Top-down labelling and modularity of term rewriting systems

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Abstract

Two kinds of top-down labelling of function symbols are proposed. For each of these labelling, a condition under which the labelling yields a sound and complete transformation of term rewriting systems is introduced. Two kinds of combinations of term rewriting systems are naturally induced from these conditions, to which various modularity results for disjoint union of term rewriting systems are lifted.

1 Introduction

In [12], H. Zantema introduced a transformation technique named *semantic labelling*. Semantic labelling yields a transformation of term rewriting systems (abbrev. TRSs) by means of suitable labelling of function symbols. It is observed that this transformation is useful to show termination of non-simply terminating TRSs, since sometimes labelled TRSs are easier to handle with simplification orderings.

In semantic labelling, any root symbol of a term is labelled by its interpretation as a term, which is defined by interpretations of its arguments and the predefined interpretation of the function symbol itself. Thus it is reasonable to regard semantic labelling a kind of bottom-up labelling. Are there any meaningful labelling for the converse strategy, i.e. a kind of *top-down labelling*? In this paper, we will answer this question.

A transformation τ of TRSs is said to be *sound* with respect to a property $\mathcal{P}$ of TRSs if $\langle \mathcal{F}, \mathcal{R} \rangle$ has the property $\mathcal{P}$ whenever so do $\langle \mathcal{F}^\tau, \mathcal{R}^\tau \rangle$. Also τ is said to be *complete* w.r.t. a property $\mathcal{P}$ of TRSs if $\langle \mathcal{F}^\tau, \mathcal{R}^\tau \rangle$ has the property $\mathcal{P}$ whenever so do $\langle \mathcal{F}, \mathcal{R} \rangle$. We introduce two kinds of top-down labelling and for each of these top-down labelling, we present a condition under which the labelling yields a sound and complete transformation of TRSs. Moreover, these labelling are useful to extend some of modularity results.

A property $\mathcal{P}$ of TRSs is said to be *modular for a combination of TRSs* if $\mathcal{P}$ is preserved for the combination. That is, for any such combination $\langle \mathcal{F}_1, \mathcal{R}_1 \rangle$ and $\langle \mathcal{F}_2, \mathcal{R}_2 \rangle$, $\langle \mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_1 \cup \mathcal{R}_2 \rangle$ has the property $\mathcal{P}$ iff so do both $\langle \mathcal{F}_1, \mathcal{R}_1 \rangle$ and $\langle \mathcal{F}_2, \mathcal{R}_2 \rangle$. Combinations introduced in this paper are a *combination with a common transparent part* and a *combination with a common lower-sorted part*. Both of these combinations allow sharing of function symbols and rules. It is shown that conditions for ensuring modular properties of disjoint union of TRSs can be lifted to these combinations in a uniform way.

In section 2, we briefly review some basic notions of TRSs and fix some notations. In section 3 and 4, two kinds of top-down labelling and combinations of TRSs are proposed. We end with reviews of our results in section 5.

2 Preliminaries

2.1 Term rewriting systems

Let $\mathcal{V}$ be a set of countably infinite variables. Given a set $\mathcal{F}$ of arity-fixed function symbols, we define the set $\mathcal{T}(\mathcal{F}, \mathcal{V})$ of all *terms* built from $\mathcal{F}$ and $\mathcal{V}$ as follows: (1) $\mathcal{V} \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V})$; (2) If $f \in \mathcal{F}_n$ and $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ then $f(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. Here $\mathcal{F}_n$ denotes the set of n arity function symbols in $\mathcal{F}$. We use $\mathcal{F}(t)$ and $\mathcal{V}(t)$ to denote the set of function symbols and variables that appear in t, respectively. Identity of terms is denoted by $\equiv$. For any term t, $root(t)$ is (1) t if $t \in \mathcal{V}$; (2) f if $t \equiv f(t_1, \ldots, t_n)$. If $\mathcal{F}$ is obvious from the context, $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is often abbreviated as $\mathcal{T}$.

A set of *rewrite rules* on $\mathcal{T}$ is $\mathcal{R} \subseteq \mathcal{T} \times \mathcal{T}$ subjects to the following constraints: (1) $l \notin \mathcal{V}$ for each $\langle l, r \rangle \in \mathcal{R}$; (2) $\mathcal{V}(r) \subseteq \mathcal{V}(l)$. We conventionally write $l \to r \in \mathcal{R}$ instead of $\langle l, r \rangle \in \mathcal{R}$. A rewrite rule $l \to r$ is said to be *collapsing* if $r \in \mathcal{V}$. A rewrite rule $l \to r$ is said to be *duplicating* if r contains more occurrences of some variable than l does.

A *Term Rewriting System* (*TRS* for short) is a pair $\langle \mathcal{F}, \mathcal{R} \rangle$ where $\mathcal{F}$ is a set of arity-fixed function symbols and $\mathcal{R}$ is a set of rewrite rules on $\mathcal{T}(\mathcal{F}, \mathcal{V})$. $\langle \mathcal{F}, \mathcal{R} \rangle$ is often abbreviated as $\mathcal{R}$, and in this case $\mathcal{F}$ is defined

to be the set of function symbols that appear in $\mathcal{R}$. For $\mathcal{F}' \subseteq \mathcal{F}$, the set $\{l \to r \in \mathcal{R} \mid \mathcal{F}(l) \cup \mathcal{F}(r) \subseteq \mathcal{F}'\}$ is written as $\mathcal{R} \mid \mathcal{F}'$.

Suppose $\Box$ be a special constant (called a *hole*). Elements of $\mathcal{T}(\mathcal{F} \cup \{\Box\}, \mathcal{V})$ is called *contexts* over $\mathcal{T}$. If $\mathcal{F}$ is obvious from the context, $\mathcal{T}(\mathcal{F} \cup \{\Box\}, \mathcal{V})$ is written just as $\mathcal{C}$. If $C \in \mathcal{C}$ has n holes and $t_1, \ldots, t_n \in \mathcal{T}$ then $C[t_1, \ldots, t_n]$ denotes the term obtained from C by replacing holes with $t_1, \ldots, t_n$ from left to right. The context that contains precisely one hole is denoted by $C[\,]$. A term s is said to be a *subterm* of t iff $t \equiv C[s]$ for some context C. For $C \in \mathcal{C}$, $\mathcal{F}(C)$ and $\mathcal{V}(C)$ are defined similarly to the case of terms.

A *substitution* σ is a mapping form $\mathcal{V}$ to $\mathcal{T}$. Substitutions are extended to homomorphisms from $\mathcal{T}$ to $\mathcal{T}$. For a substitution σ and a term t, $\sigma(t)$ is conventionally written as $t\sigma$.

Given a TRS $\mathcal{R}$, we write $s \to_{\mathcal{R}} t$ iff $s \equiv C[l\sigma]$ and $t \equiv C[r\sigma]$ for some $C \in \mathcal{C}$, $l \to r \in \mathcal{R}$ and substitution σ. We call $s \to_{\mathcal{R}} t$ a *rewrite step (from s to t)* of $\mathcal{R}$. The *redex* of this rewrite step is defined to be $l\sigma$. The relation $\to_{\mathcal{R}}$ on $\mathcal{T}$ is called the *reduction* of $\mathcal{R}$. A TRS $\mathcal{R}$ is said to be *terminating* iff there are no infinite sequences $t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} t_3 \to_{\mathcal{R}} \cdots$ of terms in $\mathcal{T}$.

The converse relation of $\to_{\mathcal{R}}$ is denoted by $\leftarrow_{\mathcal{R}}$. The transitive-reflexive closures of $\to_{\mathcal{R}}$ and $\leftarrow_{\mathcal{R}}$ are denoted by $\overset{*}{\to}_{\mathcal{R}}$ and $\overset{*}{\leftarrow}_{\mathcal{R}}$, respectively. Terms t_1 and t_2 are said to be *joinable* if there exists some term t' such that $t_1 \overset{*}{\to}_{\mathcal{R}} t' \overset{*}{\leftarrow}_{\mathcal{R}} t_2$. A term t is said to be *confluent* iff any terms t_1 and t_2 are joinable whenever $t_1 \overset{*}{\leftarrow}_{\mathcal{R}} t \overset{*}{\to}_{\mathcal{R}} t_2$. A TRS $\mathcal{R}$ is said to be confluent iff any term is confluent w.r.t. its reduction $\to_{\mathcal{R}}$. If $\mathcal{R}$ is obvious from the context, the subscript $_{\mathcal{R}}$ is often abbreviated.

