpha$ implies $\sigma(x) \in \mathcal{T}^\alpha$. Substitutions are extended to homomorphisms from $\mathcal{T}$ to $\mathcal{T}$. For a substitution σ and a term t, $\sigma(t)$ is conventionally written as $t\sigma$.

Given a STRS $\mathcal{R}$, we write $s \to_\mathcal{R} t$ iff $s \equiv C[l\sigma]$ and $t \equiv C[r\sigma]$ for some $C \in \mathcal{C}$, $l \to r \in \mathcal{R}$ and substitution σ. We call $s \to_\mathcal{R} t$ a *rewrite step (from s to t) of $\mathcal{R}$*. We also say "*s reduces to t*" or "*t is a reduct of s*" in this case. The *redex* of this rewrite step is defined to be $l\sigma$. The relation $\to_\mathcal{R}$ on $\mathcal{T}$ is called *reduction of $\mathcal{R}$* . Then the following proposition holds.

PROPOSITION 2.1. If $s \to_\mathcal{R} t$ and $s : \alpha$ then $t : \alpha$.

Proof. Easy. □

The converse relation, the symmetric closure and the reflexive closure of $\to_\mathcal{R}$ are denoted by $\leftarrow_\mathcal{R}$, $\leftrightarrow_\mathcal{R}$ and $\xrightarrow{\equiv}_\mathcal{R}$, respectively. The transitive-reflexive closures of $\to_\mathcal{R}$, $\leftarrow_\mathcal{R}$ and $\leftrightarrow_\mathcal{R}$ are denoted by $\xrightarrow{*}_\mathcal{R}$, $\xleftarrow{*}_\mathcal{R}$ and $\xleftrightarrow{*}_\mathcal{R}$, respectively. Terms t_1 and t_2 are said to be *joinable* if there exists some term t' such that $t_1 \xrightarrow{*}_\mathcal{R} t' \xleftarrow{*}_\mathcal{R} t_2$. A term t is said to be *confluent* iff any terms t_1 and t_2 are joinable whenever $t_1 \xleftarrow{*}_\mathcal{R} t \xrightarrow{*}_\mathcal{R} t_2$. A STRS $\mathcal{R}$ is said to be confluent iff any term is confluent with respect to its reduction $\to_\mathcal{R}$. If $\mathcal{R}$ is obvious from the context, the subscript $_\mathcal{R}$ is often abbreviated.

Usual (unsorted) TRSs are easily seen as special cases of STRSs, e.g. STRSs on $\mathcal{S} \equiv \{*\}$. In this case $f \in \mathcal{F}_n$ is regarded as an abbreviation for $f : \overbrace{* \times \cdots \times *}^{n\text{-times}} \to *$. Moreover for each STRS $\langle \mathcal{F}, \mathcal{R} \rangle$, the sort elimination function Θ yields a TRS $\langle \Theta(\mathcal{F}), \Theta(\mathcal{R}) \rangle$ in which every sorts are identified with $*$.

The following definition is due to [10].

DEFINITION 2.2. A property $\mathcal{P}$ of STRSs is said to be *persistent* iff

$\langle \mathcal{F}, \mathcal{R} \rangle$ has the property $\mathcal{P}$
$\Leftrightarrow$ the induced TRS $\langle \Theta(\mathcal{F}), \Theta(\mathcal{R}) \rangle$ has the property $\mathcal{P}$.

2.2 Sorting of term rewriting systems

In this subsection, we introduce the notion of *sort attachment* to usual (unsorted) TRSs.

Let $\mathcal{S}$ be a set of sorts and $\mathcal{F}$ be a set of (unsorted) arity-fixed function symbols. An *attachment* τ of $\mathcal{F}$ on $\mathcal{S}$ is a family of mapping $\bigcup_n(\mathcal{F}_n \to \mathcal{S}^{n+1}) \cup (\mathcal{V} \to \mathcal{S})$. Without loss of generality we can assume that there are countably infinite variables x with $\tau(x) = \alpha$ for each $\alpha \in \mathcal{S}$. If τ is obvious from its context, $f \in \mathcal{F}$ with $\tau(f) = \langle \alpha_1, \ldots, \alpha_n, \beta \rangle$ is written as $f : \alpha_1 \times \cdots \times \alpha_n \to \beta$. The set of τ-sorted function symbols from $\mathcal{F}$ is denoted by $\mathcal{F}^\tau$.

DEFINITION 2.3. Let t be an unsorted term and τ an attachment. Then t is said to be *well-sorted under τ with sort α* iff $t : \alpha$ is derived by the following rules:

$$\frac{}{x : \tau(x)}$$

$$\frac{\tau(f) = \langle \alpha_1, \ldots, \alpha_n, \beta \rangle \quad t_1 : \alpha_1 \quad \cdots \quad t_n : \alpha_n}{f(t_1, \ldots, t_n) : \beta.}$$

It is clear that if t is well-sorted under τ then its sort α is uniquely defined. Thus α is called *the sort of t under τ*.

The similar definition for contexts is proceeded with an additional rule:

$$\frac{}{\Box : \alpha.}$$

Again, we write $C : \alpha_1 \times \cdots \times \alpha_n \to \beta$ iff C is a context with n holes derived to be $C : \beta$ with $\Box : \alpha_1, \ldots, \Box : \alpha_n$ from left to right in it. Well-sorted terms and contexts are often treated as sorted terms and contexts, respectively.

Let $\langle \mathcal{F}, \mathcal{R} \rangle$ be a TRS. It may be assumed that the variables have been renamed appropriately so that every rewrite rules in $\mathcal{R}$ share no variables. An attachment τ of $\mathcal{F}$ on $\mathcal{S}$ is said to be *consistent with* $\mathcal{R}$ iff for any rewrite rule $\langle l, r \rangle \in \mathcal{R}$, l, r are well-sorted under τ with the restriction that l and r are derived to be of the same sort. The set of τ-sorted rewrite rules of $\mathcal{R}$ is denoted by $\mathcal{R}^\tau$. It is obvious that for given $\langle \mathcal{F}, \mathcal{R} \rangle$, an attachment τ of $\mathcal{F}$ on $\mathcal{S}$ consistent with $\mathcal{R}$ uniquely induces a STRS $\langle \mathcal{F}^\tau, \mathcal{R}^\tau \rangle$ called *the τ-sorted TRS of* $\langle \mathcal{F}, \mathcal{R} \rangle$. Then the following proposition holds.

PROPOSITION 2.4. Let $\mathcal{R}$ be a TRS and τ an attachment that is consistent with $\mathcal{R}$. For any terms s and t which are well-sorted under τ,

$$s \to_{\mathcal{R}} t \Leftrightarrow s \to_{\mathcal{R}^\tau} t.$$

Proof. Easy. □

Using attachments, persistency can be alternatively formulated as follows.

PROPOSITION 2.5. A property $\mathcal{P}$ of TRSs is persistent iff for any TRS $\langle \mathcal{F}, \mathcal{R} \rangle$ and for any attachment τ consistent with $\mathcal{R}$,

$$\langle \mathcal{F}^\tau, \mathcal{R}^\tau \rangle \text{ has property } \mathcal{P} \Leftrightarrow \langle \mathcal{F}, \mathcal{R} \rangle \text{ has property } \mathcal{P}.$$

Proof. Easy. □

We will prove persistency of confluence making use of sort attachments instead of the sort elimination function.

From now on, we assume that a set $\mathcal{S}$ of sort and a TRS $\langle \mathcal{F}, \mathcal{R} \rangle$ are given and that an attachment τ of $\mathcal{F}$ on $\mathcal{S}$ that is consistent with $\mathcal{R}$ is fixed. Then our goal is to show

$$\langle \mathcal{F}^\tau, \mathcal{R}^\tau \rangle \text{ is confluent } \Leftrightarrow \langle \mathcal{F}, \mathcal{R} \rangle \text{ is confluent.}$$

The implication from right to left immediately follows from PROPOSITION 2.1 and PROPOSITION 2.4. Thus we concentrate on the implication of the other direction. In the sequel, we assume confluence of well-sorted terms and prove confluence of unsorted terms. In the following sections unsorted terms are often abbreviated just as terms.

3 Characterization of unsorted terms

DEFINITION 3.1. [top sort] The top sort of a term t is defined by

$$top(t) = \begin{cases} \tau(t) & \text{if } t \in \mathcal{V} \\ \beta & \text{if } t \equiv f(t_1, \ldots, t_n) \text{ with } f : \alpha_1 \times \cdots \times \alpha_n \to \beta. \end{cases}$$

DEFINITION 3.2. [principal subterm] Let $t \equiv C[t_1, \ldots, t_n]$ $(n \geq 0)$ be a term with $C[, \ldots,] \not\equiv \Box$. We write $t \equiv C[\![t_1, \ldots, t_n]\!]$ iff

1. $C : \alpha_1 \times \cdots \times \alpha_n \to \beta$;
2. $top(t_i) \neq \alpha_i$ for $i = 1, \ldots, n$.

The $t_1, \ldots, t_n$ are said to be the *principal* subterms of t.

It is clear that a term t is uniquely written as $C[\![t_1, \ldots, t_n]\!]$ for some $C, t_1, \ldots, t_n$.

DEFINITION 3.3. [inner and outer rewrite step] A rewrite step $s \to t$ is said to be *inner* (written as $s \to^i t$) iff

$$s \equiv C[\![s_1, \ldots, C'[l\sigma], \ldots, s_n]\!] \to C[s_1, \ldots, C'[r\sigma], \ldots, s_n] \equiv t$$

for some $s_1, \ldots, s_n$, $l \to r$, σ and C', otherwise *outer* (written as $s \to^o t$).

DEFINITION 3.4. [destructive rewrite step] A rewrite step $s \to^o t$ is said to be *destructive at level* 1 iff $top(s) \neq top(t)$. The rewrite step $s \to t$ is said to be *destructive at level* $k+1$ iff $s \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!] \to^i C[s_1, \ldots, t_j, \ldots, s_n] \equiv t$ with $s_j \to t_j$ destructive at level k.

PROPOSITION 3.5. A rewrite step $s \to^o t$ is destructive at level 1 iff $s \equiv C[\![s_1, \ldots, \sigma(x), \ldots, s_n]\!] \to^o \sigma(x) \equiv t$ for some $s_1, \ldots, s_n, C$ and σ such that $C[s_1, \ldots, \Box, \ldots, s_n] \equiv C'\sigma$ for some $C'[x] \to x \in \mathcal{R}$.

Proof. (If)-part. Suppose $top(s) = top(t)$ i.e. $top(C[s_1, \ldots, \sigma(x), \ldots, s_n]) = top(\sigma(x))$. Since $C' \not\equiv \Box$ by definition of rewrite rules, $top(C'[x]) = top(\sigma(x))$. Hence by consistency, $top(x) = top(C'[x]) = top(\sigma(x))$. Thus $\sigma(x)$ can not be principal. (Only if)-part. Suppose $top(s) \neq top(t)$. By consistency, this can occur only in the case that this reduction is an application of a collapsing rule and whole term s is the redex of this reduction. Let the rule be $C'[x] \to x$. Then $s \equiv C'[x]\sigma$ and $t \equiv \sigma(x)$. By consistency, $C'[x]$ is well-sorted and $top(C'[x]) = top(x)$. But $top(C'[x]) = top(C'[x]\sigma) = top(s) \neq top(t) = top(\sigma(x))$ by assumption. Hence $top(x) \neq top(\sigma(x))$. Thus $\sigma(x)$ is a principal subterm of s. □

This proposition shows that a destructive rewrite step occurs only when the applied rule is collapsing, and that the reduct of a destructive rewrite step indeed come from one of the principal subterms.

We write $t \equiv C\langle\!\langle t_1, \ldots, t_n \rangle\!\rangle$ if either $t \equiv C[\![t_1, \ldots, t_n]\!]$ or $C \equiv \Box$ and $t \equiv t_1$.