We write as $\mathsf{2}_\perp$ the set $\{\perp, 1, 2\}$ with strict partial order $\prec$ such that $\perp \prec 1$ ad $\perp \prec 2$ (and no other relations). We write $i \preceq j$ iff $i \prec j$ or $i = j$.

3 Labelling via signature function

3.1 Signature function and labeling

In this subsection, we introduce a labelling via signature function. It is shown that consistent signature functions yield sound and complete transformations of TRSs w.r.t. confluence and termination.

Definition 3.1. [signature function] A *signature function* τ is a function from $\mathcal{F}$ to $\mathsf{2}_\perp$.

In the rest of this subsection, we assume that a signature function τ is fixed. We write as $\mathcal{F}_i$ the set $\{f \in \mathcal{F} \mid \tau(f) \preceq i\}$ for any $i \in \mathsf{2}_\perp$.

DEFINITION 3.2. [top signature] The *top signature* $top(t)$ of a term t is $\tau(root(t))$.

DEFINITION 3.3. [principal subterm] Let $t \equiv C[t_1, \ldots, t_n]$ $(n \geq 0)$ be a term with $C[, \ldots,] \not\equiv \Box$. We write $t \equiv C[\![t_1, \ldots, t_n]\!]$ iff

1. $\mathcal{F}(C) \subseteq \mathcal{F}_{top(t)} \cup \{\Box\}$;
2. $top(t_i) \not\trianglelefteq top(t)$ for all $i = 1, \ldots, n$.

Then $t_1, \ldots, t_n$ are said to be the *principal* subterms of t.

It is clear that a term t is uniquely written as $C[\![t_1, \ldots, t_n]\!]$ for some $C, t_1, \ldots, t_n$.

We write as $\mathcal{F}_\bot^i$ the set $\{f^i \mid f \in \mathcal{F}_\bot\}$ $(i \in \{1,2\})$. Elements of the set $\bigcup_{i\in\{1,2\}} \mathcal{F}_\bot^i$ are called *labelled symbols*. We write as $\mathcal{F}^\tau$ the set $\bigcup_{i\in\{1,2\}}(\mathcal{F}_\bot^i \cup \mathcal{F}_i)$. Terms in $\mathcal{T}(\mathcal{F}^\tau, \mathcal{V})$ are said to be *labelled terms*. Terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$ are said to be *unlabelled*. Given a term t, t^i denotes the term obtained from t by labelling all symbols in $\mathcal{F}_\bot$ with i $(i \in \{1,2\})$. This convention is also adapted for contexts.

Now top-down labelling $\Diamond_\tau$ is defined as below.

DEFINITION 3.4. Let t be an unlabelled term such that $t \equiv C[\![t_1, \ldots, t_n]\!]$. A labelled term $\Diamond_\tau t$ is defined by

$$\Diamond_\tau t \equiv \begin{cases} C^1[\Diamond_\tau t_1, \ldots, \Diamond_\tau t_n] & \text{if } root(t) \in \mathcal{F}_\bot, \\ C^i[\Diamond_\tau t_1, \ldots, \Diamond_\tau t_n] & \text{if } root(t) \in \mathcal{F}_i \ (i \in \{1,2\}). \end{cases}$$

DEFINITION 3.5. Let $\langle \mathcal{F}, \mathcal{R} \rangle$ be a TRS. Then signature function τ is said to be *consistent with* $\langle \mathcal{F}, \mathcal{R} \rangle$ iff

1. $\mathcal{F}(l) \cup \mathcal{F}(r) \subseteq \mathcal{F}_\bot$ for all $l \to r \in \mathcal{R}_\bot$; and
2. for all $l \to r \in \mathcal{R} \setminus \mathcal{R}_\bot$, $\mathcal{F}(l) \cup \mathcal{F}(r) \subseteq \mathcal{F}_i$ and $top(l) = top(r) = i$ for some $i \in \{1,2\}$;

where $\mathcal{R}_\bot = \mathcal{R} \mid \mathcal{F}_\bot$. We write as $\mathcal{R}_i$ the set $\{l \to r \in \mathcal{R} \mid root(l) \in \mathcal{F}_i\}$ $(i \in \{1,2\})$.

In the rest of this subsection, we assume that a TRS $\langle \mathcal{F}, \mathcal{R} \rangle$ *is fixed and* τ *is consistent with it.*

Now define $\mathcal{R}^\tau$ as

$$\mathcal{R}^\tau = \bigcup_{i\in\{1,2\}} \{l^i \to r^i \mid l \to r \in (\mathcal{R}_\bot \cup \mathcal{R}_i)\}.$$

It is clear that $\langle \mathcal{F}^\tau, \mathcal{R}^\tau \rangle$ is a TRS.

Our goal in this subsection is to show the THEOREM 3.10. But firstly we introduce some basic definitions.

DEFINITION 3.6. [rank] The *rank* of a term t is defined by

$$rank(t) = \begin{cases} 1 & \text{if } \mathcal{F}(t) \subseteq \mathcal{F}_i \text{ for some } i \in \{1,2\}, \\ 1 + max\{rank(t_i) \mid 1 \leq i \leq n\} & \text{if } t \equiv C[\![t_1, \ldots, t_n]\!] \text{ with } n \geq 1. \end{cases}$$

DEFINITION 3.7. [inner and outer rewrite step] A rewrite step $s \to t$ is said to be *inner* (written as $s \to^i t$) iff

$$s \equiv C[\![s_1, \ldots, C'[l\sigma], \ldots, s_n]\!] \to C[s_1, \ldots, C'[r\sigma], \ldots, s_n] \equiv t$$

for some $s_1, \ldots, s_n$, $l \to r$, σ and C', otherwise *outer* (written as $s \to^o t$).

We write $t \equiv C\langle\!\langle t_1, \ldots, t_n \rangle\!\rangle$ if either $t \equiv C[\![t_1, \ldots, t_n]\!]$ or $C \equiv \Box$ and $t \equiv t_1$.

PROPOSITION 3.8. [Case analysis]

1. If $s \to^o t$ then

$$\begin{cases} s \equiv C[\![s_1, \ldots, s_n]\!], \\ t \equiv C'\langle\!\langle s_{i_1}, \ldots, s_{i_m} \rangle\!\rangle \qquad (i_1, \ldots, i_m \in \{1, \ldots, n\}) \end{cases}$$

   for some $C, C', s_1, \ldots, s_n$ and either

   (a) $n = m = 0$ and $\mathcal{F}(s) \cup \mathcal{F}(t) \subseteq \mathcal{F}_i$ for some $i \in \{1,2\}$;

   (b) $n \neq 0$, $top(C) = \bot$, $C' \equiv \Box$ and $t \equiv s_j$ for some $1 \leq j \leq n$; or

   (c) $n \neq 0$, $top(C) = top(C')$ and $t \equiv C'[\![s_{i_1}, \ldots, s_{i_m}]\!]$.

2. If $s \to^i t$ then

$$\begin{cases} s \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!], \\ t \equiv C[\![s_1, \ldots, t_j, \ldots, s_n]\!], \\ s_j \to t_j \end{cases}$$

   for some $C, s_1, \ldots, s_n, t_j$.

Proof. Straightforward. □

PROPOSITION 3.9. Suppose that $C[\![s_1, \ldots, s_n]\!] \to^o C'\langle\!\langle s_{i_1}, \ldots, s_{i_m} \rangle\!\rangle$. Then for any $t_1, \ldots, t_n$ such that $s_i \equiv s_j$ implies $t_i \equiv t_j$ (for any $1 \leq i, j \leq n$), $C[t_1, \ldots, t_n] \to^o C'[t_{i_1}, \ldots, t_{i_m}]$.

Proof. Straightforward. □

THEOREM 3.10. For any unlabelled terms s and t, $s \to_{\mathcal{R}} t$ iff $\Diamond_\tau s \to_{\mathcal{R}^\tau} \Diamond_\tau t$.

Proof. (If)-part. It suffices to erase all labels. (Only if)-part. Our proof proceeds by induction on the rank of s. Base case is obvious. Suppose that $rank(s) > 1$.

1. Case where $s \to^o t$. Then we can assume

$$\begin{cases} s \equiv C[\![s_1, \ldots, s_n]\!], \\ t \equiv C'\langle\!\langle s_{i_1}, \ldots, s_{i_m} \rangle\!\rangle \qquad (i_1, \ldots, i_m \in \{1, \ldots, n\}) \end{cases}$$

   for some $C, C', s_1, \ldots, s_n$.