PROPOSITION 3.6. [Case analysis]

1. If $s \to^o t$ then

$$\begin{cases} s \equiv C[\![s_1, \ldots, s_n]\!], \\ t \equiv C^*\langle\!\langle s_{i_1}, \ldots, s_{i_m} \rangle\!\rangle \qquad (i_1, \ldots, i_m \in \{1, \ldots, n\}) \end{cases}$$

for some $C, C^*, s_1, \ldots, s_n$ and either

(a) $n = m = 0$ and s, t is well-sorted;

(b) $n \neq 0$ and $s \to^o t$ is destructive (at level 1) and $C^* \equiv \Box$ and $t \equiv s_j$ for some $1 \leq j \leq n$; or

(c) $n \neq 0$ and $s \to^o t$ is not destructive (at level 1) and $t \equiv C^*[\![s_{i_1}, \ldots, s_{i_m}]\!]$.

2. If $s \to^i t$ then
$$\begin{cases} s \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!], \\ t \equiv C[s_1, \ldots, t_j, \ldots, s_n], \\ s_j \to t_j \end{cases}$$
for some $C, s_1, \ldots, s_n, t_j$ and either

(a) $s_j \to t_j$ is destructive at level 1, t_j is a principal subterm of s_j and either

i. $t \equiv C[\![s_1, \ldots, t_j, \ldots, s_n]\!]$, or

ii.
$$\begin{cases} t_j \equiv C'[\![u_1, \ldots, u_l]\!] \\ t \equiv C''[\![s_1, \ldots, u_1, \ldots, u_l, \ldots, s_n]\!], \end{cases}$$
for some $C', C'', u_1, \ldots, u_l$; or

(b) $s_j \to t_j$ is not destructive at level 1 and $t \equiv C[\![s_1, \ldots, t_j, \ldots, s_n]\!]$.

Proof. Straightforward. □

The above proposition will be often used implicitly in the sequel.

DEFINITION 3.7. [rank] The *rank* of a term t is defined by

$$rank(t) = \begin{cases} 1 & \text{if } t \text{ is well-sorted,} \\ 1 + max\{rank(t_i) \mid 1 \leq i \leq n\} & \text{if } t \equiv C[\![t_1, \ldots, t_n]\!] \text{ with } n \geq 1. \end{cases}$$

PROPOSITION 3.8.

1. If $s \to t$ then $rank(s) \geq rank(t)$;

2. If a rewrite step $s \to t$ is destructive at level 1 then $rank(s) > rank(t)$.

Proof.

1. Our proof proceeds by induction on the rank of s. If $rank(s) = 0$ then s is well-sorted. By PROPOSITION 2.1, t is also well-sorted. Hence $rank(t) = 0$. Suppose $rank(s) = n(> 1)$. Then $rank(t) \leq n$ is shown as follows.

(a) Case where $s \to^o t$. If $s \to^o t$ is destructive then it follows from the definition of rank. Otherwise $s \equiv C[\![s_1, \ldots, s_n]\!] \to^o t \equiv C^*[\![s_{i_1}, \ldots, s_{i_m}]\!]$. Then

$$\begin{aligned} rank(s) &= max\{rank(s_1), \ldots, rank(s_n)\} + 1 \\ &\geq max\{rank(s_{i_1}), \ldots, rank(s_{i_m})\} + 1 \\ &= rank(t). \end{aligned}$$

(b) Case where $s \to^i t$. Then

$$\begin{cases} s \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!], \\ t \equiv C[s_1, \ldots, t_j, \ldots, s_n], \\ s_j \to t_j \end{cases}$$

for some $C, s_1, \ldots, s_n, t_j$.

i. Case where $t \equiv C[\![s_1, \ldots, t_j, \ldots, s_n]\!]$. Then

$$\begin{aligned} rank(s) &= max\{rank(s_1), \ldots, rank(s_j), \ldots, rank(s_n)\} + 1 \\ &\geq max\{rank(s_1), \ldots, rank(t_j), \ldots, rank(s_n)\} + 1 \\ &\quad \text{by induction hypothesis} \\ &= rank(t) \end{aligned}$$

ii. Otherwise $t_j \equiv C'[\![u_1, \ldots, u_l]\!]$ and $t \equiv C''[\![s_1, \ldots, u_1, \ldots, u_l, \ldots, s_n]\!]$ for some $C', C'', u_1, \ldots, u_l$. By definition, $rank(t_j) > rank(u_i)$ for all $i = 1, \ldots, l$. Then

$$\begin{aligned} rank(s) &= max\{rank(s_1), \ldots, rank(s_j), \ldots, rank(s_n)\} + 1 \\ &\geq max\{rank(s_1), \ldots, rank(t_j), \ldots, rank(s_n)\} + 1 \\ &\quad \text{by induction hypothesis} \\ &\geq max\{rank(s_1), \ldots, rank(u_1), \ldots, rank(u_l), \\ &\qquad \ldots, rank(s_n)\} + 1 \\ &= rank(t). \end{aligned}$$

2. Since $s \to t$ is destructive at level 1, $s \equiv C[\![s_1, \ldots, s_n]\!] \to s_j \equiv t$. Then the statement follows from the definition of rank. □

Note that 2 does not always hold if the level is n ($n > 1$). Also note that non-destructive steps do not alway preserve the rank.

DEFINITION 3.9. [special subterm] A term t is said to be a *level* 1 *special subterm* of t. If $t \equiv C[\![t_1, \ldots, t_n]\!]$ then level k special subterms of $t_1, \ldots, t_n$ are said to be *level* $k+1$ *special subterms* of t. The multiset of all level k special subterms of a term t is denoted by $S_k(t)$. Moreover $\sum_{k \geq 1} S_k(t)$ is denoted by $S(t)$.

Note that $S_2(t)$ denotes the multiset of principal terms of t.

PROPOSITION 3.10. Let $s \to t$ be a rewrite step that is destructive at level n. Then there exist a level n special subterm s' of s such that $s \equiv C[s']$ and a term t' such that $s' \to t'$ destructive at level 1 and $t \equiv C[t']$.

Proof. Our proof proceeds by induction on the level of destruction. If the level is 1 then it is obvious since $s \in S_1(s)$. Suppose that $s \to t$ is a rewrite step destructive at level $k+1$. Then $s \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!] \to C[s_1, \ldots, t_j, \ldots, s_n] \equiv t$ with $s' \to t'$ destructive at level k. Then by induction hypothesis, there exists a level k special subterm s' of s_j such that $s_j \equiv C'[s']$ and a term t' such that $s' \to t'$ destructive at level 1 and $t_j \equiv C'[t']$. But by definition of the special subterms, s' is a level $k+1$ special subterm of s. Moreover, $s \equiv C[\![s_1, \ldots, C'[s'], \ldots, s_n]\!]$ and $t \equiv C[s_1, \ldots, C'[t'], \ldots, s_n]$. □

PROPOSITION 3.11. Let $s \to t$ be a rewrite step. If $t' \in S(t)$ then there exists $s' \in S(s)$ such that $s' \xrightarrow{\equiv} t'$.

Proof. Let $t' \in S_n(t)$. Then our proof proceeds by induction on n. It is obvious if $n = 1$. Suppose $t' \in S_{k+1}(t)$.

1. Case where $s \to^o t$. Then $s \equiv C[\![s_1, \ldots, s_n]\!]$ and $t \equiv C^*\langle\!\langle s_{i_1}, \ldots, s_{i_m}\rangle\!\rangle$ $(i_1, \ldots, i_m \in \{1, \ldots, n\})$ for some $C, C^*, s_1, \ldots, s_n$. Then it follows since $t' \in S(s_j) \subseteq S(s)$.

2. Case where $s \to^i t$. Then

$$\begin{cases} s \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!], \\ t \equiv C[s_1, \ldots, t_j, \ldots, s_n], \\ s_j \to t_j \end{cases}$$

   for some $C, s_1, \ldots, s_n, t_j$.

   (a) Case where $t \equiv C[\![s_1, \ldots, t_j, \ldots, s_n]\!]$. If $t' \in S_k(s_i)$ then it follows from $S(s_i) \subseteq S(s)$. Otherwise $t' \in S_k(t_j)$. Then since $s_j \to t_j$ and $t' \in S_k(t_j)$, there exists $s' \in S(s_j) \subseteq S(s)$ such that $s' \xrightarrow{\equiv} t'$ by induction hypothesis.

   (b) Otherwise $t_j \equiv C'[\![u_1, \ldots, u_l]\!]$ and $t \equiv C''[\![s_1, \ldots, u_1, \ldots, u_l, \ldots, s_n]\!]$ for some $C', C'', u_1, \ldots, u_l$. If $t' \in S_k(s_i)$ then follows from $S(s_i) \subseteq S(s)$. Otherwise $t' \in S_k(u_i)$. Then it follows from $S(u_i) \subseteq S(t_j) \subseteq S(s_j) \subseteq S(s)$. □

PROPOSITION 3.12. Let $s \equiv C[s'] \to C[t'] \equiv t$ be a rewrite step with $s' \to t'$ and $s' \in S(s)$. If $s' \to t'$ is not destructive at level 1 then $t' \in S(t)$.

Proof. It immediately follows since $top(s') = top(t')$. □

DEFINITION 3.13. [preservation] A term s is said to be *preserved* iff there are no reduction sequences starting from s that contain a destructive rewrite step.

DEFINITION 3.14. [collapsing reduction] We write $s \rightarrow_c t$ iff there exist a special subterm s' of s such that $s \equiv C[s']$ and a term t' such that $s' \xrightarrow{*} t'$, $t \equiv C[t']$ and $s' \xrightarrow{*} t'$ contains a level 1 destructive rewrite step. The relation $\rightarrow_c$ is called *collapsing reduction* and s' is the *collapsing redex*.

PROPOSITION 3.15.

1. If $s \rightarrow_c t$ then $s \xrightarrow{*} t$.

2. A term is preserved iff it contains no collapsing redex.

Proof. 1. By definition, $s \equiv C[s']$, $t \equiv C[t']$ and $s' \xrightarrow{*} t'$ for some s', t', C. Since $s' \xrightarrow{*} t'$, $s \equiv C[s'] \xrightarrow{*} C[t'] \equiv t$ by the definition of reduction.

2. (If)-part. Assume there exists a reduction sequence starting from s that contains a destructive rewrite step, i.e. $s \xrightarrow{*} t \rightarrow u$ and $t \rightarrow u$ is destructive. Then by PROPOSITION 3.10, there exist a special subterm t' of t such that $t \equiv C'[t']$ and a term u' such that $t' \rightarrow u'$ is destructive at level 1 and $u \equiv C'[u']$. Then repeated applications of PROPOSITION 3.11 yield $s' \in S(s)$ with $s' \xrightarrow{*} t' \rightarrow u'$. Obviously s' is a collapsing redex in s. (Only if)-part. Suppose $s \rightarrow_c t$. Then by definition, there exists a special subterm s' of s such that $s \equiv C[s']$ and a term t' such that $s' \xrightarrow{*} t'$, $t \equiv C[t']$ and $s' \xrightarrow{*} t'$ contains a level 1 destructive rewrite step. Let the first destructive step in $s' \xrightarrow{*} t'$ be $u' \rightarrow v'$. Then since $s' \in S(s)$, $u' \in S(C[u'])$ by PROPOSITION 3.12. Therefore $C[u'] \rightarrow C[v']$ is destructive. Hence $s \equiv C[s'] \xrightarrow{*} C[u'] \rightarrow C[v'] \xrightarrow{*} C[t'] \equiv t$ contains a destructive rewrite step. □

PROPOSITION 3.16. Every term has a preserved reduct.

Proof. We show there can not be infinite collapsing reduction sequences. Then the proposition immediately follows from PROPOSITION 3.15.

For any term t, assign multiset $\|t\|$ as: $\|t\| = [rank(s) \mid s \in S(t)]$. Then we show $s \rightarrow_c t$ implies $\|s\| \gg \|t\|$. Here $\gg$ is the multiset extension of the standard ordering $>$ on the natural numbers.

Suppose $s \rightarrow_c t$. Then by definition, there exist a special subterm s' of s such that $s \equiv C[s']$ and a term t' such that $s' \xrightarrow{*} t'$ contains a level 1 destructive rewrite step and $t \equiv C[t']$. Then $rank(s') > rank(t')$ by

PROPOSITION 3.8, hence $\|s'\| \gg \|t'\|$. From this follows $\|s\| \gg \|t\|$, since this s' is a special subterm of s. But the relation $\gg$ is well-founded (e.g. see [1]) and hence there can not be infinite collapsing reduction sequences. □

4 Persistency of confluence

Let $s_1, \ldots, s_n$ and $t_1, \ldots, t_n$ be terms. We write $\langle s_1, \ldots, s_n \rangle \propto \langle t_1, \ldots, t_n \rangle$ if for any $1 \leq i, j \leq n$, $s_i \equiv s_j$ implies $t_i \equiv t_j$. Moreover, we write $\langle s_1, \ldots, s_n \rangle \infty \langle t_1, \ldots, t_n \rangle$ iff $\langle s_1, \ldots, s_n \rangle \propto \langle t_1, \ldots, t_n \rangle$ and $\langle t_1, \ldots, t_n \rangle \propto \langle s_1, \ldots, s_n \rangle$.