   (a) Case where $top(C) = \bot$. Note that this implies that the applied rule is in $\mathcal{R}_\bot$. Thus $C[\![s_1, \ldots, s_n]\!] \to^o_{\mathcal{R}_\bot} C'\langle\!\langle s_{i_1}, \ldots, s_{i_m} \rangle\!\rangle$. Since $s_i \equiv s_j$ implies $\Diamond_\tau s_i \equiv \Diamond_\tau s_j$, $C[\Diamond_\tau s_1, \ldots, \Diamond_\tau s_n] \to^o_{\mathcal{R}_\bot} C'[\Diamond_\tau s_{i_1}, \ldots, \Diamond_\tau s_{i_m}]$ by PROPOSITION 3.9. Then it follows that $C^1[\Diamond_\tau s_1, \ldots, \Diamond_\tau s_n] \to^o_{\mathcal{R}^\tau} C'^1[\Diamond_\tau s_{i_1}, \ldots, \Diamond_\tau s_{i_m}]$ by definition of $\mathcal{R}^\tau$. Hence $\Diamond_\tau s \to^o_{\mathcal{R}^\tau} \Diamond_\tau t$.

   (b) Case where $top(C) = k \in \{1, 2\}$. Note that this implies that the applied rule is in $\mathcal{R}_k \cup \mathcal{R}_\bot$. Thus $C[\![s_1, \ldots, s_n]\!] \to^o_{\mathcal{R}_k \cup \mathcal{R}_\bot} C'\langle\!\langle s_{i_1}, \ldots, s_{i_m} \rangle\!\rangle$. Since $s_i \equiv s_j$ implies $\Diamond_\tau s_i \equiv \Diamond_\tau s_j$, $C[\Diamond_\tau s_1, \ldots, \Diamond_\tau s_n] \to^o_{\mathcal{R}_k \cup \mathcal{R}_\bot} C'[\Diamond_\tau s_{i_1}, \ldots, \Diamond_\tau s_{i_m}]$ by PROPOSITION 3.9. Then it follows that $C^k[\Diamond_\tau s_1, \ldots, \Diamond_\tau s_n] \to^o_{\mathcal{R}^\tau} C'^k[\Diamond_\tau s_{i_1}, \ldots, \Diamond_\tau s_{i_m}]$ by definition of $\mathcal{R}^\tau$. Hence $\Diamond_\tau s \to^o_{\mathcal{R}^\tau} \Diamond_\tau t$.

2. Case where $s \to^i t$. Then we can assume

$$\begin{cases} s \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!], \\ t \equiv C[\![s_1, \ldots, t_j, \ldots, s_n]\!], \\ s_j \to t_j \end{cases}$$

   for some $C, s_1, \ldots, s_n, t_j$. Then $\Diamond_\tau s_j \to_{\mathcal{R}^\tau} \Diamond_\tau t_j$ by induction hypothesis. Therefore $\Diamond_\tau s \equiv C^k[\ldots, \Diamond_\tau s_j, \ldots] \to_{\mathcal{R}^\tau} C^k[\ldots, \Diamond_\tau t_j, \ldots] \equiv \Diamond_\tau t$ where $k = 1$ if $top(C) \in \{\bot, 1\}$, otherwise $k = 2$.

□

The followings are immediate corollaries of THEOREM 3.10.

COROLLARY 3.11. Let $\langle \mathcal{F}, \mathcal{R} \rangle$ be a TRS and τ a signature function that is consistent with it. Then $\mathcal{R}$ is terminating iff $\mathcal{R}^\tau$ is terminating.

COROLLARY 3.12. Let $\langle \mathcal{F}, \mathcal{R} \rangle$ be a TRS and τ a signature function that is consistent with it. Then $\mathcal{R}$ is confluent iff $\mathcal{R}^\tau$ is confluent.

Thus for a consistent signature function τ, $^\tau$ is a sound and complete transformation w.r.t. both termination and confluence.

3.2 Combination with a common transparent part

In this subsection, we introduce a combination with a common transparent part, which is naturally induced from definition of consistency. We show various modularity results for this combination.

DEFINITION 3.13. Let $\langle \mathcal{F}, \mathcal{R} \rangle$ be a TRS. Then $\mathcal{R}$ is said to be *a system with transparent symbols* $\mathcal{F}_\perp$ iff

1. $\mathcal{F}(l) \cup \mathcal{F}(r) \subseteq \mathcal{F}_\perp$ for all $l \to r \in \mathcal{R}_\perp$; and
2. $root(l), root(r) \in \mathcal{F} \setminus \mathcal{F}_\perp$ for all $l \to r \in \mathcal{R} \setminus \mathcal{R}_\perp$;

where $\mathcal{R}_\perp = \mathcal{R} \mid \mathcal{F}_\perp$. Symbols in $\mathcal{F}_\perp$ are said to be *transparent*. Note that $\mathcal{R} \setminus \mathcal{R}_\perp$ contains no collapsing rules.

DEFINITION 3.14. A combination of TRSs $\langle \mathcal{F}_1, \mathcal{R}_1 \rangle$ and $\langle \mathcal{F}_2, \mathcal{R}_2 \rangle$ such that

1. both are the systems with transparent symbols $\mathcal{F}_1 \cap \mathcal{F}_2$; and
2. $\mathcal{R}_1 \mid (\mathcal{F}_1 \cap \mathcal{F}_2) = \mathcal{R}_2 \mid (\mathcal{F}_1 \cap \mathcal{F}_2)$ (called *a common transparent part*);

is said to be a *combination with a common transparent part*. Then the system $\langle \mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_1 \cup \mathcal{R}_2 \rangle$ is said to be a *combined system with a common transparent part*.

THEOREM 3.15. Termination is modular for combinations with a common transparent part if either

1. there are no collapsing rules in the common part (hence in the combined system); or
2. there are no duplicating rules in the combined system.

Proof. Let $\langle \mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_1 \cup \mathcal{R}_2 \rangle$ be a combined system of $\langle \mathcal{F}_1, \mathcal{R}_1 \rangle$ and $\langle \mathcal{F}_2, \mathcal{R}_2 \rangle$ with a common transparent part that satisfies condition 1 or 2. We show

$$\begin{array}{ll} & \langle \mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_1 \cup \mathcal{R}_2 \rangle \text{ is terminating} \\ \Leftrightarrow & \text{both of } \langle \mathcal{F}_1, \mathcal{R}_1 \rangle \text{ and } \langle \mathcal{F}_2, \mathcal{R}_2 \rangle \text{ are terminating.} \end{array}$$

The implication from left to right immediately follows.

To prove the implication of the other direction, suppose that $\langle \mathcal{F}_1, \mathcal{R}_1 \rangle$ and $\langle \mathcal{F}_2, \mathcal{R}_2 \rangle$ are terminating. Now, define signature function τ as: $\tau(f) = \perp$ for $f \in \mathcal{F}_1 \cap \mathcal{F}_2$; $\tau(f) = 1$ for $f \in \mathcal{F}_1 \setminus \mathcal{F}_2$; $\tau(f) = 2$ for $f \in \mathcal{F}_2 \setminus \mathcal{F}_1$. Then $\{l^1 \to r^1 \mid l \to r \in \mathcal{R}_1\}$ and $\{l^2 \to r^2 \mid l \to r \in \mathcal{R}_2\}$ are terminating and disjoint. Then by modularity of termination for disjoint union of TRSs (see

e.g. [6]), $\{l^1 \to r^1 \mid l \to r \in \mathcal{R}_1\} \cup \{l^2 \to r^2 \mid l \to r \in \mathcal{R}_2\} = (\mathcal{R}_1 \cup \mathcal{R}_2)^\tau$ is terminating. Therefore $\mathcal{R}_1 \cup \mathcal{R}_2$ is terminating by COROLLARY 3.11. □

Let $\langle \mathcal{F}, \mathcal{R} \rangle$ be a TRS. Then $\mathcal{R}$ is said to be *a constructor system with constructors* $\mathcal{C}$ iff $root(l) \in \mathcal{F} \setminus \mathcal{C}$ for all $l \to r \in \mathcal{R}$. Elements of $\mathcal{C}$ and those of $\mathcal{F} \setminus \mathcal{C}$ are called *constructors* and *defined symbols*, respectively. A rewrite rule $l \to r$ of a constructor system is said to be *constructor-lifting* if $root(r) \in \mathcal{C}$.