PROPOSITION 4.1. If $C[\![s_1, \ldots, s_n]\!] \rightarrow^o C'\langle\!\langle s_{i_1}, \ldots, s_{i_m} \rangle\!\rangle$ and $\langle s_1, \ldots, s_n \rangle \propto \langle t_1, \ldots, t_n \rangle$ then $C[t_1, \ldots, t_n] \rightarrow^o C'[t_{i_1}, \ldots, t_{i_m}]$.

Proof. Straightforward. □

PROPOSITION 4.2. If rewrite steps $t_1 \stackrel{*}{\leftarrow}{}^o t \stackrel{*}{\rightarrow}{}^o t_2$ do not contain a destructive rewrite step except the last steps then t_1 and t_2 are joinable.

Proof. We may write $t \equiv C[\![s_1, \ldots, s_n]\!]$, $t_1 \equiv C_1\langle\!\langle s_{i_1}, \ldots, s_{i_m} \rangle\!\rangle$ and $t_2 \equiv C_2\langle\!\langle s_{j_1}, \ldots, s_{j_p} \rangle\!\rangle$ by PROPOSITION 3.6. Let $C : \alpha_1 \times \cdots \times \alpha_n \rightarrow \beta$. Then choose fresh variables $x_i \in \mathcal{V}_{\alpha_i} (i = 1, \ldots, n)$ such that $\langle x_1, \ldots, x_n \rangle \infty \langle s_1, \ldots, s_n \rangle$. Let $t' \equiv C[x_1, \ldots, x_n]$, $t'_1 \equiv C_1[x_{i_1}, \ldots, x_{i_m}]$ and $t'_2 \equiv C_2[x_{j_1}, \ldots, x_{j_p}]$. Then obviously t' is well-sorted. Moreover repeated applications of PROPOSITION 2.1 and PROPOSITION 4.1 yield well-sorted rewrite steps $t'_1 \stackrel{*}{\leftarrow} t' \stackrel{*}{\rightarrow} t'_2$. Therefore t'_1 and t'_2 are joinable, i.e. $t'_1 \stackrel{*}{\rightarrow} C^*[x_{k_1}, \ldots, x_{k_q}] \stackrel{*}{\leftarrow} t'_2$ for some C^*. Instantiating the valley yields $t_1 \stackrel{*}{\rightarrow}{}^o C^*\langle\!\langle s_{k_1}, \ldots, s_{k_q} \rangle\!\rangle \stackrel{*}{\leftarrow}{}^o t_2$. □

DEFINITION 4.3. [representation] Let S be a set of confluent terms. A set $\widehat{S}$ of terms *represents* S iff the following two conditions are satisfied:

1. every term s in S has a unique reduct $\widehat{s}$ in $\widehat{S}$, which will be called the *representative of* s;
2. joinable terms in S have the same representative in $\widehat{S}$.

PROPOSITION 4.4. Every finite set S of confluent terms can be represented.

Proof. Since S consists of confluent terms, joinability is an equivalence relation on S. Since S is finite, each equivalence class is also finite and has a common reduct. It is obvious that the set of such common reducts represents S. □

DEFINITION 4.5. [inner preservation] A term s is said to be *inner preserved* iff its all principal subterms are preserved.

LEMMA 4.6. Inner preserved terms are confluent.

Proof. We show that any inner preserved term t is confluent by induction on the rank of t. If $rank(t) = 1$ then t is a well-sorted term, hence confluent. Suppose that $t \equiv C[\![s_1, \ldots, s_n]\!]$ is an inner preserved term with $rank(t) = n(> 1)$ and $t_1 \stackrel{*}{\leftarrow} t \stackrel{*}{\rightarrow} t_2$.

Firstly note that destructive rewrite steps occur in $t \stackrel{*}{\rightarrow} t_i \quad (i = 1, 2)$ only once and is level 1, because of our assumption that t is inner preserved.

Define a set $\mathcal{L}(t \stackrel{*}{\rightarrow} t_i)$, say the set of lower parts of $t \stackrel{*}{\rightarrow} t_i$, $(i = 1, 2)$ as follows.

1. Suppose $t \stackrel{*}{\rightarrow} t_i$ does not contain a destructive rewrite step. Then let T be the set of all terms that appear between $t \stackrel{*}{\rightarrow} t_i$. Then define

$$\mathcal{L}(t \stackrel{*}{\rightarrow} t_i) = \{u \mid u \text{ is a principal subterm of some } s \in T\}.$$

2. Otherwise $t \stackrel{*}{\rightarrow} t_i$ contains a destructive rewrite step, i.e. there exists some t_i', t_i'' such that $t \stackrel{*}{\rightarrow} t_i' \rightarrow t_i'' \stackrel{*}{\rightarrow} t_i$, $t_i' \rightarrow t_i''$ is destructive. Note that $t \stackrel{*}{\rightarrow} t_i'$ and $t_i'' \stackrel{*}{\rightarrow} t_i$ do not contain a destructive rewrite step. Let T be the set of all terms that appear between $t \stackrel{*}{\rightarrow} t_i'$. Then define

$$\begin{aligned} \mathcal{L}(t \stackrel{*}{\rightarrow} t_i) \quad = \quad & \{u \mid u \text{ is a principal subterm of some } s \in T\} \\ \cup \quad & \{v \mid v \text{ appears between } t_i'' \stackrel{*}{\rightarrow} t_i\}. \end{aligned}$$

Let S be $\mathcal{L}(t \stackrel{*}{\rightarrow} t_1) \cup \mathcal{L}(t \stackrel{*}{\rightarrow} t_2)$. Then for any elements $s \in S$,

1. s is also inner preserved, because s is preserved; and

2. s have less rank than n by definition of rank and PROPOSITION 3.8.