A combination of TRSs $\langle \mathcal{F}_1, \mathcal{R}_1 \rangle$ and $\langle \mathcal{F}_2, \mathcal{R}_2 \rangle$ is said to be a *combination with shared constructors* iff both are constructor systems with constructors $\mathcal{F}_1 \cap \mathcal{F}_2$. Then the system $\langle \mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_1 \cup \mathcal{R}_2 \rangle$ is said to be a *combined constructor system with shared constructors*.

The following corollary is given in [2].

COROLLARY 3.16. Termination is modular for combinations of constructor systems with shared constructors if the combined system contains neither collapsing nor constructor-lifting rules.

Proof. This is nothing but a combination of TRSs with a common transparent part $\mathcal{R}_1 \mid \mathcal{C} = \mathcal{R}_2 \mid \mathcal{C} = \emptyset$. Hence modular termination follows from condition 1 of above. □

EXAMPLE 3.17. Let $\mathcal{R}$ be

$$\begin{cases} f(a, x) \to f(b, g(x)) & (r1) \\ f(x, b) \to f(x, c) & (r2) \\ g(x) \to x & (r3) \\ F(x, g(F(y, c))) \to F(x, g(F(y, a))) & (r4). \end{cases}$$

Now, take $\mathcal{R}_1 = \{(r1), (r2), (r3)\}$ and $\mathcal{R}_2 = \{(r3), (r4)\}$. Then $\mathcal{R}$ is a combined system with a common transparent part. Then putting $a > b > c$ yields termination of $\mathcal{R}_1$ by lexicographic path ordering. Also, putting $c > a$ yields termination of $\mathcal{R}_2$ by lexicographic path ordering. Moreover the combined system satisfies condition 2 of THEOREM 3.15. Hence $\mathcal{R}$ is terminating by THEOREM 3.15.

Note that by putting $\tau(f) = 1$, $\tau(g) = \tau(a) = \tau(b) = \tau(c) = \bot$ and $\tau(F) = 2$, $\mathcal{R}^\tau$ becomes

$$\begin{cases} f(a^1, x) \to f(b^1, g^1(x)) \\ f(x, b^1) \to f(x, c^1) \\ g^1(x) \to x \\ g^2(x) \to x \\ F(x, g^2(F(y, c^2))) \to F(x, g^2(F(y, a^2))). \end{cases}$$

By COROLLARY 3.11, $\mathcal{R}^\tau$ is terminating iff $\mathcal{R}$ is terminating.

Later, readers will find that THEOREM 4.17 can not deal with this example because of $(r4)$.

THEOREM 3.18. Confluence is modular for combinations with a common transparent part.

Proof. Let $\langle \mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_1 \cup \mathcal{R}_2 \rangle$ be a combined system of $\langle \mathcal{F}_1, \mathcal{R}_1 \rangle$ and $\langle \mathcal{F}_2, \mathcal{R}_2 \rangle$ with a common transparent part. We show

$$\begin{array}{ll} & \langle \mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_1 \cup \mathcal{R}_2 \rangle \text{ is confluent} \\ \Leftrightarrow & \text{both of } \langle \mathcal{F}_1, \mathcal{R}_1 \rangle \text{ and } \langle \mathcal{F}_2, \mathcal{R}_2 \rangle \text{ are confluent.} \end{array}$$

The implication from left to right immediately follows since $((\mathcal{R}_1 \cup \mathcal{R}_2) \mid \mathcal{F}_i) = \mathcal{R}_i$ for $i \in \{1, 2\}$.

To prove the implication of the other direction, suppose that $\langle \mathcal{F}_1, \mathcal{R}_1 \rangle$ and $\langle \mathcal{F}_2, \mathcal{R}_2 \rangle$ are confluent. Now, define signature function τ as: $\tau(f) = \bot$ for $f \in \mathcal{F}_1 \cap \mathcal{F}_2$; $\tau(f) = 1$ for $f \in \mathcal{F}_1 \setminus \mathcal{F}_2$; $\tau(f) = 2$ for $f \in \mathcal{F}_2 \setminus \mathcal{F}_1$. Then $\{l^1 \to r^1 \mid l \to r \in \mathcal{R}_1\}$ and $\{l^2 \to r^2 \mid l \to r \in \mathcal{R}_2\}$ are confluent and disjoint. Then by modularity of confluence for disjoint union of TRSs (e.g. see [3]), $\{l^1 \to r^1 \mid l \to r \in \mathcal{R}_1\} \cup \{l^2 \to r^2 \mid l \to r \in \mathcal{R}_2\} = (\mathcal{R}_1 \cup \mathcal{R}_2)^\tau$ is confluent. Therefore $\mathcal{R}_1 \cup \mathcal{R}_2$ is confluent by COROLLARY 3.12. □

The following corollary is given in [7].

COROLLARY 3.19. Confluence is modular for combinations of constructor systems with shared constructors if the combined system contains neither collapsing nor constructor-lifting rules.

Proof. This is nothing but a combination of TRSs with a common transparent part $\mathcal{R}_1 \mid \mathcal{C} = \mathcal{R}_2 \mid \mathcal{C} = \emptyset$. □

EXAMPLE 3.20. Let $\mathcal{R}$ be

$$\left\{ \begin{array}{ll} F(x, C(x)) \to A & (r1) \\ F(x, x) \to A & (r2) \\ C(x) \to x & (r3) \\ a \to f(C(a)) & (r4) \\ f(x) \to h(x, x) & (r5). \end{array} \right.$$

To show the confluence of this system directly seems difficult from known results, since this system is neither terminating nor left-linear. Also the modularity of confluence for disjoint union seems difficult to apply, since function symbols are overlapping among the rules. Also this is not a constructor system, hence COROLLARY 3.19 is not applicable.

Now, take $\mathcal{R}_1 = \{(r1), (r2), (r3)\}$ and $\mathcal{R}_2 = \{(r3), (r4), (r5)\}$. Then $\mathcal{R}$ is a combined system with a common transparent part. $\mathcal{R}_1$ is confluent since it is terminating and critical pairs are joinable. Also $\mathcal{R}_2$ is confluent since it is orthogonal. Hence so is $\mathcal{R}$ by THEOREM 3.18.

Note that by putting $\tau(F) = \tau(A) = 1$, $\tau(C) = \bot$ and $\tau(f) = \tau(a) = \tau(g) = 2$, $\mathcal{R}^\tau$ becomes

$$\begin{cases} F(x, C^1(x)) \to A \\ F(x, x) \to A \\ C^1(x) \to x \\ C^2(x) \to x \\ a \to f(C^2(a)) \\ f(x) \to h(x, x). \end{cases}$$

By COROLLARY 3.12, $\mathcal{R}^\tau$ is confluent iff $\mathcal{R}$ is confluent.

Later, readers will find that THEOREM 4.19 can not deal with this example because of $(r4)$.

4 Labelling via naive sort-attachment

4.1 Sorting of term rewriting systems

Let $\mathcal{S}$ be a set of sorts and $\mathcal{F}$ a set of arity-fixed function symbols. A *sort-attachment* τ of $\mathcal{F}$ on $\mathcal{S}$ is a family of mapping $\bigcup_n (\mathcal{F}_n \to \mathcal{S}^{n+1}) \cup (\mathcal{V} \to \mathcal{S})$. Without loss of generality we assume that there are countably infinite variables x with $\tau(x) = \alpha$ for each $\alpha \in \mathcal{S}$. If τ is obvious from its context, $f \in \mathcal{F}$ with $\tau(f) = \langle \alpha_1, \ldots, \alpha_n, \beta \rangle$ is written as $f : \alpha_1 \times \cdots \times \alpha_n \to \beta$.

Let τ be a sort-attachment. The set $\mathcal{T}(\mathcal{F}, \mathcal{V})^\alpha$ of terms that are *well-sorted under τ with sort α* is defined by (1) if $x \in \mathcal{V}$ and $\tau(x) = \alpha$ then $x \in \mathcal{T}(\mathcal{F}, \mathcal{V})^\alpha$; (2) if $f : \alpha_1 \times \cdots \times \alpha_n \to \alpha \in \mathcal{F}$ and $t_i \in \mathcal{T}(\mathcal{F}, \mathcal{V})^{\alpha_i}$ for $i = 1, \ldots, n$ then $f(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{F}, \mathcal{V})^\alpha$. It is clear that if t is well-sorted under τ then its sort α is uniquely defined. Thus α is called *the sort of t under τ*. The similar definition for well-sorted contexts (under τ) proceeds similarly by introducing special constants $\Box_\alpha$ with $\tau(\Box_\alpha) = \alpha$ for each $\alpha \in \mathcal{S}$. We write $C : \alpha_1 \times \cdots \times \alpha_n \to \beta$ iff $C \in \mathcal{T}(\mathcal{F} \cup \{\Box_\alpha \mid \alpha \in \mathcal{S}\}, \mathcal{V})^\alpha$ is a well-sorted context with n holes $\Box_{\alpha_1}, \ldots, \Box_{\alpha_n}$ from left to right in it.

Let $\langle \mathcal{F}, \mathcal{R} \rangle$ be a TRS. It may be assumed that the variables have been renamed appropriately so that every rewrite rules in $\mathcal{R}$ share no variables. A sort-attachment τ of $\mathcal{F}$ on $\mathcal{S}$ is said to be *consistent with* $\mathcal{R}$ iff for any rewrite rule $l \to r \in \mathcal{R}$, l and r are well-sorted under τ with the restriction that l and r are derived to be of the same sort. The sort of l (and r) is called the sort of the rewrite rule. $\langle \mathcal{F}, \mathcal{R} \rangle$ is said to be *well-sorted on* $\mathcal{S}$ iff there exists a sort-attachment of $\mathcal{F}$ on $\mathcal{S}$ that is consistent with $\mathcal{R}$.

4.2 Naive sort-attachment and labelling

In this subsection, we introduce a labelling via naive sort-attachment. It is shown that consistent naive sort-attachments yield sound and complete transformations of TRSs w.r.t. confluence and termination.

DEFINITION 4.1. A *naive* sort-attachment τ is a sort-attachment on $\mathsf{2}_\perp$ such that for any $f \in \mathcal{F}$, $f : \perp \times \cdots \times \perp \to \perp$ or $f : \alpha_1 \times \cdots \times \alpha_n \to i$ with $\alpha_1, \ldots, \alpha_n \in \{\perp, i\}$ for some $i \in \{1, 2\}$.

In the rest of this subsection, we assume that a naive sort-attachment τ on $\mathsf{2}_\perp$ is fixed. We write as $\mathcal{F}_\beta$ the set $\{f \in \mathcal{F} \mid f : \alpha_1 \times \cdots \times \alpha_n \to \beta$ for some $\alpha_1, \ldots, \alpha_n\}$ for any $\beta \in \mathsf{2}_\perp$.

DEFINITION 4.2. [top sort] The *top sort* of a term t is defined by

$$top(t) = \begin{cases} \tau(t) & \text{if } t \in \mathcal{V}, \\ \beta & \text{if } t \equiv f(t_1, \ldots, t_n) \text{ with } f : \alpha_1 \times \cdots \times \alpha_n \to \beta. \end{cases}$$

DEFINITION 4.3. [principal subterm] Let $t \equiv C[t_1, \ldots, t_n]$ $(n \geq 0)$ be a term with $C[, \ldots,] \not\equiv \Box_\alpha$. We write $t \equiv C[\![t_1, \ldots, t_n]\!]$ iff

1. $C : \alpha_1 \times \cdots \times \alpha_n \to \beta$;
2. $top(t_i) \neq \alpha_i$ for $i = 1, \ldots, n$.

The $t_1, \ldots, t_n$ are said to be the *principal* subterms of t.

It is clear that a term t is uniquely written as $C[\![t_1, \ldots, t_n]\!]$ for some $C, t_1, \ldots, t_n$.

We write as $\mathcal{F}_\perp^i$ the set $\{f^i \mid f \in \mathcal{F}_\perp\}$ $(i \in \{1, 2\})$. Elements of the set $\bigcup_{i\in\{1,2\}} \mathcal{F}_\perp^i$ are called *labelled symbols*. We write as $\mathcal{F}^\tau$ the set $\bigcup_{i\in\{1,2\}}(\mathcal{F}_\perp^i \cup \mathcal{F}_i)$. Terms in $\mathcal{T}(\mathcal{F}^\tau, \mathcal{V})$ are said to be *labelled terms*. Terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$ are said to be *unlabelled*. Given a term t, t^i denotes the term obtained from t by labelling all symbols in $\mathcal{F}_\perp$ with i $(i \in \{1, 2\})$. This convention is also adapted for contexts.

We next define top-down labelling $\triangledown_\tau^i$ $(i \in \{1, 2\})$ simultaneously.

DEFINITION 4.4. Let t be an unmarked term such that $t \equiv C[\![t_1, \ldots, t_n]\!]$ with $C : \alpha_1 \times \cdots \times \alpha_n \to \beta$. Marked terms $\triangledown_\tau^i t$ $(i \in \{1, 2\})$ are defined by

$$\triangledown_\tau^i t \equiv \begin{cases} C^i[\triangledown_\tau^i t_1, \ldots, \triangledown_\tau^i t_n] & \text{if } \beta = \perp, \\ C^\beta[\Phi_1 t_1, \ldots, \Phi_n t_n] & \text{if } \beta \in \{1, 2\}, \end{cases}$$

where

$$\Phi_j = \begin{cases} \triangledown_\tau^\beta & \text{if } \alpha_j = \perp, \\ \triangledown_\tau^i & \text{otherwise.} \end{cases}$$

In the rest of this subsection, we assume that a TRS $\langle \mathcal{F}, \mathcal{R} \rangle$ is fixed and τ is consistent with it. We write as $\mathcal{R}_i$ the set of rules in $\mathcal{R}$ of the sort i ($i \in \{1, 2\}$).

Now define $\mathcal{R}^\tau$ as

$$\mathcal{R}^\tau = \bigcup_{i \in \{1,2\}} \{l^i \to r^i \mid l \to r \in (\mathcal{R}_\perp \cup \mathcal{R}_i)\}.$$

It is clear that $\langle \mathcal{F}^\tau, \mathcal{R}^\tau \rangle$ is a TRS.

Our goal in this subsection is to show the THEOREM 4.12. But firstly we introduce some basic definitions.

DEFINITION 4.5. [rank] The *rank* of a term t is defined by

$$rank(t) = \begin{cases} 1 & \text{if } t \text{ is well-sorted,} \\ 1 + max\{rank(t_i) \mid 1 \leq i \leq n\} & \text{if } t \equiv C[\![t_1, \ldots, t_n]\!] \text{ with } n \geq 1. \end{cases}$$

DEFINITION 4.6. [inner and outer rewrite step] A rewrite step $s \to t$ is said to be *inner* (written as $s \to^i t$) iff

$$s \equiv C[\![s_1, \ldots, C'[l\sigma], \ldots, s_n]\!] \to C[s_1, \ldots, C'[r\sigma], \ldots, s_n] \equiv t$$

for some $s_1, \ldots, s_n$, $l \to r$, σ and C', otherwise *outer* (written as $s \to^o t$).

DEFINITION 4.7. [destructive rewrite step] A rewrite step $s \to^o t$ is said to be *destructive* iff $top(s) \neq top(t)$.

PROPOSITION 4.8. A rewrite step $s \to^o t$ is destructive iff $t \equiv \sigma(x)$ and $s \equiv C[\![s_1, \ldots, \sigma(x), \ldots, s_n]\!]$ for some $s_1, \ldots, s_n, C$ and σ such that $C[s_1, \ldots, \Box, \ldots, s_n] \equiv C'\sigma$ for some $C'[x] \to x \in \mathcal{R}$.

Proof. (If)-part. Suppose $top(s) = top(t)$ i.e. $top(C[s_1, \ldots, \sigma(x), \ldots, s_n]) = top(\sigma(x))$. Since $C' \not\equiv \Box$ by definition of rewrite rules, $top(C'[x]) = top(\sigma(x))$. Hence by consistency, $top(x) = top(C'[x]) = top(\sigma(x))$. From this it follows that $C[s_1, \ldots, \Box, \ldots, s_n] \equiv C'\sigma : top(x) \to top(s)$. Thus $\sigma(x)$ can not be principal. (Only if)-part. Suppose $top(s) \neq top(t)$. By consistency, this can occur only in the case that this reduction is an application of a collapsing rule and whole term s is the redex of this reduction. Let the rule be $C'[x] \to x$. Then $s \equiv C'[x]\sigma$ and $t \equiv \sigma(x)$. By consistency, $C'[x]$ is well-sorted and $top(C'[x]) = top(x)$. But $top(C'[x]) = top(C'[x]\sigma) = top(s) \neq top(t) = top(\sigma(x))$ by assumption. Hence $top(x) \neq top(\sigma(x))$. Then since $C[s_1, \ldots, \Box, \ldots, s_n] : top(x) \to top(s)$, $\sigma(x)$ is a principal subterm of s. $\Box$

This proposition shows that a destructive rewrite step occurs only when the applied rule is collapsing, and that the reduct of a destructive rewrite step indeed comes from one of the principal subterms.