Therefore we can apply induction hypothesis to each element of S, hence S is a set of confluent terms. Then by PROPOSITION 4.4, S has a representation $\widehat{S}$.

Next we define $\tilde{u}$ for each term u that appears in $t_1 \stackrel{*}{\leftarrow} t \stackrel{*}{\rightarrow} t_2$ as follows. If u is a term that appears before the destructive rewrite step occurs then define $\tilde{u}$ to be a term obtained from u by replacing all its principal subterms with its representative. Otherwise define $\tilde{u}$ to be the representative of u. Note that $u \stackrel{*}{\rightarrow} \tilde{u}$ in each case by definition of representation.

Then we can show $\tilde{u_1} \stackrel{\equiv^o}{\rightarrow} \tilde{u_2}$ for any $u_1 \rightarrow u_2$ not preceded by the destructive rewrite step and $\tilde{u_1} \equiv \tilde{u_2}$ for any $u_1 \rightarrow u_2$ that is preceded by the destructive rewrite step as follows.

1. Let $u_1 \to u_2$ be a rewrite step in $t_1 \overset{*}{\leftarrow} t \overset{*}{\rightarrow} t_2$ not preceded by destructive rewrite step.

   (a) Case where $u_1 \to^o u_2$. Then $u_1 \equiv C[\![s_1, \ldots, s_n]\!]$ and $u_2 \equiv C'\langle\!\langle s_{i_1}, \ldots, s_{i_m} \rangle\!\rangle$. It is easily seen $\tilde{u_1} \equiv C[\![\widehat{s_1}, \ldots, \widehat{s_n}]\!]$ and $\tilde{u_2} \equiv C'\langle\!\langle \widehat{s_{i_1}}, \ldots, \widehat{s_{i_m}} \rangle\!\rangle$. Since $\langle s_1, \ldots, s_n \rangle \propto \langle \widehat{s_1}, \ldots, \widehat{s_n} \rangle$ by definition of representation, $\tilde{u_1} \to^o \tilde{u_2}$ by PROPOSITION 4.1.

   (b) Case where $u_1 \to^i u_2$. Then since t is inner preserved, $u_1 \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!]$, $u_2 \equiv C[\![s_1, \ldots, s'_j, \ldots, s_n]\!]$ and $s_j \to s'_j$ by PROPOSITION 3.6. Then since $\widehat{s_j} \equiv \widehat{s'_j}$, $\tilde{u_1} \equiv \tilde{u_2}$.

2. Let $u_1 \to u_2$ be a rewrite step that is preceded by the destructive rewrite step. Then $u_1, u_2 \in S$ and $\widehat{u_1} \equiv \widehat{u_2}$ by the definition of representation, hence $\tilde{u_1} \equiv \tilde{u_2}$.

From this, we conclude that $\tilde{t_1} \overset{*}{\leftarrow}{}^o \tilde{t} \overset{*}{\rightarrow}{}^o \tilde{t_2}$ and $\tilde{t_1} \overset{*}{\leftarrow}{}^o \tilde{t} \overset{*}{\rightarrow}{}^o \tilde{t_2}$ do not contain a destructive rewrite step except the last steps. Therefore $\tilde{t_1}$ and $\tilde{t_2}$ are joinable by PROPOSITION 4.2, hence so are t_1 and t_2. See also figure 1. □

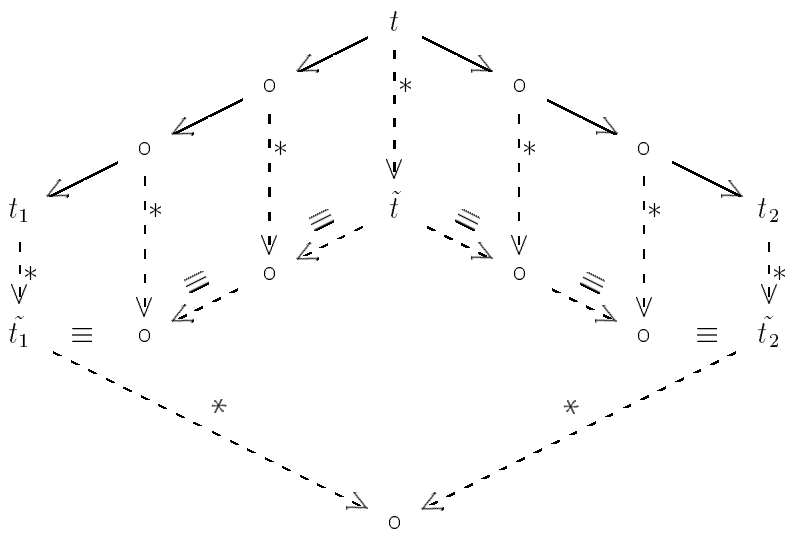

Figure 1: LEMMA 4.6

DEFINITION 4.7. [witness] Let $s \equiv C[\![s_1, \ldots, s_n]\!]$ be a term. Then a term $t \equiv C[t_1, \ldots, t_n]$ which satisfies: (1) t is inner preserved; (2) $s_i \overset{*}{\rightarrow} t_i$ for $i = 1, \ldots, n$; and (3) $\langle s_1, \ldots, s_n \rangle \propto \langle t_1, \ldots, t_n \rangle$; is called a *witness* of s.

PROPOSITION 4.8. Every term has a witness.

Proof. Let $s \equiv C[\![s_1, \ldots, s_n]\!]$. By PROPOSITION 3.16, there exists a preserved reduct t_i of s_i for any $i = 1, \ldots, n$. Taking the same preserved term for identical terms ensures $\langle s_1, \ldots, s_n \rangle \propto \langle t_1, \ldots, t_n \rangle$. Moreover any principal term of $t \equiv C[t_1, \ldots, t_n]$ is under or equal to one of t_i's. But t_i's are all preserved, so are all the principal terms of t. Thus t is a witness of s. □

In the following, an arbitrary witness of a term s is denoted by $\dot{s}$.

LEMMA 4.9. Let s, t be terms such that $s \to t$. If all the principal subterms of s are confluent then $\dot{s}$ and $\dot{t}$ are joinable.

Proof. Let $s \equiv C[\![s_1, \ldots, s_n]\!]$ and $\dot{s} \equiv C[u_1, \ldots, u_n]$ for respective reducts $u_1, \ldots, u_n$ of $s_1, \ldots, s_n$.

1. Case where $s \to^o t$. Then we may write $t \equiv C^*\langle\!\langle s_{i_1}, \ldots, s_{i_m} \rangle\!\rangle$. Suppose $\dot{t} \equiv C^*[t_{i_1}, \ldots, t_{i_m}]$ for respective reducts $t_{i_1}, \ldots, t_{i_m}$ of $s_{i_1}, \ldots, s_{i_m}$. Since $s \equiv C[\![s_1, \ldots, s_n]\!] \to^o C^*\langle\!\langle s_{i_1}, \ldots, s_{i_m} \rangle\!\rangle \equiv t$ and $\langle s_1, \ldots, s_n \rangle \propto \langle u_1, \ldots, u_n \rangle$, $\dot{s} \equiv C[u_1, \ldots, u_n] \to C^*[u_{i_1}, \ldots, u_{i_m}]$ by PROPOSITION 4.1. Now for $j = i_1, \ldots, i_m$, since $u_j \overset{*}{\leftarrow} s_j \overset{*}{\to} t_j$ and s_j is a principal subterm of s, u_j and t_j are joinable by assumption. Therefore $\dot{s}$ and $\dot{t}$ are joinable. See figure 2.

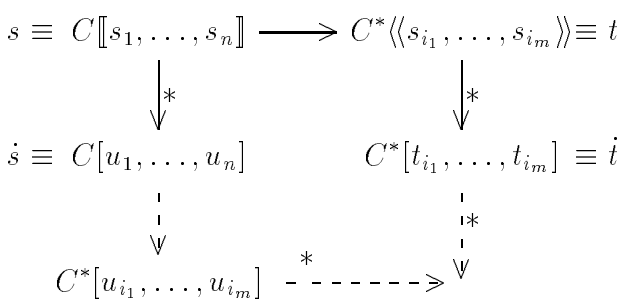

Figure 2: LEMMA 4.9 Case 1.

2. Case where $s \to^i t$. Then $t \equiv C[s_1, \ldots, s'_j, \ldots, s_n]$ and $s_j \to s'_j$. It is easily seen that $\dot{t} \equiv C[t_1, \ldots, t_n]$ for some reducts $t_1, \ldots, t_n$ of $s_1, \ldots, s'_j, \ldots, s_n$. Now for $k = 1, \ldots, n$, since $u_k \overset{*}{\leftarrow} s_k \overset{*}{\to} t_k$ and s_k is a principal subterm of s, u_k and t_k are joinable by assumption. Therefore $\dot{s} \equiv C[\![u_1, \ldots, u_n]\!]$ and $\dot{t} \equiv C[t_1, \ldots, t_n]$ are joinable. □

THEOREM 4.10. Confluence is a persistency property of TRSs.

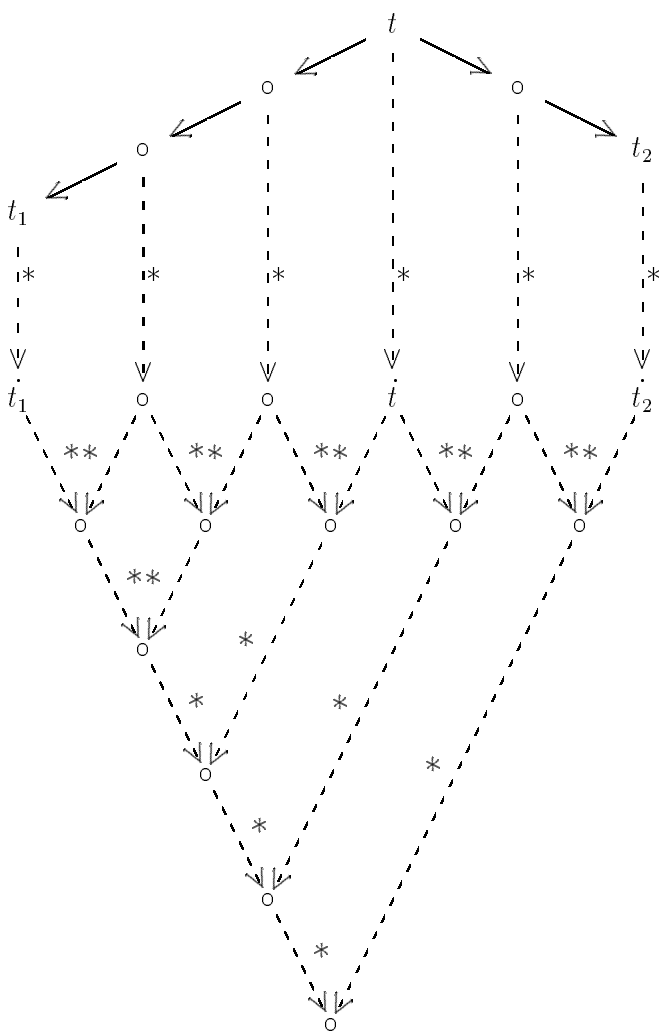

Figure 3: Theorem 4.10

Proof. Our proof proceeds by induction on the rank. If $rank(t) = 1$ then t is a well-sorted term, hence confluent. Suppose $rank(t) = n(> 1)$ and $t_1 \stackrel{*}{\leftarrow} t \stackrel{*}{\rightarrow} t_2$. Note that any term that appears in $t_1 \stackrel{*}{\leftarrow} t \stackrel{*}{\rightarrow} t_2$ has the rank less than or equal to n, and therefore its principal subterms have the ranks less than n, hence confluent by induction hypothesis. Now, by PROPOSITION 4.8, any term that appears in $t_1 \stackrel{*}{\leftarrow} t \stackrel{*}{\rightarrow} t_2$ has a witness. Then repeated applications of LEMMA 4.9 yield a conversion between $\dot{t_1}$ and $\dot{t_2}$ by way of witnesses. Since witnesses are inner preserved, we can apply LEMMA 4.6 repeatedly so that $\dot{t_1}$ and $\dot{t_2}$ are joinable. Hence so are t_1 and t_2. See figure 3. □

5 Concluding remark

Persistency of confluence is useful to infer confluence of a given system from those of its subsystems.