We write $t \equiv C\langle\langle t_1, \ldots, t_n \rangle\rangle$ if either $t \equiv C[\![t_1, \ldots, t_n]\!]$ or $C \equiv \Box$ and $t \equiv t_1$.

PROPOSITION 4.9. [Case analysis]

1. If $s \to^o t$ then

$$\begin{cases} s \equiv C[\![s_1, \ldots, s_n]\!], \\ t \equiv C'\langle\langle s_{i_1}, \ldots, s_{i_m} \rangle\rangle \qquad (i_1, \ldots, i_m \in \{1, \ldots, n\}) \end{cases}$$

   for some $C, C', s_1, \ldots, s_n$ and either

   (a) $n = m = 0$ and s, t is well-sorted;

   (b) $n \neq 0$ and $s \to^o t$ is destructive and $C' \equiv \Box$ and $t \equiv s_j$ for some $1 \leq j \leq n$; or

   (c) $n \neq 0$ and $s \to^o t$ is not destructive and $t \equiv C'[\![s_{i_1}, \ldots, s_{i_m}]\!]$.

2. If $s \to^i t$ then

$$\begin{cases} s \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!], \\ t \equiv C[s_1, \ldots, t_j, \ldots, s_n], \\ s_j \to t_j \end{cases}$$

   for some $C, s_1, \ldots, s_n, t_j$ and either

   (a) $t \equiv C[\![s_1, \ldots, t_j, \ldots, s_n]\!]$; or

   (b) $s_j \to t_j$ is destructive, t_j is a principal subterm of s_j,

$$\begin{cases} t_j \equiv C'[\![u_1, \ldots, u_l]\!], \\ t \equiv C''[\![s_1, \ldots, u_1, \ldots, u_l, \ldots, s_n]\!] \end{cases}$$

   and $C'' \equiv C[\Box, \ldots, C', \ldots, \Box]$ for some $C', u_1, \ldots, u_l$.

Proof. Straightforward. □

PROPOSITION 4.10. Suppose that $C[\![s_1, \ldots, s_n]\!] \to^o C'\langle\langle s_{i_1}, \ldots, s_{i_m} \rangle\rangle$ and $C : \alpha_1 \times \cdots \times \alpha_n \to \beta$. Then for any $t_1, \ldots, t_n$ such that $\alpha_i = \alpha_j$ and $s_i \equiv s_j$ implies $t_i \equiv t_j$ (for any $1 \leq i, j \leq n$), $C[t_1, \ldots, t_n] \to^o C'[t_{i_1}, \ldots, t_{i_m}]$.

Proof. Straightforward. □

PROPOSITION 4.11. Suppose $C[\![s_1, \ldots, s_n]\!] \to^o s_j$ be a destructive rewrite step. If $\nabla^i_\tau(C[\![s_1, \ldots, s_n]\!]) \equiv C'[\Phi_1 s_1, \ldots, \Phi_n s_n]$ then $\nabla^i_\tau = \Phi_j$. Here $C' \equiv C^{top(C)}$ if $top(C) \in \{1, 2\}$, otherwise $C' \equiv C^i$.

Proof. Suppose $C : \alpha_1 \times \cdots \times \alpha_n \to \beta$. By PROPOSITION 4.8, destructive step is an application of some $C'[x] \to x \in \mathcal{R}$ such that $C[s_1, \ldots, \Box, \ldots, s_n] \equiv C'\sigma$. Hence $\alpha_j = \beta = top(C)$. Therefore $\nabla^i_\tau = \Phi_j$ by definition. □

THEOREM 4.12. For any unlabelled terms s, t and for any $i \in \{1, 2\}$, $s \to_{\mathcal{R}} t$ iff $\nabla^i_\tau s \to_{\mathcal{R}^\tau} \nabla^i_\tau t$.

Proof. (If)-part. It suffices to erase all labels. (Only if)-part. We suppose $s \to_{\mathcal{R}} t$ and show that for any $i \in \{1, 2\}$, $\nabla^i_\tau s \to_{\mathcal{R}} \nabla^i_\tau t$. Our proof proceeds by induction on the rank of s. Base case is obvious. Suppose that $rank(s) > 1$.

1. Case where $s \to^o t$. Then

$$\begin{cases} s \equiv C[\![s_1, \ldots, s_n]\!], \\ t \equiv C'\langle\!\langle s_{i_1}, \ldots, s_{i_m} \rangle\!\rangle \qquad (i_1, \ldots, i_m \in \{1, \ldots, n\}) \end{cases}$$

   for some $C, C', s_1, \ldots, s_n$.

   (a) Case where $top(C) = \bot$. Note that the applied rule is in $\mathcal{R}_\bot$. Then $\nabla^i_\tau s \equiv C^i[\nabla^i_\tau s_1, \ldots, \nabla^i_\tau s_n]$ and $\nabla^i_\tau t \equiv C'^i[\nabla^i_\tau s_{i_1}, \ldots, \nabla^i_\tau s_{i_m}]$. Since $s_j \equiv s_k$ implies $\nabla^i_\tau s_j \equiv \nabla^i_\tau s_k$, $C[\nabla^i_\tau s_1, \ldots, \nabla^i_\tau s_n] \to_{\mathcal{R}_\bot} C'[\nabla^i_\tau s_{i_1}, \ldots, \nabla^i_\tau s_{i_m}]$ by PROPOSITION 4.10. Then $\nabla^i_\tau s \to_{\mathcal{R}^\tau} \nabla^i_\tau t$ by definition of $\mathcal{R}^\tau$.

   (b) Case where $top(C) = k \in \{1, 2\}$. Let $C : \alpha_1 \times \cdots \times \alpha_n \to k$. Note that the applied rule is in $\mathcal{R}_k \cup \mathcal{R}_\bot$. Then $\nabla^i_\tau s \equiv C^k[\Phi_1 s_1, \ldots, \Phi_n s_n]$ and $\nabla^i_\tau t \equiv C'^k[\Phi_{i_1} s_{i_1}, \ldots, \Phi_{i_m} s_{i_m}]$. Suppose $C' \equiv \Box$. Then $s \to^o t$ is destructive and $t \equiv s_j$ for some j. Then by PROPOSITION 4.11, $\nabla^i_\tau = \Phi_j$. Also since $s_l \equiv s_{l'}$ and $\alpha_l \equiv \alpha_{l'}$ implies $\Phi_l s_l \equiv \Phi_{l'} s_{l'}$, $C[\Phi_1 s_1, \ldots, \Phi_n s_n] \to_{\mathcal{R}_k \cup \mathcal{R}_\bot} \Phi_j s_j$ by PROPOSITION 4.10. Thus $\nabla^i_\tau s \equiv C^k[\Phi_1 s_1, \ldots, \Phi_n s_n] \to_{\mathcal{R}^\tau} \Phi_j s_j \equiv \nabla^i_\tau t$. Otherwise, $C' \not\equiv \Box$. Then by the similar arguments as above, $C[\Phi_1 s_1, \ldots, \Phi_n s_n] \to_{\mathcal{R}_k \cup \mathcal{R}_\bot} C'[\Phi_{i_1} s_{i_1}, \ldots, \Phi_{i_m} s_{i_m}]$ by PROPOSITION 4.10. Thus $\nabla^i_\tau s \to_{\mathcal{R}^\tau} \nabla^i_\tau t$.

2. Case where $s \to^i t$. Then

$$\begin{cases} s \equiv C[\![s_1, \ldots, s_j, \ldots, s_n]\!], \\ t \equiv C[s_1, \ldots, t_j, \ldots, s_n], \\ s_j \to t_j \end{cases}$$

   for some $C, s_1, \ldots, s_n, t_j$.

(a) Case where $t \equiv C[\![s_1, \ldots, t_j, \ldots, s_n]\!]$. Then it immediately follows from induction hypothesis.

(b) Case where $s_j \to t_j$ is destructive, t_j is a principal subterm of s_j,

$$\begin{cases} t_j \equiv C'[\![u_1, \ldots, u_l]\!], \\ t \equiv C''[\![s_1, \ldots, u_1, \ldots, u_l, \ldots, s_n]\!] \end{cases}$$

and $C'' \equiv C[\Box, \ldots, C', \ldots, \Box]$ for some $C', u_1, \ldots, u_l$.

i. Case where $top(C) = \bot$. Then $\triangledown_\tau^i s \equiv C^i[\triangledown_\tau^i s_1, \ldots, \triangledown_\tau^i s_n]$ and $\triangledown_\tau^i t \equiv C''^i[\triangledown_\tau^i s_1, \ldots, \triangledown_\tau^i u_1, \ldots, \triangledown_\tau^i u_l, \ldots, \triangledown_\tau^i s_n] \equiv C^i[\triangledown_\tau^i s_1, \ldots, C'^i[\triangledown_\tau^i u_1, \ldots, \triangledown_\tau^i u_l], \ldots, \triangledown_\tau^i s_n]$. Since $top(C') = top(C) = \bot$ by well-sortedness, $\triangledown_\tau^i t_j \equiv C'^i[\triangledown_\tau^i u_1, \ldots, \triangledown_\tau^i u_l]$. Therefore $\triangledown_\tau^i t \equiv C^i[\triangledown_\tau^i s_1, \ldots, \triangledown_\tau^i t_j, \ldots, \triangledown_\tau^i s_n]$. Then $\triangledown_\tau^i s \to_{\mathcal{R}^\tau} \triangledown_\tau^i t$ follows from induction hypothesis.

ii. Case where $top(C) = k \in \{1, 2\}$. Let $C : \alpha_1 \times \cdots \times \alpha_n \to k$. Then $\triangledown_\tau^i s \equiv C^k[\Phi_1 s_1, \ldots, \Phi_n s_n]$ and $\triangledown_\tau^i t \equiv C''^k[\Phi_1 s_1, \ldots, \Phi'_1 u_1, \ldots, \Phi'_l u_l, \ldots, \Phi_n s_n] \equiv C^k[\Phi_1 s_1, \ldots, C'^k[\Phi'_1 u_1, \ldots, \Phi'_l u_l], \ldots, \Phi_n s_n]$.

A. Case where $top(C') = \bot$. Then $C' : \bot \times \cdots \times \bot \to \bot$ by well-sortedness, and hence $\Phi'_i = \Phi_j$ for all $i = 1, \ldots, l$ by definition. Also, by well-sortedness, $\alpha_j = top(C') = \bot$ and hence $\Phi_j = \triangledown_\tau^k$ by definition. Therefore $\triangledown_\tau^i t \equiv C^k[\Phi_1 s_1, \ldots, C'^k[\triangledown_\tau^k u_1, \ldots, \triangledown_\tau^k u_l], \ldots, \Phi_n s_n] \equiv C^k[\Phi_1 s_1, \ldots, \Phi_j(C'[u_1, \ldots, u_l]), \ldots, \Phi_n s_n] \equiv C^k[\Phi_1 s_1, \ldots, \Phi_j t_j, \ldots, \Phi_n s_n]$. Then $\triangledown_\tau^i s \to_{\mathcal{R}^\tau} \triangledown_\tau^i t$ follows from induction hypothesis.

B. Case where $top(C') \in \{1, 2\}$. Then by well-sortedness, $top(C') = k$. Therefore $\triangledown_\tau^i t \equiv C^k[\Phi_1 s_1, \ldots, \triangledown_\tau^i(C'[u_1, \ldots, u_l]), \ldots, \Phi_n s_n]$. Also by well-sortedness, $k = \alpha_j$. Hence $\Phi_j = \triangledown_\tau^i$ by definition. Therefore $\triangledown_\tau^i t \equiv C[\Phi_1 s_1, \ldots, \Phi_j t_j, \ldots, \Phi_n s_n]$. Then $\triangledown_\tau^i s \to_{\mathcal{R}^\tau} \triangledown_\tau^i t$ follows from induction hypothesis.

□

The followings are immediate corollaries of THEOREM 4.12.

COROLLARY 4.13. Let $\langle \mathcal{F}, \mathcal{R} \rangle$ be a TRS and τ a naive sort-attachment that is consistent with it. Then $\mathcal{R}$ is terminating iff $\mathcal{R}^\tau$ is terminating.

COROLLARY 4.14. Let $\langle \mathcal{F}, \mathcal{R} \rangle$ be a TRS and τ a naive sort-attachment that is consistent with it. Then $\mathcal{R}$ is confluent iff $\mathcal{R}^\tau$ is confluent.

Thus for a consistent naive sort-attachment τ, $^\tau$ is a sound and complete transformation w.r.t. both termination and confluence.

4.3 Combination with a common lower-sorted part

In this subsection, we introduce a combination with a common lower-sorted part, which is naturally induced from definition of consistency. We show various modularity results for this combination.

DEFINITION 4.15. Let $\langle \mathcal{F}, \mathcal{R} \rangle$ be a TRS. Then $\mathcal{R}$ is said to be *a system with lower-sorted symbols* $\mathcal{F}_\perp$ iff $\langle \mathcal{F}, \mathcal{R} \rangle$ is well-sorted on sorts $\{\perp, i\}$ with constraints:

1. $f : \perp \times \cdots \times \perp \to \perp$ for any $f \in \mathcal{F}_\perp$;
2. $f : \alpha_1 \times \cdots \times \alpha_n \to i$ with $\alpha_1, \ldots, \alpha_n \in \{\perp, i\}$ for any $f \in \mathcal{F} \setminus \mathcal{F}_\perp$.

DEFINITION 4.16. A combination of TRSs $\langle \mathcal{F}_1, \mathcal{R}_1 \rangle$ and $\langle \mathcal{F}_2, \mathcal{R}_2 \rangle$ such that

1. both are the systems with lower-sorted symbols $\mathcal{F}_1 \cap \mathcal{F}_2$; and
2. $\mathcal{R}_1 \mid (\mathcal{F}_1 \cap \mathcal{F}_2) = \mathcal{R}_2 \mid (\mathcal{F}_1 \cap \mathcal{F}_2)$ (called *a common lower-sorted part*);

is said to be a *combination with a common lower-sorted part*. Then the system $\langle \mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_1 \cup \mathcal{R}_2 \rangle$ is said to be a *combined system with a common lower-sorted part*.

In the sequel, assume $\langle \mathcal{F}_1, \mathcal{R}_1 \rangle$ and $\langle \mathcal{F}_2, \mathcal{R}_2 \rangle$ are TRSs with lower-sorted symbols $\mathcal{F}_0 = \mathcal{F}_1 \cap \mathcal{F}_2$ such that $\mathcal{R}_\perp = \mathcal{R}_1 \mid \mathcal{F}_\perp = \mathcal{R}_2 \mid \mathcal{F}_\perp$.

THEOREM 4.17. Termination is modular for combinations with a common lower-sorted part if either

1. there are no collapsing rules in the combined system;
2. there are no duplicating rules in the combined system; or
3. there are neither collapsing rules nor duplicating rules in one of the systems.

Proof. Let $\langle \mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_1 \cup \mathcal{R}_2 \rangle$ be a combined system of $\langle \mathcal{F}_1, \mathcal{R}_1 \rangle$ and $\langle \mathcal{F}_2, \mathcal{R}_2 \rangle$ with a common lower-sorted part. Without loss of generality, we can assume variables in $\mathcal{R}_1$ and $\mathcal{R}_2$ are disjoint, hence by the condition of combination, there exists a naive sort-attachment τ of $\mathcal{F}_1 \cup \mathcal{F}_2$ on the sort $\{\perp, 1, 2\}$ whose $\mathcal{F}_1$-part is on the sort $\{\perp, 1\}$ and $\mathcal{F}_2$-part is on the sorts $\{\perp, 2\}$. We show

$$\begin{array}{l} \langle \mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_1 \cup \mathcal{R}_2 \rangle \text{ is terminating} \\ \quad \Leftrightarrow \quad \text{both of } \langle \mathcal{F}_1, \mathcal{R}_1 \rangle \text{ and } \langle \mathcal{F}_2, \mathcal{R}_2 \rangle \text{ are terminating.} \end{array}$$

The implication from left to right immediately follows.