The following example by Y. Toyama also appears in [10]. Let $\mathcal{R}$ be

$$\begin{cases} f(x,y) \to f(g(x),g(x)) & (r1) \\ g(x) \to h(x) & (r2) \\ F(g(x),x) \to F(x,g(x)) & (r3) \\ F(h(x),x) \to F(x,h(x)) & (r4). \end{cases}$$

Confluence of this system seems infeasible from known results, since this system is not terminating, not orthogonal and g or h appear every rule so that results of modularity seems difficult to apply.

Think of attachment τ:

$$\begin{cases} f : 0 \times 0 \to 1 \\ g : 0 \to 0 \\ h : 0 \to 0 \\ F : 0 \times 0 \to 2. \end{cases}$$

It is easily checked this is consistent with $\mathcal{R}$. Now, $\mathcal{R}^\tau$ is confluent iff any well-sorted term in $\mathcal{T}$ is confluent, i.e. any term in $\mathcal{T}^0$, $\mathcal{T}^1$ or $\mathcal{T}^2$ is confluent.

- Since rules applicable to terms in $\mathcal{T}^0$ are $(r2)$, they are confluent.
- Since rules applicable to terms in $\mathcal{T}^1$ are $(r1), (r2)$, they are confluent by confluence of orthogonal systems.
- Since rules applicable to terms in $\mathcal{T}^3$ are $(r2), (r3), (r4)$, they are confluent by termination and convergence of critical pairs.

Hence $\mathcal{R}^\tau$ is confluent. Then confluence of $\mathcal{R}$ follows by persistency of confluence.

Here is another example. Let $\mathcal{R}$ be

$$\begin{cases} K(x, D(x)) \to A & (r1) \\ G(x) \to G(D(x)) & (r2) \\ H(x, G(y)) \to H(D(x), x) & (r3). \end{cases}$$

Firstly an attachment τ:

$$\begin{cases} K : 0 \times 0 \to 1 \\ A : 1 \\ D : 0 \to 0 \\ G : 0 \to 0 \\ H : 0 \times 0 \to 2 \end{cases}$$

makes the system divided into three groups $\{(r2)\}$ for $\mathcal{T}^0$, $\{(r1), (r2)\}$ for $\mathcal{T}^1$ and $\{(r2), (r3)\}$ for $\mathcal{T}^2$. The first and the last systems are easily shown to be confluent. To show confluence of the second system, we now think of another attachment τ':

$$\begin{cases} K : 0 \times 0 \to 1 \\ A : 1 \\ D : 0 \to 0 \\ G : 0 \to 2. \end{cases}$$

This enable us to divide the system into two groups $\{(r1)\}$ for $\mathcal{T}^1$ and $\{(r2)\}$ for $\mathcal{T}^2$ and both are easily shown to be confluent, hence so is the whole system.

As is mentioned in the introduction, the notion of persistency is closely related to the notion of modularity. Indeed, H. Zantema showed that every component closed persistent property (like confluence, weak confluence, termination and weak normalization) is modular w.r.t. disjoint union [10]. Hence persistency of confluence implies modularity of confluence w.r.t. disjoint union. We will explain this by a simple proof sketch.

Let $\langle \mathcal{F}, \mathcal{R} \rangle$ and $\langle \mathcal{F}', \mathcal{R}' \rangle$ be the disjoint TRSs. To prove confluence of $\langle \mathcal{F} \cup \mathcal{F}', \mathcal{R} \cup \mathcal{R}' \rangle$, let us think of two sorts α and β. Then attach the sort $\overbrace{\alpha \times \cdots \times \alpha}^{n\text{-times}} \to \alpha$ to each $f \in \mathcal{F}_n$ and attach the sort $\overbrace{\beta \times \cdots \times \beta}^{n\text{-times}} \to \beta$ to each $f \in \mathcal{F}'_n$. It is obvious that this attachment is consistent with $\langle \mathcal{F} \cup \mathcal{F}', \mathcal{R} \cup \mathcal{R}' \rangle$. Then the set of rules applicable to $\mathcal{T}^\alpha$ is $\mathcal{R}$ and that of rules applicable to $\mathcal{T}^\beta$ is $\mathcal{R}'$. Thus if both are confluent then so is its union by persistency of confluence.

References

[1] Dershowitz, N. and Manna, Z., Proving termination with multiset orderings. *Communications of the ACM*, Vol. 22, No. 8, pp. 465–476, 1979.

[2] Huet, G., Confluent reductions: abstract properties and applications to term rewriting systems. *Journal of Association for Computing Machinery*, Vol. 27, No. 4, pp. 797–821, 1980.

[3] Kitahara, A., On the modularity of confluent term rewriting systems (in Japanese). Master's thesis, JAIST, 1995.

[4] Klop, J. W., Middeldorp, A., Toyama, Y. and de Vrijer, R., Modularity of confluence: A simplified proof. *Information Processing Letters*, Vol. 49, pp. 101–109, 1994.

[5] Knuth, D. E. and Bendix, P. B., Simple word problems in universal algebras. In Leech, J. (ed.), *Computational problems in abstract algebra*, pp. 263–297. Pergamon Press, 1970.

[6] Ohlebusch, E., On the modularity of confluence of constructor-sharing term rewriting systems. In *Proceedings of the 19th colloquium on Trees in Algebra and Programming, LNCS 787*, pp. 261–275, 1994.

[7] Rosen, B. K., Tree-manipulating systems and Church-Rosser theorems. *Journal of Association for Computing Machinery*, Vol. 20, pp. 160–187, 1973.

[8] Toyama, Y., On the Church-Rosser property for the direct sum of term rewriting systems. *Journal of Association for Computing Machinery*, Vol. 34, No. 1, pp. 128–143, 1987.

[9] Toyama, Y., Commutativity of term rewriting systems. In Fuchi, K. and Kott, L. (eds.), *Programming of future generation computers II*, pp. 393–407. North-Holland, Amsterdam, 1988.

[10] Zantema, H., Termination of term rewriting: interpretation and type elimination. *Journal of Symbolic Computation*, Vol. 17, pp. 23–50, 1994.