To prove the implication of the other direction, suppose that $\langle \mathcal{F}_1, \mathcal{R}_1 \rangle$ and $\langle \mathcal{F}_2, \mathcal{R}_2 \rangle$ are terminating. Then $\{l^1 \to r^1 \mid l \to r \in \mathcal{R}_1\}$ and $\{l^2 \to r^2 \mid l \to r \in \mathcal{R}_2\}$ are terminating and disjoint. Then by modularity of termination for disjoint union of TRSs (see e.g. [6]), $\{l^1 \to r^1 \mid l \to r \in \mathcal{R}_1\} \cup \{l^2 \to r^2 \mid l \to r \in \mathcal{R}_2\} = (\mathcal{R}_1 \cup \mathcal{R}_2)^\tau$ is terminating. Therefore $\mathcal{R}_1 \cup \mathcal{R}_2$ is terminating by COROLLARY 4.13. □

A property $\mathcal{P}$ of TRSs is said to be *persistent* if for any many-sorted TRS $\langle \mathcal{F}, \mathcal{R} \rangle$, $\langle \mathcal{F}, \mathcal{R} \rangle$ has the property $\mathcal{P}$ iff its underlying unsorted TRS $\langle \Theta(\mathcal{F}), \Theta(\mathcal{R}) \rangle$ has the property $\mathcal{P}$. Here Θ denotes the sort elimination function. The notion of persistency is formulated by H. Zantema [11]. He showed some of persistent properties including confluence and termination are also modular w.r.t. disjoint union. Thus the notion of persistency is thought to be an extension of the notion of modularity. Also he showed that termination is persistent for TRSs in the case of condition 1,2 above, generalizing a similar result for modularity.

Our results with conditions 1,2 are just the case that persistency of termination is applicable to infer termination of whole system from its subsystems. For, without loss of generality, we assume such system is divided into rules of sort $\bot, 1$ and 2 such that any function symbol which returns a value of sort $\bot$ takes only arguments of sort $\bot$, any function symbol which returns a value of sort 1 does not take arguments of sort 2 and vice versa. Then lower-sorted symbols are precisely those which return values of sort $\bot$. Our result with condition 3 is not in [11].

EXAMPLE 4.18. Let $\mathcal{R}$ be

$$\begin{cases} f(x, g(b, a, y)) \to f(x, g(c, y, y)) & (r1) \\ f(x, y) \to y & (r2) \\ a \to b & (r3) \\ F(c, x) \to F(a, x) & (r4). \end{cases}$$

Now, take $\mathcal{R}_1 = \{(r1), (r2), (r3)\}$ and $\mathcal{R}_2 = \{(r3), (r4)\}$. Then $\mathcal{R}$ is a combined system with a common lower-sorted part since a naive sort-attachment τ

$$\begin{cases} f : \bot \times 1 \to 1 \\ g : \bot \times \bot \times \bot \to 1 \\ a, b, c : \bot \\ F : \bot \times \bot \to 2 \end{cases}$$

is consistent with it.

Then putting $a > b > c$ yields termination of $\mathcal{R}_1$ by lexicographic path ordering. Also, putting $c > a > b$ yields termination of $\mathcal{R}_2$ by lexicographic

path ordering. Moreover the combined system satisfies condition 3 of THEOREM 4.17. Hence $\mathcal{R}$ is terminating by THEOREM 4.17.

Note that $\mathcal{R}^\tau$ becomes

$$\begin{cases} f(x, g(b^1, a^1, y)) \to f(x, g(c^1, y, y)) \\ f(x, y) \to y \\ a^1 \to b^1 \\ a^2 \to b^2 \\ F(c^2, x) \to F(a^2, x). \end{cases}$$

By COROLLARY 4.13, $\mathcal{R}^\tau$ is terminating iff $\mathcal{R}$ is terminating.

Note that THEOREM 3.15 can not deal with this example because of $(r2)$.

THEOREM 4.19. Confluence is modular for combinations with a common lower-sorted part.

Proof. Let $\langle \mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_1 \cup \mathcal{R}_2 \rangle$ be a combined system of $\langle \mathcal{F}_1, \mathcal{R}_1 \rangle$ and $\langle \mathcal{F}_2, \mathcal{R}_2 \rangle$ with a common lower-sorted part. Without loss of generality, we can assume variables in $\mathcal{R}_1$ and $\mathcal{R}_2$ are disjoint, hence by the condition of combination, there exists a naive sort-attachment τ of $\mathcal{F}_1 \cup \mathcal{F}_2$ on the sort $\{\bot, 1, 2\}$ whose $\mathcal{F}_1$-part is on the sort $\{\bot, 1\}$ and $\mathcal{F}_2$-part is on the sorts $\{\bot, 2\}$. We show

$$\begin{array}{l} \langle \mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_1 \cup \mathcal{R}_2 \rangle \text{ is confluent} \\ \quad \Leftrightarrow \quad \text{both of } \langle \mathcal{F}_1, \mathcal{R}_1 \rangle \text{ and } \langle \mathcal{F}_2, \mathcal{R}_2 \rangle \text{ are confluent.} \end{array}$$

The implication from left to right immediately follows since $((\mathcal{R}_1 \cup \mathcal{R}_2) \mid \mathcal{F}_i) = \mathcal{R}_i$ for $i \in \{1, 2\}$.

To prove the implication of the other direction, suppose that $\langle \mathcal{F}_1, \mathcal{R}_1 \rangle$ and $\langle \mathcal{F}_2, \mathcal{R}_2 \rangle$ are confluent. Then $\{l^1 \to r^1 \mid l \to r \in \mathcal{R}_1\}$ and $\{l^2 \to r^2 \mid l \to r \in \mathcal{R}_2\}$ are confluent and disjoint. Then by modularity of confluence for disjoint union of TRSs (e.g. see [3]), $\{l^1 \to r^1 \mid l \to r \in \mathcal{R}_1\} \cup \{l^2 \to r^2 \mid l \to r \in \mathcal{R}_2\} = (\mathcal{R}_1 \cup \mathcal{R}_2)^\tau$ is confluent. Therefore $\mathcal{R}_1 \cup \mathcal{R}_2$ is confluent by COROLLARY 4.14. □

The persistency of confluence is proved by the present authors [1]. The present result is just the case that persistency of confluence is applicable to infer confluence of whole system from its subsystems, as it is mentioned in the case of termination.

EXAMPLE 4.20. Let $\mathcal{R}$ be

$$\begin{cases} f(x, c) \to x & (r1) \\ f(x, h(y)) \to f(x, g(y)) & (r2) \\ g(c) \to h(c) & (r3) \\ F(g(x), x) \to F(x, g(x)) & (r4) \\ F(h(x), x) \to F(x, h(x)) & (r5). \end{cases}$$

To show the confluence of this system directly seems difficult from known results, since this system is neither terminating nor left-linear. Also modularity of confluence for disjoint union seems difficult to apply, since function symbols are overlapping among the rules. Also this is not a constructor system, hence COROLLARY 3.19 is not applicable.

Now, take $\mathcal{R}_1 = \{(r1), (r2), (r3)\}$ and $\mathcal{R}_2 = \{(r3), (r4), (r5)\}$. Then $\mathcal{R}$ is a combined system with a common lower-sorted part since a naive sort-attachment τ

$$\begin{cases} f : 1 \times \bot \to 1 \\ c : \bot \\ g : \bot \to \bot \\ h : \bot \to \bot \\ F : \bot \times \bot \to 2 \end{cases}$$

is consistent with it. $\mathcal{R}_1$ is confluent since it is orthogonal. Also $\mathcal{R}_2$ is confluent since it is terminating and critical pairs are joinable. Hence so is $\mathcal{R}$ by THEOREM 4.19

Note that $\mathcal{R}^\tau$ becomes

$$\begin{cases} f(x, c^1) \to x \\ f(x, h^1(y)) \to f(x, g^1(y)) \\ g^1(c^1) \to h^1(c^1) \\ g^2(c^2) \to h^2(c^2) \\ F(g^2(x), x) \to F(x, g^2(x)) \\ F(h^2(x), x) \to F(x, h^2(x)). \end{cases}$$

By COROLLARY 4.14, $\mathcal{R}^\tau$ is confluent iff $\mathcal{R}$ is confluent.

Note that THEOREM 3.18 can not deal with this example because of $(r1)$.

5 Concluding remarks

We have introduced two kinds of top-down labelling of function symbols: *labelling via signature function* and *labelling via naive sort-attachment*. We have presented conditions under which they yield sound and complete transformations of TRSs.

Making use of these labelling, we have shown modularity results for two kinds of combinations of TRSs: *a combination with a common transparent part* and *a combination with a common lower-sorted part*.

Our results give extensions of modularity of termination in [2], modularity application of [11] and modularity of confluence in [7]. Also it gives an alternative proof of modularity application of [1].

Although we did not present it, it is easily seen that one can similarly lift other modularity results for disjoint union of TRSs (e.g. [4], [9]) for these combinations of TRSs.

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